

ON THE AVERAGE ORDER OF THE SUM $\sum_{p \leq x} \left(\frac{p}{q} \right)$

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Throughout in this note p, q will designate odd prime numbers; (m/n) is the Legendre-Jacobi symbol (with n odd).

It is proved by H. Heilbronn (On the averages of some arithmetical functions of two variables, *Mathematika* 5 (1958), 1 - 7) that

$$\sum_{p \leq x} \sum_{q \leq x} \left(\frac{p}{q} \right) = O \left(x^{\frac{7}{4}} (\log x)^{-\frac{5}{4}} \right) \quad (x > 2),$$

so that we have in a sense

$$(*) \quad \text{average order } \sum_{p \leq x} \left(\frac{p}{q} \right) = O \left(x^{\frac{3}{4}} (\log x)^{-\frac{1}{4}} \right)$$

(and similarly for 'average order $\sum_{q \leq x} \left(\frac{p}{q} \right)$ ').

The main purpose of this note is to improve (*) to a certain extent, within the logarithmic factor.

We shall first examine the sum

$$\sum_{p \leq x} \sum_{q \leq y} \left(\frac{p}{q} \right) \quad (x, y > 2).$$

Following the argument of Heilbronn, we have

$$\begin{aligned} \left(\sum_{p \leq x} \sum_{q \leq y} \left(\frac{p}{q} \right) \right)^2 &\leq \sum_{q \leq y} 1 \cdot \sum_{q \leq y} \left(\sum_{p \leq x} \left(\frac{p}{q} \right) \right) \\ &\leq \pi(y) \sum_{p_1} \sum_{p_2} \sum_q \left(\frac{p_1 p_2}{q} \right), \end{aligned}$$

where $\pi(t)$ denotes the number of primes $p \leq t$, and

$$\begin{aligned} \left(\sum_{p_1} \sum_{p_2} \sum_q \left(\frac{p_1 p_2}{q} \right) \right)^2 &\leq \sum_{p_1} \sum_{p_2} 1 \cdot 2 \sum_{1 \leq n \leq x^2} \left(\sum_q \left(\frac{n}{q} \right) \right)^2 \\ &\leq 2\pi^2(x) \sum_{q_1} \sum_{q_2} \sum_n \left(\frac{n}{q_1 q_2} \right). \end{aligned}$$

Now

$$\begin{aligned} \sum_{1 \leq n \leq x^2} \left(\frac{n}{q_1 q_2} \right) &\leq x^2 \quad \text{if } q_1 = q_2; \\ \sum_{1 \leq n \leq x^2} \left(\frac{n}{q_1 q_2} \right) &\leq (q_1 q_2)^{\frac{1}{2}} \log q_1 q_2 \ll y \log y \quad \text{if } q_1 \neq q_2, \end{aligned}$$

so that

$$\sum_{q_1, q_2} \sum_n \left(\frac{n}{q_1 q_2} \right) \ll \pi(y)x^2 + \pi^2(y)y \log y.$$

Therefore

$$\begin{aligned} \left(\sum_{p \leq x} \sum_{q \leq y} \left(\frac{p}{q} \right) \right)^4 &\ll \pi^2(x)\pi^2(y)(x^2\pi(y) + \pi^2(y)y \log y) \\ &\ll \frac{x^4}{\log^2 x} \frac{y^3}{\log^3 y} + \frac{x^2}{\log^2 x} \frac{y^5}{\log^3 y}. \end{aligned}$$

By virtue of the quadratic reciprocity law the above argument can be repeated with p, q interchanged by each other, thus giving

$$\left(\sum_{p \leq x} \sum_{q \leq y} \left(\frac{p}{q} \right) \right)^4 \ll \frac{x^3}{\log^3 x} \frac{y^4}{\log^2 y} + \frac{x^5}{\log^3 x} \frac{y^2}{\log^2 y}.$$

Hence:

$$\sum_{p \leq x} \sum_{q \leq y} \left(\frac{p}{q} \right) = \begin{cases} O\left(\frac{x}{\log^{\frac{1}{2}} x} \frac{y^{\frac{3}{4}}}{\log^{\frac{3}{4}} y} \right) & \text{if } x \geq y, \\ O\left(\frac{x^{\frac{3}{4}}}{\log^{\frac{3}{4}} x} \frac{y}{\log^{\frac{1}{2}} y} \right) & \text{if } x \leq y. \end{cases}$$

Next, let r be a fixed positive integer and consider the sum

$$S_r = S_r(x, y) = \sum_{q \leq y} \left(\sum_{p \leq x} \left(\frac{p}{q} \right) \right)^r \quad (x, y > 2).$$

We have

$$S_r = \sum_{q \leq y} \sum_{p_1, \dots, p_r \leq x} \left(\frac{p_1 \cdots p_r}{q} \right);$$

$$S_r^2 \leq \sum_{p_1, \dots, p_r} 1 \cdot \sum_{p_1, \dots, p_r} \left(\sum_{q} \left(\frac{p_1 \cdots p_r}{q} \right) \right)^2$$

$$\leq r! \pi^r(x) \cdot \sum_{q, q'} \sum_{1 \leq n \leq x^r} \left(\frac{n}{qq'} \right)$$

$$\ll \pi^r(x) (\pi(y)x^r + \pi^2(y)y \log y)$$

$$\ll \frac{x^{2r}}{\log^r x} \frac{y}{\log y} \quad \text{if} \quad x^r \geq y^2.$$

Putting $y = x^{r/2}$ (> 2), we find

$$\text{average order} \quad \sum_{p \leq x} \left(\frac{p}{q} \right) \ll \frac{x \log^{\frac{1}{2r}} y}{\log^{\frac{1}{2}} x y^{\frac{1}{2r}}} = \frac{x^{\frac{3}{4}}}{(\log x)^{\frac{1}{2} - \frac{1}{2r}}}.$$

Since r may be chosen arbitrarily large, this proves that

$$\text{average order} \quad \sum_{p \leq x} \left(\frac{p}{q} \right) = O_{\varepsilon} \left(x^{\frac{3}{4}} (\log x)^{-\frac{1}{2} + \varepsilon} \right)$$

for any fixed $\varepsilon > 0$.

NOTE. As has been suggested by Mr. T. Hirano, our argument applies of course to the sums of the form

$$T_r = \sum_{y_1 < q \leq y} \left(\sum_{x_1 < p \leq x} \left(\frac{p}{q} \right) \right)^r \quad (r \geq 1),$$

yielding e.g. an estimate

$$T_x^2 \ll (\pi(x) - \pi(x_1))^2.$$

$$((\pi(y) - \pi(y_1))(x^r - x_1^r) + \pi^2(y)y \log y) .$$

Here, in some occasions there may be possibilities of improving this simple and weak estimate, if we appeal to a rather deep result of D. A. Burgess on character sums (cf. D. A. Burgess: On character sums and L -series. II. Proc. London Math. Soc. (3) 13 (1963), 524-536).