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Kodaira dimensions for fibre spaces.

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In the present note we shall discuss briefly Kodaira dimensions for singular varieties and fibre spaces.

§ 1. Kodaira dimension for an algebraic variety.

There are several definitions of Kodaira dimensions for non-complete algebraic varieties. (See Iitaka's and Sakai's articles in this volume.) Here we follow Sakai's definition when an algebraic variety is non-singular, and we shall show how to calculate Kodaira dimensions for complete varieties.

Let M be a non-singular algebraic variety. For a positive integer m we set $F_m(M) = \{\varphi \in H^0(M, \underline{O}(mK_M)) \mid \|\varphi\|_m < \infty\}$ where $\|\varphi\|_m$ is defined by

$$\|\varphi\|_m = \left\{ \left(\frac{-1}{2\sqrt{-1}}\right)^{\frac{n(n-3)}{2}} \int |\varphi_i|^{2/m} dz_i^1 \wedge \cdots \wedge dz_i^n \wedge d\bar{z}_i^1 \wedge \cdots \wedge d\bar{z}_i^n \right\}^{1/2},$$

$$\varphi|_{V_i} = \varphi_i(z)(dz_i^1 \wedge \cdots \wedge dz_i^n)^m .$$

$F_m(M)$ is a finite dimensional vector space. Let $\{\varphi_0, \varphi_1, \cdots, \varphi_N\}$ be a basis for $F_m(M)$. Then

$$\Phi_{F_m} : \begin{array}{ccc} M & \longrightarrow & \mathbb{P}^N \\ \cup & & \cup \\ z & \longmapsto & (\varphi_0(z):\varphi_1(z):\cdots:\varphi_N(z)) \end{array}$$

is a rational mapping. The Kodaira demension $\kappa(M)$ of M is defined by

$$\kappa(M) = \begin{cases} \max\limits_{m \in \mathbb{N}(M)} \dim \Phi_{F_m}(M), & \text{if } \mathbb{N}(M) \neq \emptyset \\ -\infty, & \text{if } \mathbb{N}(M) = \emptyset, \end{cases}$$

where $\mathbb{N}(M) = \{ m \in \mathbb{N} \mid F_m(M) \neq 0 \}$.

If M is complete and non-singular, $\kappa(M)$ is the usual Kodaira dimension of M. Let V be a non-singular complete algebraic variety and let G be a finite group of analytic automorphisms of V. We set $M = (V/G)_{reg}$ = the non-singular part of the quotient variety. Then it is easy to show that

$$F_m(M) = H^0(V, \underline{O}(mK_V))^G .$$

Hence if we know the action of G on $H^0(V, \underline{O}(mK_V))$, it is easy to calculate $\kappa(M)$. What is a relation between $\kappa(M)$ and $\kappa(V/G)$? The problem can be solved if we know how many elements of $F_m(M)$ can be extended to elements of $H^0(\bar{M}^*, \underline{O}(mK_{\bar{M}^*}))$ where $\bar{M}^*$ is a non-singular model of V/G. If G

has only isolated fixed point, the problem is reduced to the problem of local Kodaira codimensions. Let (U,x) be an isolated singular point such that U is Stein and $U-x$ is non-singular. We set

$$r_m = \dim \left\{ H^0(U-x, \underline{O}(mK_{U-x})) / H^0(U^*, \underline{O}(mK_{U^*})) \right\}$$

where U^* is a resolution of (U,x). The problem of local Kodaira codimensions is to study the asymptotic behaviour of r_m when m becomes $+\infty$. When $\dim U = 2$, there is a satisfactory theory due to Knöller [2] and Ma. Kato [1]. When $\dim U = 3$, Kuramoto is studying the problem. One of his result is the following. Let $N_{n,p,q}$ be the quotient space $\mathbb{C}^3/\langle g \rangle$, when

$$g : (z_1, z_2, z_3) \longmapsto (e_n^p z_1, e_n^q z_2, e_n z_3)$$

$$e_n = \exp(2\pi\sqrt{-1}/n), \quad (n,p)=(n,q)=1 \quad .$$

Then for the isolated singular point of $N_{n,p,q}$, we have

$$\frac{(n-p-q-1)^3}{6 \cdot pq} \leq \lim \frac{r_m}{m^3} \leqq \frac{1}{6} n^3 \quad .$$

Of course we should consider a similar problem for (U,E) where E is a singular locus of U, E is compact, and $U-E$ is non-singular. If (U,E) is a cyclic quotient singularity, there are some partial results.

Another example is a Hilbert modular variety.

Let k be a totally real algebraic member field with $[k:\mathbb{Q}] = n$. Let Γ be a discrete subgroup of $SL(2,k)$ which is commensurable with $SL(2,\underline{O})$ where $\underline{O}$ is the ring of integers of k. Γ acts H^n properly discontinuously. $V = H^n/\Gamma$ is a normal quasi-projective algebraic variety and if we add a finite member of cusps $\{\infty_1, \ldots, \infty_h\}$ to H^n/Γ, then $\bar{V} = H^n/\Gamma \cup \{\infty_1, \ldots, \infty_h\}$ is a complete normal projective variety. It is easily shown that $H^0(V_{reg}, \underline{O}(mK))$ corresponds to the space of modular forms of weight $2m$. Using the structure theorem of a cusp singularity it is not difficult to show that $F_m(V_{reg})$ corresponds to the vector space of cusp forms of weight $2m$. Thus $\kappa(V_{reg}) = n$. The local Kodaira codimension for cusp singularity has been studied by Knöller [2] when $n = 2$.

By a similar method we can study the Kodaira dimensions for S_g/Γ where S_g is the Siegel upper half plane of degree g and Γ is a discrete subgroup of $S_p(g,\mathbb{R})$ commensurable with $S_p(g,\mathbb{Z})$.

§ 2. Kodaira dimensions for fibre spaces.

By a fibre space $\varphi : V \longrightarrow W$ we mean that V and W are non-singular complete algebraic varieties defined over $\mathbb{C}$ and

φ is surjective with connected fibres. In this situation there is the following conjecture.

<u>Conjecture.</u> $\kappa(V) \geqq \kappa(W) + \kappa($a general fibre of $\varphi)$.

Very little is known about this conjecture. At the moment there are two methods to attack the conjecture. The first method is to construct element of $H^0(V,\underline{O}(mK_V))$ (see Ueno [4], Nakamura and Ueno [3]). The second method is to find a certain estimate for $P_m(V)$ (see Ueno [5]). Both methods are deeply related to each other. The first method is deeply depend on the structure of moduli spaces of algebraic varieties and degenerations of algebraic varieties. The second method is related to variation of Hodge structures.

To explain the first method we shall consider a family of principally polarized abelian varieties $\varphi : V \longrightarrow W$. We assume that φ is smooth so that the family $\varphi : V \longrightarrow W$ is obtained locally from the universal family $\varpi : \mathcal{V} \rightarrow S_g$ of principally polarized abelian varieties over the Siegel upper half plane. For simplicity we assume that $\varphi : V \longrightarrow W$ has the zero section O(i.e. for each $w \in W$, $O(w)$ is the zero of the abelian variety $A_w = \varphi^{-1}(w)$). Let $\widetilde{W}$ be the universal covering of W. Then there is a holomorphic mapping $T : \widetilde{W} \longrightarrow S_g$ called the period mapping of the family and a group representation

$$\Phi : \pi_1(W) \longrightarrow S_p(g,\mathbb{Z}) \quad \text{such that}$$

$$T(r\cdot\tilde{w}) = \Phi(r)\cdot T(\tilde{w}) .$$

$\mathbb{Z}^{2g}$ acts on $\tilde{W} \times \mathbb{C}^g$ in the form

$$\nu : \tilde{W} \times \mathbb{C}^g \longrightarrow \tilde{W} \times \mathbb{C}^g$$

$$(\tilde{w},(\zeta_1\cdot\zeta_g)) \longmapsto (\tilde{w},(\zeta_1,\ldots,\zeta_g)+\nu\binom{I_g}{T(\tilde{w})}) .$$

The action is properly discontinuous and fixed point free. We let $\tilde{V}$ be the quotient manifold $\tilde{W}\times\mathbb{C}^g/\mathbb{Z}^{2g}$. There is a natural surjective proper morphism $\tilde{\varphi} : \tilde{V} \longrightarrow \tilde{W}$. The fundamental group $\pi_1(W)$ operates on $\tilde{V}$ as follows.

$$r : \tilde{V} \longrightarrow \tilde{V}$$

$$(\tilde{w}, [\zeta]) \longrightarrow (r\tilde{w}, [\zeta\cdot f_r(\tilde{w})]) ,$$

where

$$f_r(\tilde{w}) = (C_r\ T(\tilde{w}) + D_r)^{-1}, \qquad \Phi(r) = \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix}.$$

The action is properly discontinuous and free from fixed points. Thus we obtain a quotient manifold $\hat{V} = \tilde{V}/\pi_1(W)$ and a surjective morphism $\hat{\varphi} : \hat{V} \longrightarrow W$. It is easy to show that this fibre space is isomorphic to the original fibre space $\varphi : V \longrightarrow W$. If $\varphi : V \longrightarrow W$ has no global sections, $\varphi : V \to W$ can be obtained from $\hat{\varphi} : \hat{V} \longrightarrow W$ by twisting the fibration (see Ueno [6]). Let $f(z)$ be a modular form of weight m such that

$f(T(\check{w})) \neq 0$. Such a modular form always exists if m is sufficiently large. We set

$$\tilde{\Omega} = f(T(\tilde{w}))\,\omega \wedge (d\zeta_1 \wedge \cdots \wedge d\zeta_g)^m$$

where ω is a holomorphic (or meromorphic) m-ple n-forms on W. Then $\tilde{\Omega}$ is a holomorphic (or meromorphic) m-ple (n+g)-form on $\tilde{V}$ and in view of the action of $\pi_1(W)$ on $\tilde{V}$, $\tilde{\Omega}$ induces a holomorphic (or meromorphic) m-ple (n+g)-form on $\hat{V}$ (or V). Thus we obtain

$$mK_V = \varphi^*(mK_W + F)$$

where F is an effective divisor on W defined by $f(T(\tilde{w}))=0$. This implies the affirmative answer to the above conjecture in our situation.

In general case, the morphism φ is not smooth. We let S be a subvariety of W such that φ is smooth at every point of $V' = V - \varphi^{-1}(s)$. Then by a similar method as above we can construct a holomorphic (or meromorphic) m-ple (n+g)-form Ω' on V' and the problem is reduced to extension of Ω' to the whole space V. For that purpose we need some informations about singular fibres and behaviour of modular forms near the boundary. In this way sometimes we obtain the m-th canonical bundle formula.

Let us discuss the second method. Let $\varphi: V \longrightarrow W$ be a

smooth surjective morphism of a non-singular $(n+m)$-dimensional algebraic variety V onto a non-singular m-dimensional algebraic variety W. Suppose moreover that φ has connected fibres.
We set $K_{V/W} = K_V \otimes \varphi^*(K_W^{-1})$ and $\mathcal{L} = \varphi_*(\underline{O}(K_{V/W}))$. Then $\mathcal{L}$ is a locally free sheaf of rank p where $p = p_g(V_x)$, $V_x = \varphi^{-1}(x)$, $x \in W$ and $\mathcal{L}$ is the dual sheaf of $R^m\varphi_*\underline{O}_V$. Moreover it is easy to see that $p_g(V) = \dim H^0(W, \mathcal{L} \otimes \underline{O}(K_W))$. Therefore if we know the structure of $\mathcal{L}$, we can calculate $p_g(V)$. There is an important result due to Griffiths.

Proposition. <u>The curvature of</u> $\mathcal{L}$ <u>is positive semi-difinite</u>.
This is a corollary to Griffiths' results on the theory of Hodge bundles. From this proposition if K_W is good we can caluculate $p_g(V)$. (e.g. W is a curve of genus $g \geqq 2$, W is a non-singular complete intersection with $\kappa(W) = \dim W$ etc.) Moreover there is a natural homomorphism

$$\rho_m : H^0(W, \underline{S}^m(\mathcal{L}) \otimes K_W^m) \longrightarrow H^0(V, \underline{O}(mK_V))$$

where $S^m(\mathcal{L})$ is the m-th symmetric product of $\mathcal{L}$. Therefore $p_m(V) \geqq \dim \mathrm{Im}\,\rho_m$. But the locally free sheaf $\underline{S}^m(\mathcal{L})$ has not been studied even the classical case. For example let us consider the case where $\varphi : V \longrightarrow W$ is a family of principally polarized abelian varieties with level n structure.

Put $W^* = S_g / \Gamma^{(n)}, n \geqq 3$. Let $\pi : \mathcal{A}_n \longrightarrow W^*$ be the universal family of principally polarized abelian varieties with level n structure. Then there is the period map $\varpi : W \longrightarrow W^*$ and $\varphi : V \longrightarrow W$ is isomorphic to the pull-back of the family $\pi : \mathcal{A}_n \longrightarrow W^*$ by means of ϖ. Therefore for the study of the sheaf $\underline{S}^m(\mathcal{L})$ it is enough to consider the family π. It is easily shown that the sheaf $\mathcal{L}$ is isomorphic to the dual of the normal bundle N of the image of zero section in $\mathcal{A}_n$. The line bundle $(\bigwedge^{\ell} N^*)^{\otimes m}$, $\ell = \frac{1}{2}n(n+1)$ corresponds to the sheaf of modular forms of weight m and for this line bundle we have a good theory (at least for the application to algebraic geometry). The curvature of the vector bundle $\underline{S}^m(N^*)$ is positive definete so that $\underline{S}^m(N^*)$ has sufficiently many sections for $m \gg 0$. But we don't know the way to construct these sections explicitly. If this would be possible we would have a good information on $I_m \rho_m$. Moreover if we know asymptotic behaviour of sections of $\underline{S}_m(N^*)$ near the boundary, we would be able to consider the case where $\varphi : V \longrightarrow W$ is not necessarily smooth.

As was shown above, the second method is deeply related to Griffiths' theory of variation of Hodge structure. To any family $\varphi : V \longrightarrow W$ with smooth φ we can attach the family of Griffiths' intermediate Jacobians $\varpi_1 : \underline{J}_1 \longrightarrow W$ and the family of Weil's intermediate Jacobians $\tilde{\omega}_2 : \underline{J}_2 \longrightarrow W$. $\tilde{\omega}_1$ is holomorphic but $\tilde{\omega}_2$ is in general real analytic. Hence the second method is related to construct the theory of matrix valued automorphic forms on Griffiths' period matrix domain. Of course in this case we can only expect that these automorphic forms are real analytic, and if we restrict them to certain subvarieties (the image of period mappings) they are holomorphic. If this is possible, the conjecture will be solved when the geometric genus of a general fibre of φ is positive.

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