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Boundedness and Convergence of Solutions of Duffing's Equation

Kenichi Shiraiwa

We shall discuss boundedness of solutions of the equation

(1) $x'' + f(x)x' + g(x) = e(t)$ ($' = d/dt$)

under suitable conditions. Furthermore, we shall discuss asymptotic stability of a periodic solution and convergence of solutions for the equation

(2) $x'' + cx' + g(x) = e(t)$,

where c is a positive constant and $e(t)$ is a periodic function.

This work is motivated by the work of H.Kawakami [2], which gives some numerical computations on the equation $x'' + kx' + x^3 = B \cos t$ for positive constants k and B. There are also very interesting results by C.Hayashi, Y.Ueda and H.Kawakami [1]. Also, our paper heavily depends on the work of W.S.Loud [3].

Theorem 1 In the equation (1) we assume the following conditions (a), (b) and (c).

(a) There exists a solution of (1) under any initial condition.

(b) There exist positive constants c and E such that

$$f(x) \geqq c \quad \text{and} \quad |e(t)| \leqq E .$$

(c) $g(x)$ is a differentiable function satisfying the following conditions (i), (ii) and (iii).

(i) $g(x)$ is bounded on any finite interval.

(ii) $g'(x) \geqq 0$

(iii) $\lim_{x\to\infty} g(x) > E$ and $\lim_{x\to-\infty} g(x) < -E$.

By the condition (c), $g(x)$ is a monotone increasing function, and there exist numbers x_1 and x_2 $(x_1<x_2)$ such that

$$g(x_1) = -E \quad \text{and} \quad g(x_2) = E .$$

Let $x(t)$ be any solution of (1). Then there exists a number t_0 such that

$$x_1 - 4E/c^2 \leqq x(t) \leqq x_2 + 4E/c^2 \quad \text{and}$$
$$|x'(t)| \leqq 4E/c \quad \text{for any } t \geqq t_0 .$$

Our proof is similar to that of Theorem 1 of W.S.Loud [3]. He assumed that $g'(x) \geqq b$ for some positive constant b in his paper and got an additional information.

Corollary In addition to the conditions (a), (b) and (c) of Theorem 1, we assume the following two conditions,

(d) $f(x)$ and $e(t)$ are continuous, and $f(x)$ satisfies the local Lipschitz condition.

(e) $e(t)$ is periodic of period τ($\tau>0$).

Then the equation (1) has a periodic solution of period τ .

The equation (2) is a special case of (1), and it is equivalent to the following system of equations.

$$(3)\quad \begin{cases} x' = y \\ y' = -cy - g(x) + e(t) \end{cases}$$

Theorem 2 Assume the following conditions A(i), A(ii) and A(iii).

A(i) $e(t)$ is a continuous periodic function of period τ($\tau>0$), and E is a positive constant such that $|e(t)| \leqq E$.

A(ii) $g(x)$ is of class C^1 such that

$$g'(x) \geqq 0, \quad \lim_{x\to\infty} g(x) > E, \quad \lim_{x\to-\infty} g(x) < -E, \quad \text{and } g'(x)=0$$

only on a countable subset of the real numbers.

A(iii) c is a positive constant.

Now, let n be a positive number and let $x = \phi_1(t)$, $y = \phi_2(t)$ be a non-constant periodic solution of period $n\,\tau$ for the equation (3). Suppose that $|\phi_1(t)| \leqq \beta$ for all t and $c^2 > H(\beta)$, where $H(\beta) = \sup\{g'(x)\}; -\beta \leqq x \leqq \beta\}$.

Then the periodic solution $x = \phi_1(t)$, $y = \phi_2(t)$ is asymptotically stable.

This is a generalization of Loud [3].

Corollary 1 Assume the above conditions A(i), A(ii) and A(iii). Let $A = \max\{|x_1 - 4E/c^2|, |x_2 + 4E/c^2|\}$ where $g(x_1) = -E$ and $g(x_2) = E$. Further, assume that $c^2 > H(A) = \sup\{g'(x); -A \leqq x \leqq A\}$.

Then every non-constant periodic solution of period $n\,\tau$ (n a positive integer) of the equation (3) is asymptotically stable.

Corollary 2 In addition to the assumptions of Corollary 1, we assume that e(t) is non-constant.

Then there exists a non-constant periodic solution $x = \psi_1(t)$, $y = \psi_2(t)$ of period τ for the equation (3) such that any periodic solution of period $n\,\tau$ (for a suitable positive integer n) for the equation (3) coincides with solution $x = \psi_1(t)$, $y = \psi_2(t)$.

Theorem 3 Under the same assumption of Corollary 2 of Theorem 2, there exists a unique periodic solution $x = \psi_1(t)$,

$y = \psi_2(t)$ of period τ for the equation (3) such that for any solution $x = x(t)$, $y = y(t)$ of (3) the following equalities hold.

$$\lim_{t\to\infty} |x(t) - \psi_1(t)| = \lim_{t\to\infty} |y(t) - \psi_2(t)| = 0$$

This also generalizes the results of Loud [3].

Details will appear elsewhere.

References

[1] C.Hayashi, Y.Ueda and H.Kawakami : Transformation Theory as Applied to the Solution of Non-Linear Differential Equations of Second Order, Int. J. Non-Linear Mechanics, 4 (1969), 235-255

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[3] W.S.Loud : Boundedness and Convergence of Solutions of $x'' + cx' + g(x) = e(t)$, Dnke Math. J.24. (1957), 63-72

Department of Mathematics
College of General Education
Nagoya University