<table>
<thead>
<tr>
<th>Title</th>
<th>Integrable Pluricanonical Forms and Kodaira Dimensions of Complements of Divisors (代数幾何とその近傍)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>SAKAI, FUMIO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 代数幾何とその近傍</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1976-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/105947">http://hdl.handle.net/2433/105947</a></td>
</tr>
<tr>
<td>Right</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
INTEGRABLE PLURICANONICAL FORMS
and
KODAIRA DIMENSIONS OF COMPLEMENTS OF DIVISORS

Fumio SAKAI

Let $X$ be a complex manifold (possibly non-compact) of dimension $n$ and $\omega$ a holomorphic $m$-ple $n$-form on $X$. We write $\omega$ as $\omega = \psi(w)(dw_1 \wedge \ldots \wedge dw_n)^m$, using local coordinates $(w_1, \ldots, w_n)$. We associate with $\omega$ the continuous $(n,n)$-form $(\omega \wedge \bar{\omega})^{1/m}$, given locally by $|\psi(w)|^{2/m} \prod_{i=1}^{n} (\sqrt{-1}/2\pi)dw_i \wedge d\bar{w}_i$. Then $\omega$ is called integrable ($L_2/m$-integrable) if $\int_X (\omega \wedge \bar{\omega})^{1/m} < \infty$. Let $F_m(X)$ be the set of all integrable holomorphic $m$-ple $n$-forms on $X$. When $X$ has a compactification, $F_m(X)$ becomes a vector space. Using $F_m(X)$, we shall define the Kodaira dimension $\kappa(X)$ of $X$, which is a generalization of the notion of Kodaira dimension of compact complex manifolds introduced by Iitaka [8] (cf. Ueno [19]). Here we want to discuss the properties of $\kappa(X)$ and some related aspects. Details will appear in [17].

1. Kodaira Dimension.

Let $X$ be a complex manifold of dimension $n$ and $F_m(X)$ the set of all integrable holomorphic $m$-ple $n$-forms on $X$ as above. Set $N(X) = \{m > 0 | F_m(X) \neq \{0\}\}$. If $m \in N(X)$, for a finite set of elements $\omega_0, \ldots, \omega_N \in F_m(X)$, we can define a meromorphic map $\Phi_{\{\omega_0, \ldots, \omega_N\}}: X \ni w \mapsto [\omega_0(w) : \cdots : \omega_N(w)]$ of $X$ into $\mathbb{P}_N$. Next we put $r_k = \max[\text{rank } \Phi_{\{\omega_0, \ldots, \omega_N\}}]$, where the maximum is taken over all choices of finite elements in $F_m(X)$ for $N = 0, 1, 2, \ldots$. The rank of a meromorphic map is the maximum rank of the Jacobian matrix where it is holo-
morphic. Now we define the Kodaira dimension \( \kappa(X) \) of \( X \) by

\[
\kappa(X) = \begin{cases} 
\max \{ \text{rk}_m \} & \text{if } N(X) \neq \emptyset, \\
\infty & \text{if } N(X) = \emptyset.
\end{cases}
\]

Note that \( \kappa(X) \) takes one of the values \(-\infty, 0, 1, \ldots, n\).

(1.2) **Theorem**([17]). The Kodaira dimension \( \kappa(X) \) is a bimeromorphic (in the sense of Remmert) invariant of a complex manifold \( X \).

**Proof.** Let \( X' \) be a complex manifold such that there exists a bimeromorphic map \( f:X' \to X \). Then \( f^* \) induces an isomorphism of \( F_m(X) \) onto \( F_m(X') \). To see this, take an element \( \omega \in F_m(X) \), then \( f^* \omega \) is a holomorphic \( m \)-ple \( n \)-form on \( X' \)-S(f), where S(f) is the set of points where \( f \) is not holomorphic. Since codim S(f) \( \geq 2 \), it extends holomorphically on \( X' \). Clearly \( \int_{X'} (f^* \omega \wedge f^* \omega)^{-1/m} = \int_X (\omega \wedge \bar{\omega})^{-1/m} < \infty \), which implies \( f^* \omega \in F_m(X') \). Considering the inverse map, we get the surjectivity. Consequently we have, by definitio
\[ \kappa(X') = \kappa(X). \] Q.E.D.

We list some properties of \( \kappa(X) \) (cf.[17]).

1. Let \( \mathbb{C} \) be the complex plane and \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). Then \( \kappa(\mathbb{C}) = -\infty \), \( \kappa(\mathbb{C}^*) = -\infty \). Further \( \kappa(\mathbb{C} \times Y) = -\infty \), \( \kappa(\mathbb{C}^* \times Y) = -\infty \), for any complex manifold \( Y \).

2. Let \( X, Y \) be complex manifolds of the same dimension such that \( X \subseteq Y \). Then \( \kappa(X) \geq \kappa(Y) \). In particular, if \( \kappa(X) = -\infty \), we get \( \kappa(Y) = -\infty \).

3. Let \( X \) be a complex manifold and \( Z \) an analytic subset of \( X \) with codim \( Z \geq 2 \). Then \( \kappa(X - Z) = \kappa(X) \).

4. Let \( X, Y \) be complex manifolds of the same dimension.
Suppose that there is a surjective proper meromorphic map \( f: X \to Y \). Then \( \kappa(X) \geq \kappa(Y) \).

In case \( X \) is a complex space, we define \( \kappa(X) \) to be \( \kappa(X^*) \), using a desingularization \( X^* \) of \( X \).

2. Complements of Divisors.

In this section, we deal with the case in which \( X \) has a compactification \( \overline{X} \). We assume that \( \overline{X} \) is a smooth compactification in the sense that \( \overline{X} \) is a compact complex manifold and \( D=\overline{X}-X \) is a divisor of normal crossings. Let \( K_{\overline{X}} \) be the canonical bundle of \( \overline{X} \) and \([D]\) the line bundle determined by \( D \). In this case, we have

\[(2.1) \text{Theorem([17])}. \quad F_m(X) \cong H^0(\overline{X}, O(mK_{\overline{X}}+\langle m-1 \rangle [D])).\]

The proof is based on the fact that if \( f(z)(dz)^m \) is integrable on the punctured disc \( \Delta^* \), then the Laurent expansion of \( f(z) \) becomes as \( \sum_{j=-\langle m-1 \rangle}^{\infty} a_j z^j \) ([14], Appendix).

\[(2.2) \text{Definition}. \quad \gamma_m(X)=\dim F_m(X)=\dim H^0(\overline{X}, O(mK_{\overline{X}}+\langle m-1 \rangle [D])).\]

We can redefine the Kodaira dimension as follows.

\[(2.3) \text{Definition}. \quad \text{Let } \psi_0, \ldots, \psi_N \text{ be a basis of } H^0(\overline{X}, O(mK_{\overline{X}}+\langle m-1 \rangle [D])). \text{ Let } \phi_m \text{ be the meromorphic map defined by } [\psi_0: \ldots: \psi_N] \text{ of } \overline{X} \text{ into } \mathbb{P}_n. \text{ Put } N(X)=\{m>0 | \dim H^0(\overline{X}, O(mK_{\overline{X}}+\langle m-1 \rangle [D]))>0 \}. \text{ Then }\]

\[\kappa(X)=\begin{cases} \max \{\dim \phi_m(\overline{X})\} & \text{if } N(X)\neq \emptyset, \\ -\infty & \text{if } N(X)=\emptyset. \end{cases}\]

\[(2.4) \text{Example}. \quad \text{Let } D \text{ be a hypersurface of degree } d \text{ in } \mathbb{P}_n \text{ which has at most normal crossings. Then } \kappa(\mathbb{P}_n-D)=n \text{ if } d>n+1 \text{ and } \kappa(\mathbb{P}_n-D) \]
\(=-\infty\) if \(d\leq n+1\). Next put \(U_a=\{z_1^{a_1}+\ldots+z_n^{a_n+1}=1\}\) in \(\mathbb{P}^{n+1}\). Then \(\kappa(U_a)\)
\(=n\) if \(\sum a_i<1\). Here we represent a classification of complements of finite points on compact curves.

<table>
<thead>
<tr>
<th>(\kappa)</th>
<th>(\gamma_1=g)</th>
<th>(\gamma_m) ((m\geq 2))</th>
<th>structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty)</td>
<td>0</td>
<td>0</td>
<td>(\mathbb{P}^1, \mathbb{P}^1-{a_1}, \mathbb{P}^1-{a_1}-{a_2})</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>elliptic curve</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(m(k-2)-k+1) (except (k=3, m=2))</td>
<td>(\mathbb{P}^1-\bigcup_{i=1}^{k}{a_k}, k\geq 3)</td>
</tr>
<tr>
<td>(\geq 2)</td>
<td>1</td>
<td>(mk-k)</td>
<td>elliptic curve-(\bigcup_{i=1}^{k}{a_k}, k\geq 1)</td>
</tr>
<tr>
<td></td>
<td>(g\geq 2)</td>
<td>(m(k+2g-2)-k+1-g)</td>
<td>curve of genus (\geq 2-\bigcup_{i=1}^{k}{a_k}, k\geq 0)</td>
</tr>
</tbody>
</table>

Let \(X\) be again a complex manifold of dimension \(n\) and \(\overline{X}\) a smooth compactification of \(X\) with \(D=\overline{X}-X\). We remark that \(F_m(X)\) has an invariant Hermitian metric. So \(F_m(X)\) is a finite dimensional Hilbert space ([17]). The Kodaira dimension \(\kappa(X)\) has the following relation with \(\kappa(K_X+D, \overline{X})\).

(2.5) Proposition. If \(\kappa(X)\geq 0\), then \(\kappa(X)=\kappa(K_X+D, \overline{X})\). Further \(\kappa(X)\)
\(=n\) if and only if \(\kappa(K_X+D, \overline{X})=n\).

The first part of this relation also holds without the assumption that \(D\) has normal crossings (See [17], Appendix).

(2.6) Remark. When \(\overline{X}\) is a smooth compactification of \(X\), Iitaka calls \(\kappa(K_X+D, \overline{X})\) the logarithmic Kodaira dimension of \(X\) and writes it by \(\overline{\kappa}(X)\) ([9]). He proves that \(\overline{\kappa}(X)\) is a proper birational invariant of \(X\). From Theorem (1.2) and Proposition (2.5), it follows that if \(\kappa(X)\geq 0\), then \(\overline{\kappa}(X)\) is a bimeromorphic invariant. But the following examples show that in case \(\kappa(X)=-\infty\), \(\overline{\kappa}(X)\) need not be a bimeromorphic invariant of \(X\). We consider several
compactifications of $\mathbb{C}^2$. We have, by (1.3) $\kappa(\mathbb{C}^2)=-\infty$.

1) $\mathbb{C}^2=\mathbb{P}^2-\cup_{i=1}^3 H_i$, with three lines $H_1$, $H_2$, $H_3$ in general position.

In this case $\overline{\kappa}(\mathbb{P}^2-\cup_{i=1}^3 H_i)=0$.

2) $\mathbb{C}^2=\mathbb{P}^1 \times \mathbb{P}^1-\cup_{i=1}^4 L_i$, where $L_i=a_i \times \mathbb{P}^1$, $i=1,2$ and $L_i=\mathbb{P}^1 \times b_i$, $i=3,4$. We have $\overline{\kappa}(\mathbb{P}^1 \times \mathbb{P}^1-\cup_{i=1}^4 L_i)=0$.

3) $\mathbb{C}^2=S-E$, where $S$ is a Hopf surface given by $S=\mathbb{C}^2-\{0\}/\{g\}$ with $g:(z_1,z_2)\mapsto (\alpha^p z_1+\lambda z_2^p,\alpha z_2)$, $\lambda \neq 0$, $0<|\alpha|<1$, for a positive integer $p$ and $E$ is an elliptic curve given by $E=\{(\mathbb{C}^2-\{0\})\cap \{z_2=0\} \}/\{g\}$(See [7], for details). In this case, we have $K_S=-(p+1)[E]$ and then $\overline{\kappa}(S-E)=-\infty$.

4) $\mathbb{C}^2=F-D$, where $F$ is a $\mathbb{P}^1$-bundle over an elliptic curve constructed by Serre ([5], p232) and $D$ is a section with $D^2=0$. We also have $\overline{\kappa}(F-D)=-\infty$.

In case $X$ is given by $X=X-D$ with a singular divisor $D$ on a compact complex manifold $X$, it is not so easy to calculate $\kappa(X)$. Here we give a method. According to Hironaka, there exists a desingularization $\pi:X^* \longrightarrow X$ such that $\pi^{-1}(D)=D^*$ has normal crossings. Let $\pi^{-1}(\text{Sing } D)=\bigcup_{i=1}^n S_i$ be the irreducible decomposition of the exceptional set of $\pi$. Let $R_\pi$ be the ramification divisor of $\pi$. Set $E_D=\pi^*D-D^*\cdot R_\pi$. We can write $E_D=\sum_{i=1}^n t_i S_i$ with integers $t_i$.

(2.7) Definition (Shiffman [18]). Let $A$ be a divisor on $X$ passing through the non-normal crossing points of $D$. If we write $\pi^*A=\overline{A}+\sum_{i=1}^n p_i A_i$, where $\overline{A}$ is the strict transform of $A$, then $p_i A_i \geq 1$. Define $\gamma_{A,D}=\max_{i} \{ t_i / p_i A_i \}$, where $x^+$ means $\max (x,0)$.

(2.8) Proposition. We have

$$\gamma_m(X) \geq \dim H^0(X, O(mK_X + (m-1)([D] - \gamma_{A,D}[A])))$$

$$\overline{\kappa}(X)=\kappa(K_X+D^*) \geq \kappa(K_X+D-\gamma_{A,D}[A], X).$$
Proof. Note that \([\mathcal{C}_D] = \pi^*(K_X + [D]) - (K_X + [D^*]).\) We have, by definition \(\kappa(\gamma_{A,D}, \pi^*A - \mathcal{C}_D, X^*) \geq 0.\) The assertion follows from this.

(2.9) Corollary. Let \(D\) be a singular hypersurface of degree \(d\) in \(\mathbb{P}_n.\) Let \(A\) be a hypersurface of degree \(a\) in \(\mathbb{P}_n\) passing through the non-normal crossing points of \(D.\) If \((d - n - 1 - \gamma_{A,D}a) > 0,\) then \(\kappa(\mathbb{P}_n - D) = \kappa(\mathbb{P}_n - D) = n.\)

3. Quasi-Projective Manifolds with \(\kappa(X) = \dim X.\)

A complex manifold is called a quasi-projective manifold if it is given as a complement of an analytic subset of a projective algebraic manifold. In [17], we prove the following facts.

(3.1) Theorem. Let \(X\) be a quasi-projective manifold of dimension \(n.\) Assume that \(\kappa(X) = n.\) Then \(X\) satisfies

1. Any non-degenerate holomorphic map \(f: \Delta^*_A \times \Delta^{n-1} \to X\) can be extended to a meromorphic map from \(\Delta^n\) to any compactification of \(X.\) Here \(\Delta\) is the unit disc and \(\Delta^*_A = \Delta - \{0\}.\) An equidimensional holomorphic map is called non-degenerate if the Jacobian does not vanish identically.

2. Every biholomorphic transformation of \(X\) extends as a meromorphic transformation of any compactification of \(X.\)

3. Let \(\text{Aut}(X)\) be the group of biholomorphic transformations of \(X.\) Then \(\text{Aut}(X)\) is a finite group.

4. \(X\) is measure-hyperbolic.

4'. Every holomorphic map \(f: \mathbb{C} \times \Delta^{n-1} \to X\) degenerates.

These properties show that in this case \(X\) behaves like a projective algebraic manifold of general type.
4. **Concluding Remarks.**

A. Let \( \mathcal{Q} \) be a bounded symmetric domain of dimension \( n \) and \( \Gamma \) a totally discontinuous group operating on \( \mathcal{Q} \) such that \( X=\mathcal{Q}/\Gamma \) has a compactification. Let \( \pi: \mathcal{Q} \rightarrow X \) be the projection. In many cases, the space \( \pi^*F_m(X) \) corresponds to the vector space of cusp forms on \( \mathcal{Q} \) (For instance, see [6],[10]). So it is expected that this phenomenon holds in general. Moreover we have the following question: Let \( X \) be a complex manifold of dimension \( n \). If the universal covering manifold of \( X \) is a bounded domain in \( \mathbb{C}^n \), is it true that \( \kappa(X)=n? \)

B. Let \( Y \) be a complex manifold of dimension \( n \) and \( Z \) an analytic subset of \( Y \). We set \( \tilde{F}_m^Z(Y)=\{ \omega \in H^0(Y-Z,0(mK))/H^0(Y,0(mK)) \mid \omega \text{ is locally integrable across } Z \text{, i.e., for every point } x \in Z, \text{ there is a neighborhood } U \text{ of } x \text{ in } Y \text{ such that } \omega \text{ is integrable on } U-Z \cap U. \} \) If \( \text{codim } Z \leq 2 \), then \( \tilde{F}_m^Z(Y)=\{0\} \). In case \( Z \) is a divisor \( D \) having normal crossings, then we obtain

\[
\tilde{F}_m^D(Y) \cong H^0(Y,0(mK+-(m-1)[D]))/H^0(Y,0(mK))
\]

(cf. Theorem (2.1)). Take neighborhoods \( U, U' \) of \( Z \) in \( Y \). If \( U \supseteq U' \), then we have an inclusion \( F_m^Z(U) \hookrightarrow F_m^Z(U') \). Hence we can define \( \hat{F}_m^Z(Y)=\lim_{U \nearrow Y} F_m^Z(U) \). Put \( \gamma_m^Z(Y)=\dim \tilde{F}_m^Z(Y) \) and \( \gamma_m^Z(Y)=\dim \hat{F}_m^Z(Y) \). Then \( \gamma_m^Z(Y) \leq \gamma_m^Z(Y) \leq 0(m^N) \). Further we can define \( \kappa^Z(Y) \) and \( \kappa_m^Z(Y) \) in a similar manner as in (1.1).

Next in case \( Y \) is a complex space, letting \( \pi:Y^* \rightarrow Y \) be a desingularization of \( Y \), we put \( \gamma_m^Z(Y)=\gamma_m^Z(Y^*) \), \( \gamma_m^Z(Y)=\gamma_m^Z(Y^*) \) with \( Z^*=\pi^{-1}(Z \cup Y) \).

(4.1) **Proposition.** Let \( \bar{X} \) be a compact complex manifold of dimension \( n \) and \( D \) an effective divisor on \( \bar{X} \). Put \( X=\bar{X}-D \). Then

\[
P_m(X) \leq \gamma_m(X) \leq P_m(\bar{X}) + \hat{F}_m^D(\bar{X}).
\]
Proof. This follows from the exact sequence
\[ 0 \rightarrow H^0(\bar{X}^*,0(mK_{\bar{X}^*})) \rightarrow H^0(\bar{X}^*,0(mK_{\bar{X}^*}+(m-1)[D^*])) \rightarrow F^D_m(\bar{X}) \rightarrow 0, \]
where \( \bar{X}^*,D^* \) is a desingularization of \( X \), \( D \) such that \( \bar{X}^* \) is a smooth compactification of \( X \). Q.E.D.

We consider the special case in which \( Z=y \) is an isolated singularity of an \( n \)-dimensional complex space \( Y \). For simplicity, put \( \gamma_m = \gamma_m^Y(Y) \). Let \( \pi:Y^* \rightarrow Y \) be a desingularization of \( Y \).
For a neighborhood \( U \) of \( y \), put \( U^* = \pi^{-1}(U) \). Define
\[ r_m = \text{dim} \lim_{U} H^0(U-y,0(mK))/H^0(U^*,0(mK)). \]
Put \( \sigma_m = r_m - \gamma_m \). Then \( \sigma_m > 0 \) and \( \gamma_1 = 0 \). It is easily seen that if \( y \) is a quotient singularity, then \( \sigma_m = 0 \) for all \( m \) (cf.\([1],[2]) \). Question: When \( \sigma_m = 0 \)? When \( \gamma_m = 0 \)?

(4.2) Example. Suppose that \( \pi^{-1}(y) = E \) is \( \mathbb{P}_{n-1} \) and \( E|E \sim (-e) \), where \( (1) \) means the hyperplane bundle on \( \mathbb{P}_{n-1} \). In this case, we get easily that \( \sigma_m = 0 \) for all \( m \) and if \( e \leq n \), then \( \gamma_m = 0 \) for all \( m \).

In case \( \dim Y = 2 \), Laufer showed in \([15]\) that \( \gamma_1 = 0 \) if and only if \( y \) is a rational singularity. Precisely he proved \( \dim R^1\pi_*O_{U^*} = \sigma_1 \). Knöller \([11]\) calculates \( r_m \) and \( \lim_{m \to \infty} r_m/m^2 \) for several singularities. In particular, the condition \( r_m = 0 \) for all \( m \) characterizes the rational double points (See also \([12]\), for an application).

References


[9] ______: On logarithmic Kodaira dimension of algebraic varieties. to appear


[17] ______.: Kodaira dimensions of complements of divisors. To appear
