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| 引用      | 数理解析研究所講究録 |}

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<th>発行年月</th>
<th>1976-05</th>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/105947">http://hdl.handle.net/2433/105947</a></td>
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<td>部門</td>
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Kyoto University
INTEGRABLE PLURICANONICAL FORMS
and
KODAIRA DIMENSIONS OF COMPLEMENTS OF DIVISORS
Fumio SAKAI

Let $X$ be a complex manifold (possibly non-compact) of dimension $n$ and $\omega$ a holomorphic $m$-ple $n$-form on $X$. We write $\omega$ as $\omega = \psi(w) (dw_1 \wedge \cdots \wedge dw_n)^m$, using local coordinates $(w_1, \ldots, w_n)$. We associate with $\omega$ the continuous $(n,n)$-form $(\omega \wedge \overline{\omega})^{1/m}$, given locally by $|\psi(w)|^2/m \prod_{i=1}^n (\sqrt{-1}/2\pi) dw_i \wedge d\overline{w}_i$. Then $\omega$ is called integrable ($L^2/m$-integrable) if $\int_X (\omega \wedge \overline{\omega})^{1/m} < \infty$. Let $F_m(X)$ be the set of all integrable holomorphic $m$-ple $n$-forms on $X$. When $X$ has a compactification, $F_m(X)$ becomes a vector space. Using $F_m(X)$, we shall define the Kodaira dimension $\kappa(X)$ of $X$, which is a generalization of the notion of Kodaira dimension of compact complex manifolds introduced by Iitaka [8] (cf. Ueno [19]). Here we want to discuss the properties of $\kappa(X)$ and some related aspects. Details will appear in [17].

1. Kodaira Dimension.

Let $X$ be a complex manifold of dimension $n$ and $F_m(X)$ the set of all integrable holomorphic $m$-ple $n$-forms on $X$ as above. Set $N(X) = \{ m > 0 | F_m(X) \neq \{0\} \}$. If $m \in N(X)$, for a finite set of elements $\omega_0, \ldots, \omega_N \in F_m(X)$, we can define a meromorphic map $\phi_{\{\omega_0, \ldots, \omega_N\}} : X \rightarrow [\omega_0(w) : \cdots : \omega_N(w)]$ of $X$ into $\mathbb{P}^N$. Next we put $r_k = \max \{ \text{rank } \phi_{\{\omega_0, \ldots, \omega_N\}} \}$, where the maximum is taken over all choices of finite elements in $F_m(X)$ for $N=0,1,2,\ldots$. The rank of a meromorphic map is the maximum rank of the Jacobian matrix where it is holo-
morphic. Now we define the Kodaira dimension $\kappa(X)$ of $X$ by
\[
\kappa(X) = \begin{cases} 
\max \{ \text{rk}_m \} & \text{if } N(X) \neq \emptyset, \\
-\infty & \text{if } N(X) = \emptyset.
\end{cases}
\]
Note that $\kappa(X)$ takes one of the values $-\infty, 0, 1, \ldots, n$.

(1.2) Theorem([17]). The Kodaira dimension $\kappa(X)$ is a bimeromor-
phic (in the sense of Remmert) invariant of a complex manifold $X$.

Proof. Let $X'$ be a complex manifold such that there exists
a bimeromorphic map $f : X' \to X$. Then $f^*$ induces an isomorphism of
$F_m(X)$ onto $F_m(X')$. To see this, take an element $\omega \in F_m(X)$, then
$f^*\omega$ is a holomorphic $m$-ple $n$-form on $X'$-S(f), where $S(f)$ is the
set of points where $f$ is not holomorphic. Since codim $S(f) \geq 2$,
it extends holomorphically on $X'$. Clearly $\int_X (f^*\omega \wedge f^*\omega)^{1/m} =
\int_X (\omega \wedge \bar{\omega})^{1/m} < \infty$, which implies $f^*\omega \in F_m(X')$. Considering the inverse
map, we get the surjectivity. Consequently we have, by definitio
$\kappa(X') = \kappa(X)$. Q.E.D.

We list some properties of $\kappa(X)$ (cf.[17]).

1. Let $\mathbb{C}$ be the complex plane and $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then $\kappa(\mathbb{C}) = -\infty$,
$\kappa(\mathbb{C}^*) = -\infty$. Further $\kappa(\mathbb{C} \times Y) = -\infty$, $\kappa(\mathbb{C}^* \times Y) = -\infty$, for any
complex manifold $Y$.

2. Let $X, Y$ be complex manifolds of the same dimension such
that $X \subset Y$. Then $\kappa(X) \leq \kappa(Y)$. In particular, if $\kappa(X) = -\infty$,
we get $\kappa(Y) = -\infty$.

3. Let $X$ be a complex manifold and $Z$ an analytic subset of
$X$ with codim $Z \geq 2$. Then $\kappa(X - Z) = \kappa(X)$.

4. Let $X, Y$ be complex manifolds of the same dimension.
Suppose that there is a surjective proper meromorphic map \( f: X \to Y \). Then \( \kappa(X) \geq \kappa(Y) \).

In case \( X \) is a complex space, we define \( \kappa(X) \) to be \( \kappa(X^*) \), using a desingularization \( X^* \) of \( X \).

2. Complements of Divisors.

In this section, we deal with the case in which \( X \) has a compactification \( \overline{X} \). We assume that \( \overline{X} \) is a smooth compactification in the sense that \( \overline{X} \) is a compact complex manifold and \( D = \overline{X} - X \) is a divisor of normal crossings. Let \( K_{\overline{X}} \) be the canonical bundle of \( \overline{X} \) and \([D]\) the line bundle determined by \( D \). In this case, we have

(2.1) Theorem ([17]). \( F_m(X) \cong H^0(\overline{X}, O(mK_{\overline{X}} + (m-1)[D])) \).

The proof is based on the fact that if \( f(z)(dz)^m \) is integrable on the punctured disc \( \Delta^* \), then the Laurent expansion of \( f(z) \) becomes as \( \sum_{j=-(m-1)}^{\infty} a_j z^j \) ([14], Appendix).

(2.2) Definition. \( \gamma_m(X) = \dim F_m(X) = \dim H^0(\overline{X}, O(mK_{\overline{X}} + (m-1)[D])) \).

We can redefine the Kodaira dimension as follows.

(2.3) Definition. Let \( \psi_0, \ldots, \psi_N \) be a basis of \( H^0(\overline{X}, O(mK_{\overline{X}} + (m-1)[D])) \). Let \( \phi_m \) be the meromorphic map defined by \([\psi_0: \ldots: \psi_N]\) of \( \overline{X} \) into \( \mathbb{P}_N \).

Put \( N(X) = \{m > 0 | \dim H^0(\overline{X}, O(mK_{\overline{X}} + (m-1)[D])) > 0\} \). Then

\[
\kappa(X) = \begin{cases} 
\max \{\dim \phi_m(\overline{X})\} & \text{if } N(X) \neq \emptyset, \\
-\infty & \text{if } N(X) = \emptyset.
\end{cases}
\]

(2.4) Example. Let \( D \) be a hypersurface of degree \( d \) in \( \mathbb{P}_n \) which has at most normal crossings. Then \( \kappa(\mathbb{P}_n - D) = n \) if \( d > n + 1 \) and \( \kappa(\mathbb{P}_n - D) \)
=-\infty$ if $d\leq n+1$. Next put $U_a=\{z_1^{a_1}+\cdots+z_{n+1}^{a_{n+1}}=1\}$ in $\mathbb{P}^{n+1}$. Then $\kappa(U_a)=n$ if $\sum \frac{1}{a_i}<1$. Here we represent a classification of complements of finite points on compact curves.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\gamma_1=g$</th>
<th>$\gamma_m$ ($m\geq 2$)</th>
<th>structure</th>
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<tr>
<td>$-\infty$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{P}_1, \mathbb{P}_1-{a_1}, \mathbb{P}_1-{a_1}-{a_2}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>elliptic curve</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$m(k-2)-k+1$ (except $k=3$, $m=2$)</td>
<td>$\mathbb{P}<em>1-\bigcup</em>{i=1}^{k}{a_k}$, $k\geq 3$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$mk-k$</td>
<td>elliptic curve-$\bigcup_{i=1}^{k}{a_k}$, $k\geq 1$</td>
</tr>
<tr>
<td>$g\geq 2$</td>
<td>$m(k+2g-2)-k+1-g$</td>
<td></td>
<td>curve of genus $\geq 2-\bigcup_{i=1}^{k}{a_k}$, $k\geq 0$</td>
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Let $X$ be again a complex manifold of dimension $n$ and $\bar{X}$ a smooth compactification of $X$ with $D=\bar{X}-X$. We remark that $F_m(X)$ has an invariant Hermitian metric. So $F_m(X)$ is a finite dimensional Hilbert space ([17]). The Kodaira dimension $\kappa(X)$ has the following relation with $\kappa(K_X+D, \bar{X})$.

(2.5) Proposition. If $\kappa(X)\geq 0$, then $\kappa(X)=\kappa(K_X+D, \bar{X})$. Further $\kappa(X)=n$ if and only if $\kappa(K_X+D, \bar{X})=n$.

The first part of this relation also holds without the assumption that $D$ has normal crossings (See [17], Appendix).

(2.6) Remark. When $\bar{X}$ is a smooth compactification of $X$, Iitaka calls $\kappa(K_X+D, \bar{X})$ the logarithmic Kodaira dimension of $X$ and writes it by $\overline{\kappa}(X)$ ([9]). He proves that $\overline{\kappa}(X)$ is a proper birational invariant of $X$. From Theorem (1.2) and Proposition (2.5), it follows that if $\kappa(X)\geq 0$, then $\overline{\kappa}(X)$ is a bimeromorphic invariant. But the following examples show that in case $\kappa(X)=-\infty$, $\overline{\kappa}(X)$ need not be a bimeromorphic invariant of $X$. We consider several
compactifications of $\mathbb{C}^*$. We have, by (1.3) $\kappa(\mathbb{C}^*^2)=-\infty$.

1) $\mathbb{C}^*^2=\mathbb{P}^2_2-\bigcup_{i=1}^3 H_i$, with three lines $H_1$, $H_2$, $H_3$ in general position. In this case $\kappa(\mathbb{P}^2_2-\bigcup_{i=1}^3 H_i)=0$.

2) $\mathbb{C}^*^2=\mathbb{P}_1^4 \times \mathbb{P}_1^4 - \bigcup_{i=1}^4 L_i$, where $L_i=a_i \times \mathbb{P}_1^1$, $i=1,2$ and $L_i=b_i \times \mathbb{P}_1^1$, $i=3,4$. We have $\kappa(\mathbb{P}_1^4 \times \mathbb{P}_1^4 - \bigcup_{i=1}^4 L_i)=0$.

3) $\mathbb{C}^*^2=S-E$, where $S$ is a Hopf surface given by $S=\mathbb{C}^2-\{0\}/\langle g \rangle$ with $g:(z_1,z_2) \mapsto (\alpha z_1^{\ast}+\lambda z_2^{\ast},az_2)$, $\lambda \neq 0$, $0<|\alpha|<1$, for a positive integer $p$ and $E$ is an elliptic curve given by $E=\{(\mathbb{C}^2-\{0\})\cap \{z_2=0\}\}/\langle g \rangle$ (See [7], for details). In this case, we have $\kappa_S=-(p+1)[E]$ and then $\kappa(S-E)=-\infty$.

4) $\mathbb{C}^*^2=F-D$, where $F$ is a $\mathbb{P}_1$-bundle over an elliptic curve constructed by Serre ([5], p232) and $D$ is a section with $D^2=0$. We also have $\kappa(F-D)=-\infty$.

In case $X$ is given by $X=\bar{X}-D$ with a singular divisor $D$ on a compact complex manifold $\bar{X}$, it is not so easy to calculate $\kappa(X)$.

Here we give a method. According to Hironaka, there exists a desingularization $\pi:X^* \longrightarrow X$ such that $\pi^{-1}(D)=D^*$ has normal crossings. Let $\pi^{-1}(\text{Sing } D)=\bigcup_{i=1}^r S_i$ be the irreducible decomposition of the exceptional set of $\pi$. Let $R_\pi$ be the ramification divisor of $\pi$. Set $E_D=\pi^*D-D^*-R_\pi$. We can write $E_D=\sum_{i=1}^r t_i S_i$ with integers $t_i$.

(2.7) Definition (Shiffman [18]). Let $A$ be a divisor on $\bar{X}$ passing through the non-normal crossing points of $D$. If we write $\pi^*A=\bar{A}+\sum_{i=1}^r p^A_i S_i$, where $\bar{A}$ is the strict transform of $A$, then $p^A_i \geq 1$. Define $\gamma_{A,D}=\max_{i=1}^r \frac{t_i}{p^A_i}$, where $x^+$ means $\max(x,0)$.

(2.8) Proposition. We have

$$\gamma_m(X) \leq \dim H^0(\bar{X},O(mK_D)([D]-\gamma_{A,D}[A])))$$

$$\kappa(X)=\kappa(K_{\bar{X}}^{\ast}+D^{\ast},X^{\ast}) \geq \kappa(K_{\bar{X}}^{\ast}+D^{\ast}-\gamma_{A,D}[A],\bar{X})$$.
Proof. Note that \([\mathcal{E}_D] = \pi^*(K_X + [D]) - (K_{Y^*} + [D^*])\). We have, by definition \(\kappa(\gamma_{A,D^*}^n - \mathcal{E}_D, \mathcal{X}^*) \geq 0\). The assertion follows from this.

(2.9) Corollary. Let \(D\) be a singular hypersurface of degree \(d\) in \(\mathbb{P}_n\). Let \(A\) be a hypersurface of degree \(a\) in \(\mathbb{P}_n\) passing through the non-normal crossing points of \(D\). If \((d - n - 1 - \gamma_{A,D^a}) > 0\), then 
\[ \kappa(\mathbb{P}_n - D) = \kappa(\mathbb{P}_n - D) = n. \]

3. Quasi-Projective Manifolds with \(\kappa(X) = \dim X\).

A complex manifold is called a quasi-projective manifold if it is given as a complement of an analytic subset of a projective algebraic manifold. In [17], we prove the following facts.

(3.1) Theorem. Let \(X\) be a quasi-projective manifold of dimension \(n\). Assume that \(\kappa(X) = n\). Then \(X\) satisfies

1. Any non-degenerate holomorphic map \(f: \Delta^* \times \Delta^{n-1} \to X\) can be extended to a meromorphic map from \(\Delta^n\) to any compactification of \(X\). Here \(\Delta\) is the unit disc and \(\Delta^* = \Delta - \{0\}\).
   An equidimensional holomorphic map is called non-degenerate if the Jacobian does not vanish identically.
2. Every biholomorphic transformation of \(X\) extends as a meromorphic transformation of any compactification of \(X\).
3. Let \(\text{Aut}(X)\) be the group of biholomorphic transformations of \(X\). Then \(\text{Aut}(X)\) is a finite group.
4. \(X\) is measure-hyperbolic.
4'. Every holomorphic map \(f: \mathbb{C} \times \Delta^{n-1} \to X\) degenerates.

These properties show that in this case \(X\) behaves like a projective algebraic manifold of general type.

A. Let $\mathfrak{Q}$ be a bounded symmetric domain of dimension $n$ and $\Gamma$ a totally discontinuous group operating on $\mathfrak{Q}$ such that $X=\mathfrak{Q}/\Gamma$ has a compactification. Let $\pi: \mathfrak{Q} \rightarrow X$ be the projection. In many cases, the space $\pi^* F_m(X)$ corresponds to the vector space of cusp forms on $\mathfrak{Q}$ (for instance, see [6], [10]). So it is expected that this phenomenon holds in general. Moreover we have the following question: Let $X$ be a complex manifold of dimension $n$. If the universal covering manifold of $X$ is a bounded domain in $\mathbb{C}^n$, is it true that $\kappa(X)=n$?

B. Let $Y$ be a complex manifold of dimension $n$ and $Z$ an analytic subset of $Y$. We set $F_m^Z(Y) = \{ \omega \in H^0(Y-Z, O(mK))/H^0(Y, O(mK)) | \omega \text{ is locally integrable across } Z, \text{ i.e., for every point } x \in Z, \text{ there is a neighborhood } U \text{ of } x \text{ in } Y \text{ such that } \omega \text{ is integrable on } U-Z \cap U \}$. If codim $Z \geq 2$, then $F_m^Z(Y) = \{ 0 \}$. In case $Z$ is a divisor $D$ having normal crossings, then we obtain

$$F_m^D(Y) \cong H^0(Y, O(mK+(m-1)[D]))/H^0(Y, O(mK))$$

(cf. Theorem (2.1)). Take neighborhoods $U, U'$ of $Z$ in $Y$. If $U \supset U'$, then we have an inclusion $F_m^Z(U) \hookrightarrow F_m^Z(U')$. Hence we can define $\hat{F}_m^Z(Y) = \lim_{U \supset U'} F_m^Z(U)$. Put $\gamma_m^Z(Y) = \dim F_m^Z(Y)$ and $\gamma_m^\Lambda (Y) = \dim \hat{F}_m^Z(Y)$. Then $\gamma_m^Z(Y) \leq \gamma_m^\Lambda (Y) \leq O(m^n)$. Further we can define $\kappa^Z(Y)$ and $\kappa^\Lambda (Y)$ in a similar manner as in (1.1).

Next in case $Y$ is a complex space, letting $\pi: Y^* \rightarrow Y$ be a desingularization of $Y$, we put $\gamma_m^Z(Y) = \gamma_m^Y(Y^*)$, $\gamma_m^\Lambda (Y) = \gamma_m^Y(Y^*)$ with $Z^* = \pi^{-1}(Z)$.

(4.1) Proposition. Let $\overline{X}$ be a compact complex manifold of dimension $n$ and $D$ an effective divisor on $\overline{X}$. Put $X = \overline{X} - D$. Then

$$P_m(X) \leq \gamma_m(X) \leq P_m(\overline{X}) + \gamma_m^D(\overline{X}).$$
Proof. This follows from the exact sequence
\[ 0 \longrightarrow H^0(\overline{X}^*, O(mK_{\overline{X}^*})) \longrightarrow H^0(\overline{X}^*, O(mK_{\overline{X}^*}+D^*)) \longrightarrow F^D_m(\overline{X}) \longrightarrow 0, \]
where $\overline{X}^*, D^*$ is a desingularization of $X$, $D$ such that $\overline{X}^*$ is a smooth compactification of $X$. Q.E.D.

We consider the special case in which $Z=y$ is an isolated singularity of an $n$-dimensional complex space $Y$. For simplicity, put $\gamma_m = \gamma_m^Y(Y)$. Let $\pi: Y^* \longrightarrow Y$ be a desingularization of $Y$.

For a neighborhood $U$ of $y$, put $U^* = \pi^{-1}(U)$. Define
\[ r_m = \dim \lim_{U} \big( H^0(U \setminus y, O(mK))/H^0(U^*, O(mK)) \big). \]
Put $\sigma_m = r_m - \gamma_m$. Then $\sigma_m > 0$. and $\gamma_1 = 0$. It is easily seen that if $y$ is a quotient singularity, then $\sigma_m = 0$ for all $m$ (cf. [1],[2]). Question: When $\sigma_m = 0$? When $\gamma_m = 0$?

(4.2) Example. Suppose that $\pi^{-1}(y) = E$ is $\mathbb{P}_{n-1}$ and $E|E \sim (-e)$, where (1) means the hyperplane bundle on $\mathbb{P}_{n-1}$. In this case, we get easily that $\sigma_m = 0$ for all $m$ and if $e \leq n$, then $\gamma_m = 0$ for all $m$.

In case $\dim Y = 2$, Laufer showed in [15] that $\gamma_1 = 0$ if and only if $y$ is a rational singularity. Precisely he proved $\dim R^1\pi_* O_{U^*} = \sigma_1$. Knöller [11] calculates $r_m$ and $\lim_{m \to \infty} r_m/m^2$ for several singularities. In particular, the condition $r_m = 0$ for all $m$ characterizes the rational double points (See also [12], for an application).

References


[9] ______: On logarithmic Kodaira dimension of algebraic varieties. to appear


[17] ______.: Kodaira dimensions of complements of divisors. to appear