数理解析研究所講究録
278巻 1976年 85-99

Dissipations and Derivations

A. Kishimoto

Department of Physics, Kyoto University

§1. Introduction.

Recently various authors have studied unbounded derivations of C^*algebras [2,3,4,6,7,10,11,13]. In particular Powers and Sakai [10] have studied unbounded derivations of UHF algebra.

The purpose of the present note is to show a usefulness of the notion of "dissipative operators" [9,17] in the study of derivations of C^*algebras.

Our first result is that an everywhere defined "dissipation" is bounded, which implies the well-known theorem concerning derivations [5,12].

Our second result is about a normal *derivation of UHF algebra satisfying a special condition discussed in [1,10,14,15]. For such a *derivation, we prove that its closure is a generator of a one-parameter group of *automorphisms. As its application we consider one-dimensional lattice system.

To appear in Commun. math. Phys.

§2. Bounded derivation

Let $\mathfrak{A}$ be a Banach space. For each $x \in \mathfrak{A}$ there is at least one non-zero element f of the dual Banach space $\mathfrak{A}^*$ such that $\langle x, f \rangle = \|x\| \cdot \|f\|$ by the Hahn-Banach theorem. An f_x denotes one of them throughout this note.

Definition 1. [9] A linear map γ with domain $\mathcal{D}(\gamma)$ in a Banach space is called dissipative if there is an f_x such that

$$\mathrm{Re} \langle \gamma x, f_x \rangle \leq 0$$

for each $x \in \mathcal{D}(\gamma)$.

Definition 2. A linear map δ with domain $\mathcal{D}(\delta)$ in a Banach space is called derivative if there is an f_x such that

$$\mathrm{Re} \langle \delta x, f_x \rangle = 0$$

for each $x \in \mathcal{D}(\delta)$.

Let $\mathfrak{A}$ be a C^*-algebra. A linear map δ of $\mathfrak{A}$ is called a derivation if it satisfies

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for $x, y \in \mathcal{D}(\delta)$, where $\mathcal{D}(\delta)$, the domain of δ, is a *-subalgebra in $\mathfrak{A}$. A derivation δ is a *-derivation if $\delta(x)^* = \delta(x^*)$ for $x \in \mathcal{D}(\delta)$. In the following we will be concerned with only *-derivation and so omit *.

A linear map δ of $\mathfrak{A}$ is a derivation if δ and $-\delta$ are dissipations whose definition is:

Definition 3. [8] A linear map γ of a C^*-algebra $\mathfrak{A}$ is called a dissipation if it satisfies

$$\gamma(x)^* = \gamma(x^*)$$
$$\gamma(x^*x) \geq \gamma(x^*)x + x^*\gamma(x)$$

for each $x \in \mathcal{D}(\gamma)$, where $\mathcal{D}(\gamma)$, the domain of γ, is a *-subalgebra.

Remark 1. Call γ an "n-dissipation" if $\gamma \otimes \iota ; \mathfrak{A} \otimes F_n \to \mathfrak{A} \otimes F_n$ is a dissipation where F_n is an algebra of all $n \times n$ matrices and ι is an identity map. If γ is a 2n-dissipation of a C*-algebra with identity and $\mathcal{D}(\gamma) \ni 1$, then γ' defined by $\gamma'(x) = \gamma(x) - \frac{1}{2}\{\gamma(1)x + x\gamma(1)\}$ is an n-dissipation. Note $\gamma(1) \leq 0$ and $\gamma'(1) = 0$. (See [8] for the arguments of bounded complete dissipations; a complete dissipation is defined to be an n-dissipation for all n.)

Lemma 1. Let γ be a dissipation with domain $\mathcal{D}(\gamma)$. Suppose that for any positive $x \in \mathcal{D}(\gamma)$ there is an f_x such that $\mathrm{Re}\langle \gamma x, f_x \rangle \leq 0$. Then γ is dissipative.

Proof. Note that f_x is positive for a positive $x \in \mathfrak{A}$ [12]. If we define fx^* and xf in $\mathfrak{A}^*$ for $x \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$ by $\langle a, fx^* \rangle = \langle x^* a, f \rangle$ and $\langle a, xf \rangle = \langle ax, f \rangle$ ($a \in \mathfrak{A}$), then $x f_{x^*x} = f_{x^*}$ and $f_{x^*x} x^* = f_x$. For any $x \in \mathcal{D}(\gamma)$, there is an $f = f_{x^*x}$ such that $\langle \gamma(x^*x), f \rangle \leq 0$. Then we have

$$\begin{aligned} 0 &\geq \langle \gamma(x^*x), f \rangle \\ &\geq \langle \gamma x^*, xf \rangle + \langle \gamma x, fx^* \rangle \\ &= 2\,\mathrm{Re}\langle \gamma x, fx^* \rangle . \end{aligned}$$

Lemma 2. (Lemmas 3.3 and 3.4 in [9]) A dissipative operator with dense domain in a Banach space is closable and its closure is also dissipative.

Sketch of the proof. Let γ be the dissipative operator. Let $x_n \in \mathcal{D}(\gamma)$ with $x_n \to 0$ and $\gamma x_n \to y$. For any $a \in \mathcal{D}(\gamma)$ and $\lambda \in \mathbb{R}$, let $f_{n,\lambda} = f_{a+\lambda x_n}$ with $\|f_{n,\lambda}\| = 1$ and $\mathrm{Re}\langle \gamma(a+\lambda x_n), f_{n,\lambda} \rangle \leq 0$. We may suppose

$f_{n,\lambda} \to f_\lambda$ $(n \to \infty)$ and $f_\lambda \to f'$ $(\lambda \to \infty)$. Then we have $f' = f_a$ and $\mathrm{Re}\langle y, f'\rangle \leq 0$. We may suppose $f' \to f$ $(a \to y)$. Then $f = f_y$ and $\|y\| = \mathrm{Re}\langle y, f\rangle \leq 0$, i.e. $y = 0$. The rest of the proof is easy.

In the rest of this section we will treat only everywhere defined operators.

Theorem 1. A dissipation γ of a C*-algebra $\mathfrak{A}$ $(= \mathcal{D}(\gamma))$ is dissipative and bounded.

Proof. We suppose $\mathfrak{A} \ni 1$. If $\mathfrak{A} \not\ni 1$, we can consider a dissipation γ_1 of $\mathfrak{A}_1 = \mathfrak{A} + \mathbb{C}\cdot 1$ defined by $\gamma_1(x + \lambda 1) = \gamma(x)$ $(x \in \mathfrak{A}, \lambda \in \mathbb{C})$.

Let $x \in \mathfrak{A}$ be positive. Setting $h = (\|x\| 1 - x)^{1/2}$, we have for $f = f_x$,

$$\begin{aligned} \langle \gamma x, f\rangle &\leq \langle \gamma(x - \|x\|\cdot 1), f\rangle \\ &= -\langle \gamma h^2, f\rangle \\ &\leq -\langle (\gamma h) h, f\rangle - \langle h \gamma h, f\rangle \\ &= 0 \end{aligned}$$

where we have used the Schwartz inequality and the fact $\langle h^2, f\rangle = 0$ and $f \geq 0$. Hence γ is dissipative by Lemma 1 and closed by Lemma 2. An everywhere defined closed operator is bounded by the closed graph theorem.

Corollary. A derivation of a C*-algebra is derivative and bounded.

Proof. The proof is quite similar to the above. Or it follows from the above theorem by the following remark.

Remark 2. From the proof of Theorem 1 we can conclude that if γ is a dissipation, for any f_x, $\mathrm{Re}\langle \gamma x, f_x\rangle \leq 0$. It is immediate for $x \geq 0$. For a general $x \in \mathfrak{A}$, any f_x is equal to fx^* where $f = f_{x^*x} = \|x\|^{-1}|f_x|$. (Let $x = u|x|$ be the polar decomposition of in the enveloping von Neumann algebra of $\mathfrak{A}$. Then $|f_x| = f_x u$, from which we can deduce $|f_x| = f_{|x|} = f_{x^*x}$.) The same situation prevails for derivations. (See Remark 2 in [9])

Remark 3. [6] A dissipation γ generates a uniformly continuous one-parameter semi-group of positive contractions $\Phi_t = e^{t\gamma}$. Lindblad showed the equivalence of (i) and (ii);

(i) Φ_t is uniformly continuous, $\Phi_t(1) = 1$ and
$\Phi_t(x^*)\Phi_t(x) \leq \Phi_t(x^*x)$.

(ii) γ is a dissipation with $\gamma(1) = 0$.

Finally we remark the following property of a derivation δ . Let x be self-adjoint and $\mathcal{C}(x)$ be the commutative C^* subalgebra generated by x and 1 . Let φ be a character of $\mathcal{C}(x)$ and $\bar{\varphi}$ be any norm-preserving extension of φ ($\bar{\varphi}$ is a state). Then $\langle \delta x, \bar{\varphi} \rangle = 0$ which is considered as generalization of derivativeness (see [5]).

This is easily seen; if a polynomial $P(x)$ of x satisfies $\langle P'(x), \varphi \rangle = P'(\langle x, \varphi \rangle) = 0$, then $\langle \delta P(x), \bar{\varphi} \rangle = 0$ The set of such $P(x)$ is dense in $\mathcal{C}(x)$ and so $\langle \delta x, \bar{\varphi} \rangle = 0$ by the continuity of δ .

§3. Unbounded derivations

In the following the domain of a derivation or dissipation of a C^*-algebra is a dense *-subalgebra.

Theorem 2. Let γ be a dissipation of a C^*-algebra $\mathfrak{A}$. If $\mathcal{D}(\gamma)$ is closed under the square root operation of positive elements, then γ is dissipative and hence closable.

Proof. [4,10] The proof that γ is dissipative is quite similar to that of Theorem 1. By Lemma 2 it is closable.

Let $\mathfrak{A}$ be a uniformly hyperfinite C^*algebra (UHF algebra). A derivation δ in $\mathfrak{A}$ is said to be normal [10] if $\mathcal{D}(\delta)$ is the union of an increasing sequence of finite type I subfactors $\{\mathfrak{A}_n \mid n=1,2,\cdots\}$ in $\mathfrak{A}$.

Corollary. A normal derivation of a UHF algebra is derivative and hence closable. Its closure is also a derivative derivation.

Let τ be a unique tracial state on a UHF algebra $\mathfrak{A}$. A derivation δ in $\mathfrak{A}$ is said to be regular [10] if $\langle \delta(a), \tau\rangle = 0$ for $a \in \mathcal{D}(\delta)$.

Let δ be a normal derivation. Since $\langle ab, \tau\circ\delta\rangle = \langle ba, \tau\circ\delta\rangle$ for $a, b \in \mathcal{D}(\delta) \equiv \bigcup \mathfrak{A}_n$ and $\langle 1, \tau\circ\delta\rangle = 0$, $\tau\circ\delta / \mathfrak{A}_n = 0$ for any n. Hence δ is regular [10].

Theorem 3. If a derivation δ in a UHF algebra is regular, then δ is derivative.

Proof. Let $L^2(\mathfrak{A}, \tau)$ be a Hilbert space completion of a UHF algebra $\mathfrak{A}$ with inner product $\langle x, y\rangle_\tau = \langle y^* x, \tau\rangle$. Let x be a positive element of $\mathcal{D}(\delta)$ and $L^2(C(x), \tau)$ be the closed subspace spaned by $C(x)$. Let E_x be the orthogonal projection onto $L^2(C(x), \tau)$. If δ is regular,

$$
\begin{aligned}
0 &= \langle x^n, \tau \circ \delta \rangle \\
&= n \langle x^{n-1} \delta(x), \tau \rangle \\
&= n \langle \delta(x), x^{n-1} \rangle_\tau
\end{aligned}
$$

Hence $E_x \delta(x) = 0$. Let φ be a character of $C(x)$ and $\hat{\varphi}$ be any norm-preserving extension of φ into $L^\infty(C(x), \tau)^*$. Since $E_x : \mathfrak{A} \equiv L^\infty(\mathfrak{A}, \tau) \to L^\infty(C(x), \tau)$ is a contraction, $\bar{\varphi} = \hat{\varphi} \circ E_x$ is an element of $\mathfrak{A}^*$. Let φ be a character such that $\langle x, \varphi \rangle = \|x\| \, \|\varphi\| = \|x\|$ and let $\bar{\varphi} = \hat{\varphi} \circ E_x$. Then $\bar{\varphi} = f_x$ and $\langle \delta x, \bar{\varphi} \rangle = 0$. Now the proof is completed by Lemma 1.

Let δ be a normal derivation in $\mathfrak{A}$. Let $\tilde{\delta}$ be the greatest linear extension of δ in all linear extensions γ satisfying

$$
\begin{aligned}
&\gamma(axb) = \delta(a)xb + a\gamma(x)b + ax\delta(b) \\
&\langle x, \tau \circ \gamma \rangle = 0 \quad , \quad a, b \in \mathcal{D}(\delta), \; x \in \mathcal{D}(\gamma) .
\end{aligned}
$$

$\tilde{\delta}$ is called the greatest regular extension of a normal derivation δ [10].

<u>Theorem 4</u>. Let δ be a normal derivation. Suppose that $\tilde{\delta}$ is a derivation (or $\tilde{\delta}$ is derivative) and that there is an infinitesimal generator δ_1 of a strongly continuous group of *-automorphisms such that $\delta_1 \supseteq \delta$. Then $\delta_1 = \tilde{\delta}$.

Proof. Since δ_1 is regular [10], $\delta_1 \subseteq \tilde{\delta}$. As $(1 \pm \tilde{\delta})\mathcal{D}(\tilde{\delta}) \supseteq (1 \pm \delta_1)\mathcal{D}(\delta_1) = \mathfrak{A}$ and $\tilde{\delta}$ is derivative by Theorem 3, $\tilde{\delta}$ is an infinitesimal generator by the following theorem and remark. Hence $\delta_1 = \tilde{\delta}$.

Theorem 5. Let δ be a derivation of a C^*-algebra $\mathfrak{A}$. If δ is derivative and closed and $(1\pm\delta)\mathcal{D}(\delta)$ is dense in $\mathfrak{A}$, then δ is an infinitesimal generator of a strongly continuous group of $*$-automorphisms.

Proof. If f_x satisfies $\mathrm{Re}\langle\delta x, f_x\rangle = 0$ and $\|f_x\| = 1$,

$$\|(\delta+\lambda)x\| \geq \pm\mathrm{Re}\langle(\delta+\lambda)x, f_x\rangle = \pm\mathrm{Re}\lambda\,\|x\|$$

$$\text{i.e. } \|(\delta+\lambda)x\| \geq |\mathrm{Re}\lambda|\cdot\|x\|$$

The rest of the proof is standard [2, 3, 4].

Remark 4. The assumption that δ is a derivation in Theorem 5 can be replaced as follows: Let δ be a linear operator with dense domain $\mathcal{D}(\delta)$ such that $\mathcal{D}(\delta)\ni 1$ and $\delta(1)=0$. It is shown as follows: By a result in the Hill-Yosida semi-group theory [17] δ generates a strongly continuous group of contractions ρ_t on $\mathfrak{A}$. Since $\rho_t(1)=1$ (by the assumption $\delta(1)=0$) and $\|\rho_t\| = 1$ they are positive contractions. As they form a group, they are order-isomorphisms. Thus ρ_t is a strongly continuous one-parameter group of Jordan automorphisms. Hence we have to show that any strongly continuous group of Jordan automorphisms of the C^*-algebra $\mathfrak{A}$ is a group of $*$-automorphisms. Let π be any irreducible representation of $\mathfrak{A}$ and $\mathcal{H}_\pi$ be its representation space. Then the Jordan homomorphism $\pi\circ\rho_t$ of $\mathfrak{A}$ onto $\pi(\mathfrak{A})\subset\mathcal{B}(\mathcal{H}_\pi)$ is a homomorphism or an anti-homomorphism [16, Theorem 5.1]. Let

$$H = \{t\in\mathbb{R}\ ;\ \pi\circ\rho_t \text{ is a homomorphism}\}$$

$$A = \{t\in\mathbb{R}\ ;\ \pi\circ\rho_t \text{ is an anti- homomorphism}\}$$

Then H and A are both closed subsets of $\mathbb{R}$ as easily shown:
Let $\{t_n\}$ be a sequence in H such that $t_n \to t$. Then

$$\begin{aligned}\pi \circ \rho_t(xy) &= \lim \pi \circ \rho_{t_n}(xy) \\ &= \lim \pi(\rho_{t_n}(x)\rho_{t_n}(y)) \\ &= \pi \circ \rho_t(x)\, \pi \circ \rho_t(y)\end{aligned}$$

Hence $t \in H$. (Similarly for A.) Now if $\dim \mathcal{H}_\pi = 1$, $H = \mathbb{R}$. If $\dim \mathcal{H}_\pi \geq 2$, then $\mathbb{R} = H \cup A$ and $H \cap A = \phi$. This shows H and A are both open and closed subsets of $\mathbb{R}$. Since $\mathbb{R}$ is connected and H is non-empty $(H \ni 0)$, we have $H = \mathbb{R}$. Since the direct sum of all irreducible representations of the C^*-algebra $\mathfrak{A}$ is faithful, ρ_t is a homomorphism and hence a $*$-automorphism. (This remark cannot be applied in th case of von Neumann algebras. O. Bratteli showed me an example of a σ-weakly continuous group of Jordan automorphisms of a von Neumann algebra which is not a group of $*$-automorphisms.)

Remark 5. $\tilde{\delta}$ is in general not a derivation (see Problem 1 of [10]). For if δ is a normal derivation which has more than two different extensions to infinitesimal generators, then $\tilde{\delta}$ is not a derivation, as easily shown by using Theorem 4. (We can construct such δ. See Remark 3 of [10].)

Let P_n be the canonical conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}_n$. Let h_n be a self-adjoint element of $\mathfrak{A}$ such that $\delta(a) = [ih_n, a] \equiv \delta_{ih_n}(a)$ for all $a \in \mathfrak{A}_n$. Then $P_n\tilde{\delta}(x) = P_n\delta_{ih_n}(x)$ for $x \in \mathfrak{D}(\tilde{\delta})$ [10]. For if $a \in \mathfrak{A}_n$,

$$\langle aP_n \tilde{\delta}(x), \tau \rangle = \langle a \tilde{\delta}(x), \tau \rangle$$
$$= \langle ax, \tau \circ \tilde{\delta} \rangle - \langle (\delta a)x, \tau \rangle$$
$$= - \langle (\delta_{ih_n} a) x, \tau \rangle$$
$$= \langle a \delta_{ih_n} x, \tau \rangle$$
$$= \langle a P_n \delta_{ih_n} x, \tau \rangle .$$

In [10] $W \subset \mathcal{D}(\tilde{\delta})$ is defined by

$$W \equiv \{ x \in \mathcal{D}(\tilde{\delta}) ; \lim P_n \tilde{\delta}(1-P_n)x = 0 \}.$$

If we set $P_n(h_n) = k_n$,

$$W = \{ x \in \mathcal{D}(\tilde{\delta}) ; \lim \delta_{ik_n} P_n x = \tilde{\delta}(x) \}.$$

In [6] an operator $\text{ex-}\lim \delta_{ik_n}$ (the extended limit of the $\delta_{ik_n}|\mathfrak{A}_n$) is defined, whose graph is the set of $(x, y) \in \mathfrak{A} \times \mathfrak{A}$ such that there is a sequence $x_n \in \mathfrak{A}_n$, with $\|x_n - x\| \to 0$ and $\|\delta_{ik_n}(x_n) - y\| \to 0$.

In [7] an operator $\hat{\delta}$ (the graph limit of the δ_{ik_n}) is defined, whose graph is the set of $(x, y) \in \mathfrak{A} \times \mathfrak{A}$ such that there is a sequence $x_n \in \mathfrak{A}$, with $\|x_n - x\| \to 0$ and $\|\delta_{ik_n}(x_n) - y\| \to 0$.

Then

$$\delta \subset \tilde{\delta}|W \subset \text{ex-}\lim \delta_{ik_n} \subset \hat{\delta} \subset \tilde{\delta}.$$

Theorem 6. $\hat{\delta}$ is derivative.

Proof. Let $x \in \mathcal{D}(\hat{\delta})$ and $\{x_n\}$ be a sequence such that $x_n \to x$ and $\delta_{ik_n}(x_n) \to \hat{\delta}(x)$. Let $f_n = f_{x_n}$ be of norm 1. We may suppose $f_n \to f$. Then $f = f_x$ and

$$Re \langle \hat{\delta}x, f \rangle = \lim Re \langle \delta_{ik_n} x_n, f_n \rangle$$
$$= 0$$

where we have used Remark 2.

Remark 6. [6,7] $\hat{\delta}$ and ex-$\lim \delta_{ik_n}$ are closed derivations.

Lemma 3. If $\{\|h_n-k_n\|\}$ is uniformly bounded, $\tilde{\delta}$ is derivative.

Proof. Let $x\in\mathcal{D}(\tilde{\delta})$ and $f_n=f_{P_nx}$ with $\|f_n\|=1$. We may suppose $f_n\to f$. Then $f=f_x$ and

$$\begin{aligned}\mathrm{Re}<\tilde{\delta}x, f> &= \lim \mathrm{Re}<P_n\tilde{\delta}x, f_n>\\ &= \lim \mathrm{Re}<P_n\tilde{\delta}(1-P_n)x, f_n>\\ &= \lim \mathrm{Re}<P_n\delta_{ih_n-ik_n}(1-P_n)x, f_n>\end{aligned}$$

where we have used $\mathrm{Re}<P_n\tilde{\delta}P_nx, f_n>=0$, $P_n\delta_{ik_n}(1-P_n)=0$ and $\delta_{ih_n-ik_n}=\delta_{ih_n}-\delta_{ik_n}$. The last term is dominated by

$$2\|h_n-k_n\|\cdot\|(1-P_n)x\|$$

which tends to zero as $n\to\infty$.

Theorem 7. Let δ be a normal derivation. If $\{\|h_n-k_n\|\}$ is uniformly bounded, $\bar{\delta}$, the closure of δ , is an infinitesimal generator of a strongly continuous group of *-automorphisms and $\bar{\delta}=\tilde{\delta}$.

Proof. Suppose that $(1+\delta)\mathcal{D}(\delta)$ is not dense in $\mathfrak{A}$. Then there is an element f in $\mathfrak{A}^*$ such that $\|f\|=1$ and $<x+\delta x, f>=0$ for all $x\in\mathcal{D}(\delta)$. There are $x_n\in\mathfrak{A}_n\subset\mathcal{D}(\delta)\equiv\cup\mathfrak{A}_n$ such that $<x_n,f>=\|x_n\|\,\|f|\mathfrak{A}_n\|=\|f|\mathfrak{A}_n\|$. Then

$$\begin{aligned}0 &= \lim \mathrm{Re}\{<x_n, f>+<\delta x_n, f>\}\\ &= \lim \mathrm{Re}\{\|f|\mathfrak{A}_n\|+<\delta_{ih_n}x_n, f>\}\\ &= \|f\| + \lim \mathrm{Re}<\delta_{ih_n-ik_n}x_n, f>\\ &\geq 1-\overline{\lim}\, 2\cdot\|h_n-k_n\|\end{aligned}$$

where we have used $\mathcal{R}e\langle \delta i k_n x_n, f\rangle = 0$. Suppose $\| h_n - k_n \| < 1/2 - \varepsilon \ (\varepsilon > 0)$. Then it is a contradiction and hence $(1+\delta)\mathcal{D}(\delta)$ is dense in $\mathfrak{A}$. Quite similarly we can conclude that $(1-\delta)\mathcal{D}(\delta)$ is dense in $\mathfrak{A}$. Since $\bar{\delta}$ is derivative by Corollary of Theorem 3, $\bar{\delta}$ is an infinitesimal generator by Theorem 5. If $\| h_n - k_n \| < C$ for any n, we may consider $\delta/3C$ instead of δ. $\bar{\delta} = \tilde{\delta}$ follows from Theorem 4 and Lemma 3.

Remark 7. Under the assumption of Theorem 7 the one-parameter group ρ_t generated by $\bar{\delta}$ satisfies

$$\rho_t(x) = \lim e^{t\delta i k_n}(x), \quad x \in \mathfrak{A}$$

where the convergence is uniform in t on every compact subset of $(-\infty, \infty)$. This follows from Theorem 7 combined with Theorems 6 and 8 in [10] (cf. the proof of Theorem 8 below).

As an application of Theorem 7, we consider one-dimensional lattice system. Let $\{\mathfrak{A}_j : j \in Z\}$ be a family of type I finite factors and let $\mathfrak{A} = \bigotimes_{j\in Z} \mathfrak{A}_j$ be the infinite tensor product of them. Let Φ be a map from the family $P_f(Z)$ of finite subsets of Z into $\mathfrak{A}$ such that $\Phi(\phi) = 0$ and $\Phi(\Lambda)$ is a self-adjoint element of $\mathfrak{A}(\Lambda) = \bigotimes_{j\in\Lambda} \mathfrak{A}_j$. Put

$$\|\Phi\|_\alpha = \sum_{k=1}^{\infty} e^{\alpha k} \sup_j \sum_{\Lambda \ni j,\, N(\Lambda)=k} \| N(\Lambda) \|$$

where $N(\Lambda)$ denotes the number of points in Λ and $\alpha \geq 0$.

It is known [cf. 1] that if $\|\Phi\|_\alpha < \infty$ for $\alpha > 0$, there exists a one-parameter group of *-automorphisms such that

$$\rho_t(Q) = \lim_{\Lambda} e^{itU(\Lambda)} Q e^{-itU(\Lambda)} = \lim e^{t\delta_{iU(\Lambda)}} Q, \quad Q \in \mathfrak{A}$$

$$U(\Lambda) = \sum_{J \subset \Lambda} \Phi(J).$$

Now we give another sufficient condition for the existence of the above automorphism group:

Theorem 8. Suppose that (i) $\|\Phi\|_0 < \infty$ and (ii) there is an increasing sequence $\{\Lambda_n\} \subset \mathcal{P}_f(Z)$ such that $\cup \Lambda_n = Z$ and the following element $W(\Lambda_n)$ of $\mathfrak{A}$ is bounded in norm uniformly in n :

$$W(\Lambda_n) = \sum_{J} \{ \Phi(J) \;;\; J \in \mathcal{P}_f(Z),\ J \cap \Lambda \neq \phi,\ J \cap \Lambda^c \neq \phi \}$$

where Λ^c denotes the complement of Λ in Z. Then there exists a strongly continuous one-parameter group of *-automorphisms such that

$$\rho_t(Q) = \lim_{n} e^{t\delta_n}(Q) \qquad (*)$$

where $\delta_n = \delta_{iU(\Lambda_n)}$ and the convergence is uniformly in t on every compact interval of t.

Proof. By (i), $W(\Lambda_n)$ is well-defined. Let $\mathfrak{A}_n = \mathfrak{A}(\Lambda_n)$ and let $h_n = U(\Lambda_n) + W(\Lambda_n)$. Let δ be the normal derivation such that

$$\delta \,|\, \mathfrak{A}_n = \delta_{ih_n}, \quad \mathcal{D}(\delta) = \cup \mathfrak{A}_n$$

Then [1]

$$\begin{aligned} \|h_n - k_n\| &\leq \|h_n - U(\Lambda_n)\| + \|U(\Lambda_n) - k_n\| \\ &\leq 2\,\|W(\Lambda_n)\| \end{aligned}$$

where $k_n = P_n(h_n)$. Hence $\bar{\delta}$ is an infinitesimal generator by Theorem 7. Now the proof of the convergence in (*) follows as in [10]: It is shown by (i) that $\lim \delta_n = \delta$ on $\mathcal{D}(\delta)$. Then for $x \in \mathcal{D}(\delta)$

$$\begin{aligned}
&\| \{ (1 \pm \delta_n)^{-1} - (1 \pm \bar{\delta})^{-1} \} (1 \pm \bar{\delta}) x \| \\
&= \| (1 \pm \delta_n)^{-1} \{ (1 \pm \bar{\delta}) x - (1 \pm \delta_n) x \} \| \\
&\leq \| (1 \pm \bar{\delta}) x - (1 \pm \delta_n) x \| \\
&\leq \| \bar{\delta} x - \delta_n x \| \\
&\longrightarrow 0 \quad \text{as} \quad n \to \infty ,
\end{aligned}$$

where we have used $\| (1 \pm \delta_n)^{-1} \| \leq 1$.
Hence $\lim (1 \pm \delta_n)^{-1} = (1 \pm \bar{\delta})^{-1}$ since $(1 \pm \delta)\mathcal{D}(\delta)$ is dense in $\mathfrak{A}$. By the Trotter-Kato theorem [cf. 17] we get (*).

Finally we remark that the assumption (i) can be weakened by (i') $\sum_{\Lambda \ni j} \| \Phi(\Lambda) \| < \infty$ for any $j \in Z$.

Acknowledgements. The author is deeply indebted to Professor H. Araki for his encouragement and useful suggestions. He would like to thank Professor H. Hasegawa for his encouragement.

References

1. Araki, H.: On uniqueness of KMS states of one-dimensional quantum lattice systems.

2. Bratteli, O.: Self-adjointness of unbounded derivations of C^*algebras.

3. Bratteli, O., Robinson, D. W.: Unbounded derivations of C^*algebras.

4. Bratteli, O., Robinson, D. W.: Unbounded derivations of C^*algebras II.

5. Diximier, J.: Les algebres d'operateurs dans l'espace hilbertien. Gauthier-Villars, Paris (1957)

6. Herman, R.H.: Unbounded derivations.

7. Herman, R.H.: Unbounded derivations II.

8. Lindblad, G.: On the generator of quantum dynamical semi-groups.

9. Lumer, G., Phillips, R.S.: Pacific J. Math. 11, 679-698(1961)

10. Powers, R.T., Sakai, S.: J. Func. Anal. 19, 81-95 (1975)

11. Powers, R.T., Sakai, S.: Commun. math. Phys. 39, 273-288(1975)

12. Sakai, S.: C^*algebras and W^*algebras. Springer Verlag, Berlin (1970)

13. Sakai, S.: On one parameter groups of *automorphisms on operator algebras and the corresponding unbounded derivations (to appear in Amer. J. Math.)

14. Sakai, S.: Commutative normal *derivations II.

15. Sakai, S.: Commutative normal *derivations III.

16. Størmer, E.: In: Lecture notes in Physics 29, Springer Verlag, Berlin (1974)

17. Yosida, K.: Functional Analysis. Springer Verlag, Berlin (1965)