INTEGRABLE PLURICANONICAL FORMS
and
KODAIRA DIMENSIONS OF COMPLEMENTS OF DIVISORS

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Let $X$ be a complex manifold (possibly non-compact) of dimension $n$ and $\omega$ a holomorphic $m$-ple $n$-form on $X$. We write $\omega$ as $\omega = \psi(w)(dw_1 \wedge \cdots \wedge dw_n)^m$, using local coordinates $(w_1, \ldots, w_n)$. We associate with $\omega$ the continuous $(n,n)$-form $(\omega \wedge \overline{\omega})^{1/m}$, given locally by $|\psi(w)|^{2/m} \prod_{i=1}^{n} (i-1/2\pi) dw_i \wedge d\overline{w}_i$. Then $\omega$ is called integrable ($L_2/m$-integrable) if $\int_X (\omega \wedge \overline{\omega})^{1/m} < \infty$. Let $F_m(X)$ be the set of all integrable holomorphic $m$-ple $n$-forms on $X$. When $X$ has a compactification, $F_m(X)$ becomes a vector space. Using $F_m(X)$, we shall define the Kodaira dimension $\kappa(X)$ of $X$, which is a generalization of the notion of Kodaira dimension of compact complex manifolds introduced by Iitaka [8] (cf. Ueno [19]). Here we want to discuss the properties of $\kappa(X)$ and some related aspects. Details will appear in [17].

1. Kodaira Dimension.

Let $X$ be a complex manifold of dimension $n$ and $F_m(X)$ the set of all integrable holomorphic $m$-ple $n$-forms on $X$ as above. Set $N(X) = \{m \geq 0 | F_m(X) \neq \{0\}\}$. If $m \in N(X)$, for a finite set of elements $\omega_0, \ldots, \omega_N \in F_m(X)$, we can define a meromorphic map $\Phi_{\omega_0, \ldots, \omega_N}: X \to \mathbb{P}_N[r_m = \max\{\text{rank} \Phi_{\omega_0, \ldots, \omega_N}\}]$, where the maximum is taken over all choices of finite elements in $F_m(X)$ for $N=0,1,2,\ldots$. The rank of a meromorphic map is the maximum rank of the Jacobian matrix where it is holo-
morphic. Now we define the Kodaira dimension $\kappa(X)$ of $X$ by

\[
\kappa(X) = \begin{cases} 
\max_{m \in \mathbb{N}} \{ \text{rk}_m \} & \text{if } \mathcal{N}(X) \neq \emptyset, \\
-\infty & \text{if } \mathcal{N}(X) = \emptyset. 
\end{cases}
\]

Note that $\kappa(X)$ takes one of the values $-\infty, 0, 1, \ldots, n$.

(1.2) **Theorem ([17])**. The Kodaira dimension $\kappa(X)$ is a bimeromorphic (in the sense of Remmert) invariant of a complex manifold $X$.

**Proof.** Let $X'$ be a complex manifold such that there exists a bimeromorphic map $f : X' \to X$. Then $f^*$ induces an isomorphism of $\mathcal{F}_m(X)$ onto $\mathcal{F}_m(X')$. To see this, take an element $\omega \in \mathcal{F}_m(X)$, then $f^*\omega$ is a holomorphic $m$-ple $n$-form on $X' - S(f)$, where $S(f)$ is the set of points where $f$ is not holomorphic. Since $\text{codim } S(f) \geq 2$, it extends holomorphically on $X'$. Clearly $\int_{X}(f^*\omega \wedge f^*\omega)^{1/m} = \int_{X}(\omega \wedge \omega)^{1/m} < \infty$, which implies $f^*\omega \in \mathcal{F}_m(X')$. Considering the inverse map, we get the surjectivity. Consequently we have, by definitio $\kappa(X') = \kappa(X)$. Q.E.D.

We list some properties of $\kappa(X)$ (cf. [17]).

1. Let $\mathbb{C}$ be the complex plane and $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then $\kappa(\mathbb{C}) = -\infty$, $\kappa(\mathbb{C}^*) = -\infty$. Further $\kappa(\mathbb{C} \times Y) = -\infty$, $\kappa(\mathbb{C}^* \times Y) = -\infty$, for any complex manifold $Y$.

2. Let $X$, $Y$ be complex manifolds of the same dimension such that $X \subseteq Y$. Then $\kappa(X) \geq \kappa(Y)$. In particular, if $\kappa(X) = -\infty$, we get $\kappa(Y) = -\infty$.

3. Let $X$ be a complex manifold and $Z$ an analytic subset of $X$ with $\text{codim } Z \geq 2$. Then $\kappa(X - Z) = \kappa(X)$.

4. Let $X$, $Y$ be complex manifolds of the same dimension.
Suppose that there is a surjective proper meromorphic map \( f: X \to Y \). Then \( \kappa(X) \geq \kappa(Y) \).

In case \( X \) is a complex space, we define \( \kappa(X) \) to be \( \kappa(X^*) \), using a desingularization \( X^* \) of \( X \).

2. Complements of Divisors.

In this section, we deal with the case in which \( X \) has a compactification \( \overline{X} \). We assume that \( \overline{X} \) is a smooth compactification in the sense that \( \overline{X} \) is a compact complex manifold and \( D=\overline{X}-X \) is a divisor of normal crossings. Let \( K_{\overline{X}} \) be the canonical bundle of \( \overline{X} \) and \( [D] \) the line bundle determined by \( D \). In this case, we have

\[(2.1) \text{Theorem([17])}. \quad F_m(X) \cong H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D])).\]

The proof is based on the fact that if \( f(z)(dz)^m \) is integrable on the punctured disc \( \Delta^* \), then the Laurent expansion of \( f(z) \) becomes as \( \sum_{j=-(m-1)a}^{\infty} a_j z^j \) ([14], Appendix).

\[(2.2) \text{Definition}. \quad \gamma_m(X) = \dim F_m(X) = \dim H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D])).\]

We can redefine the Kodaira dimension as follows.

\[(2.3) \text{Definition}. \quad \text{Let } \psi_0, \ldots, \psi_N \text{ be a basis of } H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D])). \text{ Let } \phi_m \text{ be the meromorphic map defined by } [\psi_0: \ldots: \psi_N] \text{ of } \overline{X} \text{ into } \mathbb{P}^N. \text{ Put } N(X) = \{m > 0 | \dim H^0(\overline{X}, O(mK_{\overline{X}}+(m-1)[D])) > 0\}. \text{ Then}
\]
\[
\kappa(X) = \begin{cases} 
\max \{\dim \phi_m(\overline{X})\} & \text{if } N(X) \neq \emptyset, \\
-\infty & \text{if } N(X) = \emptyset.
\end{cases}
\]

\[(2.4) \text{Example}. \quad \text{Let } D \text{ be a hypersurface of degree } d \text{ in } \mathbb{P}^n \text{ which has at most normal crossings. Then } \kappa(\mathbb{P}^n-D) = n \text{ if } d > n+1 \text{ and } \kappa(\mathbb{P}^n-D)\]
\[\gamma_1 = 1, \frac{\gamma_1}{a_1} < 1.\] Here we represent a classification of complements of finite points on compact curves.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\gamma_1 = g$</th>
<th>$\gamma_m$ (m&gt;2)</th>
<th>structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{P}_1, \mathbb{P}_1 - {a_1}, \mathbb{P}_1 - {a_1} - {a_2}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>elliptic curve</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$m(k-2) - k + 1$ (except $k=3, m=2$)</td>
<td>$\mathbb{P}<em>1 - \bigcup</em>{i=1}^{k} {a_k}, k \geq 3$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$mk - k$</td>
<td>elliptic curve - $\bigcup_{i=1}^{k} {a_k}, k \geq 1$</td>
</tr>
<tr>
<td>$g \geq 2$</td>
<td>$m(k+2g-2) - k + 1 - g$</td>
<td>curve of genus $\geq 2 - \bigcup_{i=1}^{k} {a_k}, k \geq 0$</td>
<td></td>
</tr>
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</table>

Let X be again a complex manifold of dimension n and $\overline{X}$ a smooth compactification of X with $D=\overline{X}-X$. We remark that $F_m(X)$ has an invariant Hermitian metric. So $F_m(X)$ is a finite dimensional Hilbert space ([17]). The Kodaira dimension $\kappa(X)$ has the following relation with $\kappa(K_X + D, \overline{X})$.

(2.5) **Proposition.** If $\kappa(X) \geq 0$, then $\kappa(X) = \kappa(K_X + D, \overline{X})$. Further $\kappa(X) = n$ if and only if $\kappa(K_X + D, \overline{X}) = n$.

The first part of this relation also holds without the assumption that D has normal crossings (See [17], Appendix).

(2.6) **Remark.** When $\overline{X}$ is a smooth compactification of X, Iitaka calls $\kappa(K_X + D, \overline{X})$ the logarithmic Kodaira dimension of X and writes it by $\overline{\kappa}(X)$ ([9]). He proves that $\overline{\kappa}(X)$ is a proper birational invariant of X. From Theorem (1.2) and Proposition (2.5), it follows that if $\kappa(X) \geq 0$, then $\overline{\kappa}(X)$ is a bimeromorphic invariant. But the following examples show that in case $\kappa(X) = -\infty$, $\overline{\kappa}(X)$ need not be a bimeromorphic invariant of X. We consider several
compactifications of $\mathbb{C}^{*2}$. We have, by (1.3) $\kappa(\mathbb{C}^{*2})=\kappa(\mathbb{C}^{*2})=-\infty$. 1) $\mathbb{C}^{*2}=\mathbb{P}_2-\cup_{i=1}^{3} H_i$, with three lines $H_1$, $H_2$, $H_3$ in general position. In this case $\kappa(\mathbb{P}_2-\cup_{i=1}^{3} H_i)=0$. 2) $\mathbb{C}^{*2}=\mathbb{P}_1 \times \mathbb{P}_1-\cup_{i=1}^{4} L_i$, where $L_i=a_i \times \mathbb{P}_1$, $i=1,2$ and $L_i=\mathbb{P}_1 \times b_i$, $i=3,4$. We have $\kappa(\mathbb{P}_1 \times \mathbb{P}_1-\cup_{i=1}^{4} L_i)=0$. 3) $\mathbb{C}^{*2}=S-E$, where $S$ is a Hopf surface given by $S=\mathbb{C}^{2}-\{0\}/\{g\}$ with $g:(z_1,z_2) \rightarrow (\alpha z_1^p+\lambda z_2^p,az_2)$, $\lambda \neq 0$, $0<|\alpha|<1$, for a positive integer $p$ and $E$ is an elliptic curve given by $E=(\mathbb{C}^{2}-\{0\})\cap\{z_2=0\}/\{g\}$. (See [7], for details). In this case, we have $\kappa(S-E)=-(p+1)[E]$ and then $\kappa(S-E)=-\infty$. 4) $\mathbb{C}^{*2}=F-D$, where $F$ is a $\mathbb{P}_1$-bundle over an elliptic curve constructed by Serre ([5], p232) and $D$ is a section with $D^2=0$. We also have $\kappa(F-D)=-\infty$.

In case $X$ is given by $X=\mathbb{C}^{*}D$ with a singular divisor $D$ on a compact complex manifold $\mathbb{C}^{*}$, it is not so easy to calculate $\kappa(X)$. Here we give a method. According to Hironaka, there exists a desingularization $\pi:X^{*} \rightarrow X$ such that $\pi^{-1}(D)=D^{*}$ has normal crossings. Let $\pi^{-1}(\text{Sing} D)=\cup_{i=1}^{r} S_{i}$ be the irreducible decomposition of the exceptional set of $\pi$. Let $R_{\pi}$ be the ramification divisor of $\pi$. Set $\tilde{C}_{D}=\pi^{*}D-D^{*}+R_{\pi}$. We can write $\tilde{C}_{D}=\sum_{i=1}^{r} t_{i} S_{i}$ with integers $t_{i}$.

(2.7) Definition (Shiffman [18]). Let $A$ be a divisor on $\mathbb{C}^{*}$ passing through the non-normal crossing points of $D$. If we write $\pi^{*}A=\tilde{A}+\sum_{i=1}^{r} p_{i}^{A} S_{i}$, where $\tilde{A}$ is the strict transform of $A$, then $p_{i}^{A}\geq 1$. Define $\gamma_{A,D}=\max_{i}(t_{i}^{+}/p_{i}^{A})$, where $x^{+}$ means $\max(x,0)$.

(2.8) Proposition. We have

$$\gamma_{m}(X) \geq \dim H^{0}(\mathbb{C}^{*},O(mK_{\mathbb{C}^{*}}+m-1)([D]-\gamma_{A,D}[A])).$$

$$\kappa(X) \geq \kappa(K_{\mathbb{C}^{*}}+D^{*},\mathbb{C}^{*}) \geq \kappa(K_{\mathbb{C}^{*}}+D^{*}-\gamma_{A,D}[A],\mathbb{C}^{*}).$$

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Proof. Note that \([\hat{c}_D^*]=\pi^*(K_{\hat{X}}+[D])-(K_{\hat{X}}+[D^*]).\) We have, by definition \(\kappa(\gamma_{A,D,D},\pi^*A-\hat{c}_D^*,\hat{X})\geq 0.\) The assertion follows from this.

(2.9) Corollary. Let \(D\) be a singular hypersurface of degree \(d\) in \(\mathbb{P}_n.\) Let \(A\) be a hypersurface of degree \(a\) in \(\mathbb{P}_n\) passing through the non-normal crossing points of \(D.\) If \((d-n-1-\gamma_{A,D,a})>0,\) then \(\kappa(\mathbb{P}_n-D)=\kappa(\mathbb{P}_n-D)=n.\)

3. Quasi-Projective Manifolds with \(\kappa(X)=\dim X.\)

A complex manifold is called a quasi-projective manifold if it is given as a complement of an analytic subset of a projective algebraic manifold. In [17], we prove the following facts.

(3.1) Theorem. Let \(X\) be a quasi-projective manifold of dimension \(n.\) Assume that \(\kappa(X)=n.\) Then \(X\) satisfies

1. Any non-degenerate holomorphic map \(f:\Delta^*\times \Delta^{n-1}\to X\) can be extended to a meromorphic map from \(\Delta^n\) to any compactification of \(X.\) Here \(\Delta\) is the unit disc and \(\Delta^*\Delta-\{0\}.\)

An equidimensional holomorphic map is called non-degenerate if the Jacobian does not vanish identically.

2. Every biholomorphic transformation of \(X\) extends as a meromorphic transformation of any compactification of \(X.\)

3. Let \(\text{Aut}(X)\) be the group of biholomorphic transformations of \(X.\) Then \(\text{Aut}(X)\) is a finite group.

4. \(X\) is measure-hyperbolic.

4'. Every holomorphic map \(f:\mathbb{C}\times \Delta^{n-1}\to X\) degenerates.

These properties show that in this case \(X\) behaves like a projective algebraic manifold of general type.
4. **Concluding Remarks.**

A. Let $\mathcal{D}$ be a bounded symmetric domain of dimension $n$ and $\Gamma$ a totally discontinuous group operating on $\mathcal{D}$ such that $X=\mathcal{D}/\Gamma$ has a compactification. Let $\pi: \mathcal{D} \to X$ be the projection. In many cases, the space $\pi^* F_m(X)$ corresponds to the vector space of cusp forms on $\mathcal{D}$ (For instance, see [6],[10]). So it is expected that this phenomenon holds in general. Moreover we have the following question: Let $X$ be a complex manifold of dimension $n$. If the universal covering manifold of $X$ is a bounded domain in $\mathbb{C}^n$, is it true that $\kappa(X)=n$?

B. Let $Y$ be a complex manifold of dimension $n$ and $Z$ an analytic subset of $Y$. We set $F_m^Z(Y) = \{ \omega \in H^0(Y-Z, \mathcal{O}(mK))/H^0(Y, \mathcal{O}(mK)) \mid \omega \text{ is locally integrable across } Z, \text{ i.e., for every point } x \in Z, \text{ there is a neighborhood } U \text{ of } x \text{ in } Y \text{ such that } \omega \text{ is integrable on } U-Z\cap U \}. \text{ If } \text{codim } Z \geq 2, \text{ then } F_m^Z(Y) = \{0\}. \text{ In case } Z \text{ is a divisor } D \text{ having normal crossings, then we obtain }

$$F_m^D(Y) \cong H^0(Y, \mathcal{O}(mK+(m-1)[D]))/H^0(Y, \mathcal{O}(mK))$$

(cf. Theorem (2.1)). Take neighborhoods $U, U'$ of $Z$ in $Y$. If $U \supset U'$, then we have an inclusion $F_m^Z(U) \supset F_m^Z(U')$. Hence we can define $\hat{F}_m^Z(Y) = \lim_{U \supset U'} F_m^Z(U)$. Put $\gamma_m^Z(Y) = \dim F_m^Z(Y)$ and $\gamma_m^Z(Y) = \dim F_m^Z(Y)$. Then $\gamma_m^Z(Y) \leq \gamma_m^Z(Y) \leq 0(m^N)$. Further we can define $\kappa_m^Z(Y)$ and $\kappa_m^Z(Y)$ in a similar manner as in (1.1).

Next in case $Y$ is a complex space, letting $\pi: Y^* \to Y$ be a desingularization of $Y$, we put $\gamma_m^Z(Y) = \gamma_m^Z(Y^*)$, $\gamma_m^Z(Y) = \gamma_m^Z(Y^*)$ with $Z^* = \pi^{-1}(Z)$ (4.1) Proposition. Let $\overline{X}$ be a compact complex manifold of dimension $n$ and $D$ an effective divisor on $\overline{X}$. Put $X = \overline{X} - D$. Then

$$P_m(\overline{X}) \leq \gamma_m(X) \leq P_m(\overline{X}) + \gamma_m^D(\overline{X})$$.
Proof. This follows from the exact sequence

$$0 \rightarrow H^0(\mathbb{X}^*, O(mK_{\mathbb{X}^*})) \rightarrow H^0(\mathbb{X}^*, O(mK_{\mathbb{X}^*} + (m-1)[D^*])) \rightarrow \mathbb{F}_m(\mathbb{X}) \rightarrow 0,$$

where $\mathbb{X}^*, D^*$ is a desingularization of $\mathbb{X}$, $D$ such that $\mathbb{X}^*$ is a smooth compactification of $\mathbb{X}$. Q.E.D.

We consider the special case in which $Z=y$ is an isolated singularity of an $n$-dimensional complex space $Y$. For simplicity, put $\gamma_m = \gamma_m^Y(Y)$. Let $\pi: Y^* \rightarrow Y$ be a desingularization of $Y$.

For a neighborhood $U$ of $y$, put $U^* = \pi^{-1}(U)$. Define

$$r_m = \dim \lim_{U^*} H^0(U^*-y, O(mK))/H^0(U^*, O(mK)).$$

Put $\sigma_m = r_m - \gamma_m$. Then $\sigma_m > 0$.

and $\gamma_1 = 0$. It is easily seen that if $y$ is a quotient singularity, then $\sigma_m = 0$ for all $m$ (cf. [1], [2]). Question: When $\sigma_m = 0$? When $\gamma_m = 0$?

(4.2) Example. Suppose that $\pi^{-1}(y) = E$ is $\mathbb{P}_{n-1}$ and $E|E \sim (-e)$, where $e$ means the hyperplane bundle on $\mathbb{P}_{n-1}$. In this case, we get easily that $\sigma_m = 0$ for all $m$ and if $e \leq n$, then $\gamma_m = 0$ for all $m$.

In case $\dim Y = 2$, Laufer showed in [15] that $\sigma_1 = 0$ if and only if $y$ is a rational singularity. Precisely he proved $\dim R^1\pi_*O_{U^*} = \sigma_1$. Knöller [11] calculates $r_m$ and $\lim_{m \to \infty} r_m/m^2$ for several singularities. In particular, the condition $r_m = 0$ for all $m$ characterizes the rational double points (See also [12], for an application).

References


[9] ______: On logarithmic Kodaira dimension of algebraic varieties. to appear


[17] ______: Kodaira dimensions of complements of divisors. To appear