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Hypoellipticity for the second order partial differential equations

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First we shall consider a sufficient condition for the second order partial differential equations to be hypoelliptic. Secondly we shall give some applications of our method to prove the propagation of regularity for the solutions of generalized Tricomi equations.
As is well known, very general results concerning with hypoellipticity of the second order equations with real coefficients have been obtained by Hörmander [1] and Oleinik & Radkevič [5], where the assumption that the coefficients are real is crucial. Our first aim is to consider the equations with complex coefficients by a method which may be considered as an generalization of usual variational method.

Let R^N be N-dimensional Euclidean space regarded as a direct product of three Euclidean spaces:

$$R^N = R_x^m \times R_y^n \times R_t^p \qquad (m+n+p=N),$$

and generic point of R^N will be denoted by

$$(x, y, t) = (x_1, \ldots, x_m, y_1, \ldots, y_n, t_1, \ldots, t_p) \in R^N.$$

Now, we consider a partial differential equation of the form:

$$(*) \quad L(x,y,t,D)u = -\sum_{k,j=1}^{m} (a^{kj} u_{x_k})_{x_j} - \sum_{k,j=1}^{n} (a_1^{kj} u_{y_k})_{y_j}$$

$$-2\sum_{k=1}^{m}\sum_{j=1}^{n} (g^{kj} u_{x_k})_{y_j} + \sum_{k=1}^{m} b^k u_{x_k} + \sum_{j=1}^{n} b_1^j u_{y_j}$$

$$+\sum_{l=1}^{p} d^l u_{t_l} + cu = f \quad \text{in } \Omega \subset R^N,$$

where the coefficients a^{kj}, a_1^{kj}, g^{kj}, b^k, b_1^j and c are complex valued C^∞-functions defined on Ω, and only d^l, $l=1,\ldots,p$, are supposed to be real valued C^∞-functions on Ω. We consider this equation in the neighbourhood of the origin: $0 \in \Omega$.

Let A, A_1 and G denote the matrices

$$A = (a^{kj}(x,y,t))_{1 \leq k,j \leq m},$$

$$A_1 = (a_1^{kj})_{1 \leq k,j \leq n},$$

$$G = (g^{kj})_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}}.$$

The equation (*) will be considered under the following nine conditions among which two assumptions (4) and (6) are essential.

(1) $\quad \mathrm{Re}\, A \geq 0 \quad \forall (x,y,t) \in \Omega$;

that is, $\mathrm{Re} \sum_{k,j=1}^{m} a^{kj} \xi_k \bar{\xi}_j \geq 0 \quad (x,y,t) \in \Omega,\ \xi \in R^m$.

(2) $\quad a^{kk}(0) = 0 \quad , k = 1, \dots, m.$

(3) $\quad \mathrm{Re}\, A_1 \geq \alpha \cdot I$ in Ω, $\alpha > 0$;

that is, $\mathrm{Re} \sum_{k,j=1}^{n} a_1^{kj} \eta_k \eta_j \geq \alpha |\eta|^2$ in Ω, $\eta \in R^n$,

which means the operator L is strongly elliptic in the direction y.

$$(4)\quad \sum_{\sigma=1}^{m}[\mathcal{R}e\,A_{x_\sigma}]^2+\sum_{l=1}^{p}[\mathcal{R}e\,A_{t_l}]^2 \leqq C\cdot A \text{ in } \Omega,\ C>0,$$

where A_{x_σ} stands for a matrix each element of which is differentiated by x_σ.

$$(5)\quad \left|\mathcal{I}m\sum_{k,j=1}^{m}a^{kj}\xi_k\xi_j\right| \leqq C\,\mathcal{R}e\sum_{k,j=1}^{m}a^{kj}\xi_k\xi_j \text{ in } \Omega,\ \xi\in R^m,\ C>0.$$

$$(6)\quad \|v\|_{(\varepsilon)} \leqq C(|||v|||+|||Q_0v|||')\quad v\in C_0^\infty(\Omega)$$

for some ε, $0<\varepsilon\leqq 1$. Here $\|\cdot\|_{(\varepsilon)}$ means the usual Sobolev space norm and

$$|||v|||^2 \equiv \sum_{k,j=1}^{m}\int_\Omega \mathcal{R}e\,a^{kj}v_{x_k}\overline{v_{x_j}}\,dx+\sum_{j=1}^{n}\|v_{y_j}\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)},$$

$$|||v|||' = \sup_{w\in C_0^\infty(\Omega)}\frac{\left|\int_\Omega v\,w\,dx\,dy\,dt\right|}{|||w|||},$$

$$Q_0=Q_0(x,y,t,D)=\sum_{k=1}^{m}\mathcal{R}e\,b^k\frac{\partial}{\partial x_k}+\sum_{l=1}^{p}d^l\frac{\partial}{\partial t_l}.$$

$$(7)\quad \mathcal{R}e\begin{bmatrix}\mu A & tG\\ G & \mu A_1\end{bmatrix}\geqq 0 \text{ in } \Omega$$

for some μ, $0<\mu<1$.

$$(8)\quad \sum_{j=1}^{n}\left|\sum_{k=1}^{m}g^{kj}\xi_k\right|^2 \leqq C\,\mathcal{R}e\,a^{kj}\xi_k\xi_j \text{ in } \Omega,\ \xi\in R^m,\ C>0.$$

(9) $\sum_{k=1}^{m} |\mathrm{Im}\, b^k \xi_k|^2 \leqq C\, \mathrm{Re}\, a^{kj} \xi_k \xi_j$ in Ω, $\xi \in R^m$, $C > 0$.

Theorem 1. Under the assumptions given above the equation (*) is hypoelliptic in Ω.

The a priori estimate (6) may be considered as generalization of Gårding's inequality for the second order strongly elliptic equations. Our method of the proof of Theorem 1 is a generalization of usual variational method (cf. [3], [4]) and we can prove the hypoellipticity at the boundary for the Dirichlet and Neumann problem for the equation (*), where the estimate (6) plays similar role as Gårding's inequality in the variational method.

Let us examine the assumptions (4) and (6) by some examples. First consider the following equation in two variables:

(a) $\quad Lu = u_t - a(x,t)u_{xx} + \cdots = f$ in $\Omega \subset R^2_{x,t}$,

$$\mathcal{R}e\, a(x,t) \geqq 0.$$

Then the following inequality holds:

$$[\mathcal{R}e\, a_x(x,t)]^2 + [\mathcal{R}e\, a_t(x,t)]^2 \leqq C\, \mathcal{R}e\, a(x,t)$$

if $a(x,t) \in C^2(\bar{\Omega}_0)$, Ω_0 compact $\subset \Omega$ (cf. [5]),

which means the condition (4).

Next consider a degenerate parabolic equation

$$\text{(b)}\quad Lu = u_t - \sum_1^m (a^{kj} u_{x_k})_{x_j} - \sum_1^n (a_1^{kj} u_{y_k})_{y_j} - 2\sum_{j=1}^n \sum_{k=1}^m (g^{kj} u_{x_k})_{y_j} + \sum_{k=1}^m b^k u_{x_k} + \sum_1^n b_1^j u_{y_j} + cu = f.$$

If every vector field $\frac{\partial}{\partial x_j}$, $j=1,\dots,m$ (on Ω), is generated by the vector fields

$$Q_0 = \sum_{k=1}^m \mathcal{R}e\, b^k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial t},$$

$$Q_j = \sum_{k=1}^m \mathcal{R}e\, a^{kj} \frac{\partial}{\partial x_k}, \quad j=1,\dots,m,$$

that is, if every vector field $\frac{\partial}{\partial x_j}$ $(j=1,\dots,m)$ can be expressed as a linear combination (with C^∞ coefficients) of $Q_0, Q_1, \dots Q_m, \dots, \frac{\partial}{\partial y_j}, \dots, [Q_k, Q_j], \dots$

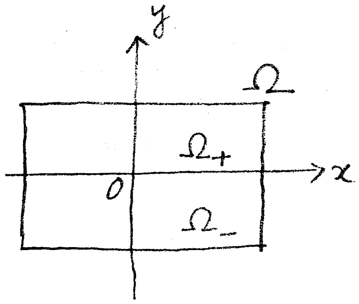
then by virtue of Hörmander's results, we can derive the inequality (6) (cf. [4]).

Now we shall give an application of our method for a generalized Tricomi equation. We denote by

$$\Omega = \{|x| < A\} \times \{|y| < B\},$$

$$\Omega_+ = \{(x,y) \in \Omega;\ y > 0\},$$

$$\Omega_- = \{(x,y) \in \Omega;\ y < 0\}.$$



For the simplicity we consider the equation in two variables:

$$(C) \qquad Lu = u_{yy} + (a(x,y)u_x)_x + b(x,y)u_x + b^{2}u(x,y)u_y + c(x,y)u = f \text{ in } \Omega,$$

where all the coefficients are real valued smooth functions in Ω satisfying

1) $a(x,y) \geqq 0$ in Ω_+, $a(x,y) \leqq 0$ in Ω_-,

$(a(x,0) = 0)$

2) vector fields $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ are generated by $\{\frac{\partial}{\partial y},\ a(x,y)\frac{\partial}{\partial x}\}$ on Ω.

3) $\alpha|y|\,b^2(x,y) \leqq -(A+\frac{1}{\alpha|y|})\,a(x,y) + a_y(x,y)$ in Ω_-

for some constants $A>0$ and $\alpha>1$.

Under the above three assumptions we have the following theorem:

<u>Theorem 2.</u> Let $u \in \mathcal{D}'(\Omega)$ be a solution of the equation (C) in Ω with $f(x,y) \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$ if it is in $C^\infty(\Omega_-)$.

We shall give a sketch of the proof which is devided in three steps.

First we consider u as a solution of the Cauchy problem in Ω_- with Cauchy data $u(x,-T)$, $u_y(x,-T)$ $(T>0$, small$)$. We recall that the last condition 3) is a particular case of the sufficient condition for the Cauchy problem for weakly hyperbolic equations to be well posed (cf.[5]).

Thus we can see $u \in C^\infty(\bar{\Omega}_-)$, in particular

$$u(x,0) \in C^\infty(-A<x<A).$$

We remark that the trace $u(x,0)$ is uniquely determined in the sense of distribution according to the partial hypoellipticity of the operator L in the direction y.

Next in Ω_+ we can consider u as a distribution solution of the Dirichlet problem with boundary data

$$u(x,0) \in C^\infty(-A<x<A).$$

Then as I have pointed out, our method applies to prove hypoellipticity at the boundary, that is, we can show $u(x,y)$ is smooth up to the boundary:

$$u(x,y) \in C^\infty(\bar{\Omega}_+). \quad (\text{cf. } [3]).$$

Final step is to prove $u(x,y) \in C^\infty(\Omega)$, which is obtained by a direct computation combining the above results.

An example satisfying such conditions is given by

$$(d) \quad Lu = u_{yy} + y(x^2+y^2)u_{xx} + byu_x + b^0u_y + cu = f,$$

where b, b^0 and c are arbitrary real valued smooth functions in Ω.

References.

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