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INTEGRABLE PLURICANONICAL FORMS and KODAIRA DIMENSIONS OF COMPLEMENTS OF DIVISORS

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Let X be a complex manifold (possibly non-compact) of dimension n and ω a holomorphic m-ple n-form on X. We write ω as $\omega=\psi(w)(dw_1\wedge\cdots\wedge dw_n)^m$, using local coordinates $(w_1,\ldots,w_n)$. We associate with ω the continuous (n,n)-form $(\omega\wedge\bar{\omega})^{1/m}$, given locally by $|\psi(w)|^{2/m}\Pi_{i=1}^{n}(\sqrt{-1}/2\pi)dw_i\wedge d\bar{w}_i$. Then ω is called <u>integrable</u> ($L_{2/m}$-integrable) if $\int_X(\omega\wedge\bar{\omega})^{1/m}<\infty$. Let $F_m(X)$ be the set of all integrable holomorphic m-ple n-forms on X. When X has a compactification, $F_m(X)$ becomes a vector space. Using $F_m(X)$, we shall define the <u>Kodaira dimension</u> $\kappa(X)$ of X, which is a generalization of the notion of Kodaira dimension of compact complex manifolds introduced by Iitaka [8] (cf.Ueno [19]). Here we want to discuss the properties of $\kappa(X)$ and some related aspects. Details will appear in [17].

1. Kodaira Dimension.

Let X be a complex manifold of dimension n and $F_m(X)$ the set of all integrable holomorphic m-ple n-forms on X as above. Set $N(X)=\{m>0\,|\,F_m(X)\neq\{0\}\}$. If $m\in N(X)$, for a finite set of elements $\omega_0,\ldots,\omega_N\in F_m(X)$, we can define a meromorphic map $\Phi_{\{\omega_0,\ldots,\omega_N\}}$: $X\ni w\longrightarrow[\omega_0(w):\cdots:\omega_N(w)]$ of X into $\mathbb{P}_N$. Next we put $rk_m=\max[\mathrm{rank}\ \Phi_{\{\omega_0,\ldots,\omega_N\}}]$, where the maximum is taken over all choices of finite elements in $F_m(X)$ for N=0,1,2,.... The rank of a meromorphic map is the maximum rank of the Jacobian matrix where it is holomorphic. Now we define the Kodaira dimension $\kappa(X)$ of X by

(1.1) $$\kappa(X)=\begin{cases}\max\limits_{m\in N(X)}\{rk_m\} & \text{if } N(X)\neq\emptyset,\\ -\infty & \text{if } N(X)=\emptyset.\end{cases}$$

Note that $\kappa(X)$ takes one of the values $-\infty,0,1,\dots,n$.

(1.2) Theorem ([17]). The Kodaira dimension $\kappa(X)$ is a bimeromorphic (in the sense of Remmert) invariant of a complex manifold X.

Proof. Let X' be a complex manifold such that there exists a bimeromorphic map $f:X'\dashrightarrow X$. Then f* induces an isomorphism of $F_m(X)$ onto $F_m(X')$. To see this, take an element $\omega\in F_m(X)$, then $f^*\omega$ is a holomorphic m-ple n-form on $X'-S(f)$, where S(f) is the set of points where f is not holomorphic. Since codim $S(f)\geqq 2$, it extends holomorphically on X'. Clearly $\int_{X'}(f^*\omega\wedge\overline{f^*\omega})^{1/m}=\int_X(\omega\wedge\bar{\omega})^{1/m}<\infty$, which implies $f^*\omega\in F_m(X')$. Considering the inverse map, we get the surjectivity. Consequently we have, by definitio $\kappa(X')=\kappa(X)$. Q.E.D.

We list some properties of $\kappa(X)$ (cf. [17]).

1. Let $\mathbb{C}$ be the complex plane and $\mathbb{C}^*=\mathbb{C}-\{0\}$. Then $\kappa(\mathbb{C})=-\infty$, $\kappa(\mathbb{C}^*)=-\infty$. Further $\kappa(\mathbb{C}\times Y)=-\infty$, $\kappa(\mathbb{C}^*\times Y)=-\infty$, for any complex manifold Y.
2. Let X, Y be complex manifolds of the same dimension such that $X\subset Y$. Then $\kappa(X)\geqq\kappa(Y)$. In particular, if $\kappa(X)=-\infty$, we get $\kappa(Y)=-\infty$.

(1.3)

3. Let X be a complex manifold and Z an analytic subset of X with codim $Z\geqq 2$. Then $\kappa(X-Z)=\kappa(X)$.
4. Let X, Y be complex manifolds of the same dimesion.

Suppose that there is a surjective proper meromorphic map $f:X \longrightarrow Y$. Then $\kappa(X) \geqq \kappa(Y)$.

In case X is a complex space, we define $\kappa(X)$ to be $\kappa(X^*)$, using a desingularization X^* of X.

2. Complements of Divisors.

In this section, we deal with the case in which X has a compactification $\overline{X}$. We assume that $\overline{X}$ is a smooth compactification in the sense that $\overline{X}$ is a compact complex manifold and $D=\overline{X}-X$ is a divisor of normal crossings. Let $K_{\overline{X}}$ be the canonical bundle of $\overline{X}$ and [D] the line bundle determined by D. In this case, we have

(2.1) Theorem([17]). $F_m(X) \cong H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D]))$.

The proof is based on the fact that if $f(z)(dz)^m$ is integrable on the punctured disc Δ^*, then the Laurent expansion of f(z) becomes as $\sum_{j=-(m-1)}^{\infty} a_j z^j$ ([14], Appendix).

(2.2) Definition. $\gamma_m(X) = \dim F_m(X) = \dim H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D]))$.

We can redefine the Kodaira dimension as follows.

(2.3) Definition. Let $\psi_0, \ldots, \psi_N$ be a basis of $H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D]))$. Let Φ_m be the meromorphic map defined by $[\psi_0 : \cdots : \psi_N]$ of $\overline{X}$ into $\mathbf{P}_N$. Put $N(X) = \{m>0 \mid \dim H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D]))>0\}$. Then

$$\kappa(X) = \begin{cases} \max_{m \in N(X)} \{\dim \Phi_m(\overline{X})\} & \text{if } N(X) \neq \emptyset, \\ -\infty & \text{if } N(X) = \emptyset. \end{cases}$$

(2.4) Example. Let D be a hypersurface of degree d in $\mathbb{P}_n$ which has at most normal crossings. Then $\kappa(\mathbf{P}_n - D) = n$ if $d > n+1$ and $\kappa(\mathbb{P}_n - D)$

$=-\infty$ if $d \leqq n+1$. Next put $U_a = \{z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}} = 1\}$ in $\mathbb{C}^{n+1}$. Then $\kappa(U_a) = n$ if $\sum_i 1/a_i < 1$. Here we represent a classification of complements of finite points on compact curves.

κ	$\gamma_1 = g$	γ_m $(m \geqq 2)$	structure
$-\infty$	0	0	$\mathbb{P}_1$, $\mathbb{P}_1 - \{a_1\}$, $\mathbb{P}_1 - \{a_1\} - \{a_2\}$
0	1	1	elliptic curve
1	0	$m(k-2)-k+1$ (except $k=3$, $m=2$)	$\mathbb{P}_1 - \bigcup_{i=1}^{k} \{a_k\}$, $k \geqq 3$
	1	$mk-k$	elliptic curve $- \bigcup_{i=1}^{k} \{a_k\}$, $k \geqq 1$
	$g \geqq 2$	$m(k+2g-2)-k+1-g$	curve of genus $\geqq 2 - \bigcup_{i=1}^{k} \{a_k\}$, $k \geqq 0$

Let X be again a complex manifold of dimension n and $\bar{X}$ a smooth compactification of X with $D = \bar{X} - X$. We remark that $F_m(X)$ has an invariant Hermitian metric. So $F_m(X)$ is a finite dimensional Hilbert space ([17]). The Kodaira dimension $\kappa(X)$ has the following relation with $\kappa(K_{\bar{X}} + D, \bar{X})$.

(2.5) Proposition. If $\kappa(X) \geqq 0$, then $\kappa(X) = \kappa(K_{\bar{X}} + D, \bar{X})$. Further $\kappa(X) = n$ if and only if $\kappa(K_{\bar{X}} + D, \bar{X}) = n$.

The first part of this relation also holds without the assumption that D has normal crossings (See [17], Appendix).

(2.6) Remark. When $\bar{X}$ is a smooth compactification of X, Iitaka calls $\kappa(K_{\bar{X}} + D, \bar{X})$ the logarithmic Kodaira dimension of X and writes it by $\bar{\kappa}(X)$ ([9]). He proves that $\bar{\kappa}(X)$ is a proper birational invariant of X. From Theorem (1.2) and Proposition (2.5), it follows that if $\kappa(X) \geqq 0$, then $\bar{\kappa}(X)$ is a bimeromorphic invariant. But the following examples show that in case $\kappa(X) = -\infty$, $\bar{\kappa}(X)$ need not be a bimeromorphic invariant of X. We consider several compactifications of $\mathbb{C}^{*2}$. We have, by (1.3) $\kappa(\mathbb{C}^{*2})=-\infty$. 1) $\mathbb{C}^{*2}=\mathbb{P}_2-\cup_{i=1}^3 H_i$, with three lines H_1, H_2, H_3 in general position. In this case $\bar{\kappa}(\mathbb{P}_2-\cup_{i=1}^3 H_i)=0$. 2) $\mathbb{C}^{*2}=\mathbb{P}_1\times\mathbb{P}_1-\cup_{i=1}^4 L_i$, where $L_i=a_i\times\mathbb{P}_1$, $i=1,2$ and $L_i=\mathbb{P}_1\times b_i$, $i=3,4$. We have $\bar{\kappa}(\mathbb{P}_1\times\mathbb{P}_1-\cup_{i=1}^4 L_i)=0$. 3) $\mathbb{C}^{*2}=S-E$, where S is a Hopf surface given by $S=\mathbb{C}^2-\{0\}/\{g\}$ with $g:(z_1,z_2)\longrightarrow(\alpha^p z_1+\lambda z_2^p,\alpha z_2)$, $\lambda\neq 0$, $0<|\alpha|<1$, for a positive integer p and E is an elliptic curve given by $E=(\mathbb{C}^2-\{0\})\cap\{z_2=0\}$ /{g}(See [7], for details). In this case, we have $K_S=-(p+1)[E]$ and then $\bar{\kappa}(S-E)=-\infty$. 4) $\mathbb{C}^{*2}=F-D$, where F is a $\mathbb{P}_1$-bundle over an elliptic curve constructed by Serre ([5], p232) and D is a section with $D^2=0$. We also have $\bar{\kappa}(F-D)=-\infty$.

In case X is given by $X=\bar{X}-D$ with a singular divisor D on a compact complex manifold $\bar{X}$, it is not so easy to claculate $\kappa(X)$. Here we give a method. According to Hironaka, there exists a desingularization $\pi:X^*\longrightarrow X$ such that $\pi^{-1}(D)=D^*$ has normal crossings. Let $\pi^{-1}(\mathrm{Sing}\ D)=\sum_i S_i$ be the irreducible decomposition of the exceptional set of π. Let R_π be the ramification divisor of π. Set $\mathcal{E}_D=\pi^*D-D^*-R_\pi$. We can write $\mathcal{E}_D=\sum_i t_i S_i$ with integers t_i.

(2.7)<u>Definition</u> (Shiffman [18]). Let A be a divisor on $\bar{X}$ passing through the non-normal crossing points of D. If we write $\pi^*A=\bar{A}+\sum_i p_i^A S_i$, where $\bar{A}$ is the strict transform of A, then $p_i^A\geqq 1$. Define

$$\gamma_{A,D}=\max_i\{t_i^+/p_i^A\},\qquad \text{where } x^+ \text{ means } \max(x,0).$$

(2.8)<u>Proposition</u>. We have

$$\gamma_m(X)\geqq\dim H^0(\bar{X},\mathcal{O}(mK_{\bar{X}}+(m-1)\{[D]-\gamma_{A,D}[A]\})).$$

$$\bar{\kappa}(X)=\kappa(K_{\bar{X}^*}+D^*,\bar{X}^*)\geqq\kappa(K_{\bar{X}}+D-\gamma_{A,D}A,\bar{X}).$$

Proof. Note that $[\mathcal{E}_D]=\pi^*(K_{\overline{X}}+[D])-(K_{\overline{X}^*}+[D^*])$. We have, by definition $\kappa(\gamma_{A,D}\pi^*A-\mathcal{E}_D,\overline{X}^*)\geqq 0$. The assertion follows from this.

(2.9) Corollary. Let D be a singular hypersurface of degree d in $\mathbb{P}_n$. Let A be a hypersurface of degree a in $\mathbb{P}_n$ passing through the non-normal crossing points of D. If $(d-n-1-\gamma_{A,D}a)>0$, then $\kappa(\mathbb{P}_n-D)=\overline{\kappa}(\mathbb{P}_n-D)=n$.

3. Quasi-Projective Manifolds with $\kappa(X)=\dim X$.

A complex manifold is called a quasi-projective manifol if it is given as a complement of an analytic subset of a projective algebraic manifold. In [17], we prove the following facts.

(3.1) Theorem. Let X be a quasi-projective manifold of dimension n. Assume that $\kappa(X)=n$. Then X satisfies

1. Any non-degenerate holomorphic map $f:\Delta^*\times\Delta^{n-1}\longrightarrow X$ can be extended to a meromorphic map from Δ^n to any compactification of X. Here Δ is the unit disc and $\Delta^*=\Delta-\{0\}$. An equidimensional holomorphic map is called non-degenerate if the Jacobian does not vanish identically.
2. Every biholomorphic transformation of X extends as a meromorphic transformation of any compactification of X.
3. Let Aut(X) be the group of biholomorphic transformations of X. Then Aut(X) is a finite group.
4. X is measure-hyperbolic.

4'. Every holomorphic map $f:\mathbb{C}\times\Delta^{n-1}\longrightarrow X$ degenerates.

These properties show that in this case X behaves like a projective algebraic manifold of general type.

4. Concluding Remarks.

A. Let $\mathfrak{D}$ be a bounded symmetric domain of dimension n and Γ a totally discontinuous group operating on $\mathfrak{D}$ such that $X=\mathfrak{D}/\Gamma$ has a compactification. Let $\pi:\mathfrak{D}\longrightarrow X$ be the projection. In many cases, the space $\pi^*F_m(X)$ corresponds to the vector space of cusp forms on $\mathfrak{D}$ (For instance, see [6], [10]). So it is expected that this phenomenon holds in general. Moreover we have the following question: Let X be a complex manifold of dimension n. If the universal covering manifold of X is a bounded domain in $\mathbb{C}^n$, is it true that $\kappa(X)=n$?

B. Let Y be a complex manifold of dimesion n and Z an analytic subset of Y. We set $F_m^Z(Y)=\{\omega\in H^0(Y-Z,\mathcal{O}(mK))/H^0(Y,\mathcal{O}(mK))\mid$ ω is locally integrable across Z, i.e., for every point $x\in Z$, there is a neighborhood U of x in Y such that ω is integrable on $U-Z\cap U.\}$ If codim $Z\geqq 2$, then $F_m^Z(Y)=\{0\}$. In case Z is a divisor D having normal crossings, then we obtain

$$F_m^D(Y)\cong H^0(Y,\mathcal{O}(mK+(m-1)[D]))/H^0(Y,\mathcal{O}(mK))$$

(cf. Theorem (2.1)). Take neighborhoods U, U' of Z in Y. If $U\supset U'$, then we have an inclusion $F_m^Z(U)\hookrightarrow F_m^Z(U')$. Hence we can define $\hat{F}_m^Z(Y)=\varinjlim_{U} F_m^Z(U)$. Put $\gamma_m^Z(Y)=\dim F_m^Z(Y)$ and $\hat{\gamma}_m^Z(Y)=\dim \hat{F}_m^Z(Y)$. Then $\gamma_m^Z(Y)\leqq\hat{\gamma}_m^Z(Y)\leqq O(m^n)$. Further we can define $\kappa^Z(Y)$ and $\hat{\kappa}^Z(Y)$ in a similar manner as in (1.1).

Next in case Y is a complex space, letting $\pi:Y^*\longrightarrow Y$ be a desingularization of Y, we put $\gamma_m^Z(Y)=\gamma_m^{Z^*}(Y^*)$, $\hat{\gamma}_m^Z(Y)=\hat{\gamma}_m^{Z^*}(Y^*)$ with $Z^*=\pi^{-1}(Z$

(4.1) Proposition. Let $\bar{X}$ be a compact complex manifold of dimension n and D an effective divisor on $\bar{X}$. Put $X=\bar{X}-D$. Then

$$P_m(\bar{X})\leqq\gamma_m(X)\leqq P_m(\bar{X})+\hat{\gamma}_m^D(\bar{X}).$$

Proof. This follows from the exact sequence

$$0 \longrightarrow H^0(\overline{X}^*, O(mK_{\overline{X}^*})) \to H^0(\overline{X}^*, O(mK_{\overline{X}^*}+(m-1)[D^*])) \to F_m^D(\overline{X}) \longrightarrow 0,$$

where $\overline{X}^*, D^*$ is a desingularization of $\overline{X}$, D such that $\overline{X}^*$ is a smooth compactification of X. Q.E.D.

We consider the special case in which Z=y is an isolated singularity of an n-dimensional complex space Y. For simplicity, put $\gamma_m = \hat{\gamma}^y_m(Y)$. Let $\pi : Y^* \longrightarrow Y$ be a desingulariztion of Y. For a neighborhood U of y, put $U^* = \pi^{-1}(U)$. Define $r_m = \dim \varinjlim_U H^0(U-y, O(mK))/H^0(U^*, O(mK))$. Put $\sigma_m = r_m - \gamma_m$. Then $\sigma_m \geqq 0$. and $\gamma_1 = 0$. It is easily seen that if y is a quotient singularity, then $\sigma_m = 0$ for all m (cf. [1],[2]). Question: When $\sigma_m = 0$? When $\gamma_m = 0$?

(4.2) Example. Suppose that $\pi^{-1}(y) = E$ is $\mathbb{P}_{n-1}$ and $E|E \sim (-e)$, where (1) means the hyperplane bundle on $\mathbb{P}_{n-1}$. In this case, we get easily that $\sigma_m = 0$ for all m and if $e \leqq n$, then $\gamma_m = o$ for all m.

In case dim Y=2, Laufer showed in [15] that $\sigma_1 = 0$ if and only if y is a rational singularity. Precisely he proved $\dim R^1\pi_* O_{U^*} = \sigma_1$. Knöller [11] calculates r_m and $\lim_{m\to\infty} r_m/m^2$ for several singularities. In particular, the condition $r_m = 0$ for all m characterizes the rational double points (See also [12], for an application).

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