数理解析研究所講究録
282巻 1976年 17-25

A categorical consideration of parallel flows

Kobe University Jirō Egawa

In this lecture, we reinvestigate parallelizability of dynamical systems from a categorical point of view, and classify parallel flows on a fixed phase space by isomorphisms. Finally, some remarks related to isomorphisms are given.

§1. Notations and definitions

R denotes the set of real numbers with the usual topology. D(X) denotes the set of dynamical systems on a topological space X. $C_\pi(x)$ denotes the orbit of π through $x \in X$. S_π denotes the set of singular points of π. $J^+_\pi(x)$ denotes the positive prolongational limit set of x with respect to π. Let $\pi \in D(X)$, and $\rho \in D(Y)$. A homeomorphism h of X onto Y is called an NS-isomorphism of π onto ρ iff we have $h(C_\pi(x)) = C_\rho(h(x))$ for all $x \in X$. In this case, we say that π and ρ are NS-isomorphic or topologically equivalent. Let π and ρ be topologically equivalent. Then it is known that there exists a mapping ϕ of $X \times R$ into R, which is called the reparametrization for h, such that $h((x, t)) = \rho(h(x), \phi(x, t))$ for all $(x, t) \varepsilon X \times R$ and ϕ is continuous on $X \times R - S_\pi \times R$.

When we consider an NS-isomorphism h with the reparametrization ϕ for it, we say, in acoordance with T.Ura, that (h, ϕ) is a GH-isomorphism of π onto ρ. Further, if ϕ satisfies the following condition (n), then (h, ϕ) is said to be of GH(n)-isomorphism of π onto ρ.

(3) ϕ is continuous on $X \times R$.

(2) There exists a continuous function c on X such that $\phi(x, t) = c(x)t$ for $(x, t) \varepsilon X \times R$.

(1) (h, ϕ) is of type 2 and c is constant.

(0) (h, ϕ) is of type 1 and $c = 1$.

Let $\pi \varepsilon D(X)$. $\Xi \subset X$ is called a section of π iff for every $x \varepsilon X$ there exists a unique number $\tau(x) \varepsilon R$ such that $\pi(x, \tau(x)) \varepsilon \Xi$ and τ is continuous on X. π is called a parallel flow with the section Ξ iff $X = \Xi \times R$ and for each $(\xi, r) \varepsilon \Xi \times R$ and $t \varepsilon R$, we have $\pi((\xi, r), t) = (\xi, r+t)$.

π is said to be parallelizable iff there exists a section of π. In order to consider parallelizability of dynamical systems from a categorical point of view, we introduce the following notions: π is said to be NS-parallelizable (GH(n)-parallelizable) iff there exists a parallel flow NS-isomorphic (GH(n)-isomorphic) to π.

§2. Parallelizability and Parallel Flows

The following is well known.

Proposition 1. Let X be locally compact and separable. Then $\pi \in D(X)$ is parallelizable iff for each $x \in X$ we have $J_\pi^+(x) = \phi$. (Such a flow is usually said to be dispersive.)

The following theorem holds for parallelizability introduced above.

Theorem 1 ([4]). The followings are equivalent.

(1) π is parallelizable.

(2) π is NS-parallelizable.

(3) π is GH(n)-parallelizable.

Next we consider the problem of the classification of parallel flows. Let π and ρ be parallel flows on $\Xi \times R$ and $H \times R$, respectively. Then we have

Theorem 2 ([4]). The followings are equivalent.

(1) π and ρ are NS-isomorphic.

(2) π and ρ are GH(3)-isomorphic

(3) π and ρ are GH(0)-isomorphic

(4) Ξ and H are homeomorphic.

By the above theorem, every isomorphism considered in this lecture is equivalent, and the problem of the classification of parallel flows on a fixed topological space X is completely reduced to a topological one: How can the space X be represented in the form $X = W \times R$? As an example, we consider parallel flows on $R^n (n \geqq 4)$. It is known that there exist uncountable many open manifolds $\{W_\lambda\}$ such that all of them are not mutually homeomorphic, and but $W_\lambda \times R$ is homeomorphic to R^n. Using this result and Theorm 2 we can assert that there exist uncountable many parallel flows on R^n which are not mutually isomorphic.

3. Remarks on topologically equivalent flows

By Theorem 1 and Theorem 2, isomorphisms considered in this lecture are equivalent for parallelizability or parallel flows. In other words, the reparametrization does not play any roles for them. But there are several notions in the theory of dynamical systems, in which the reparametrization plays an important role. It is important to clarify them. We shall exhibit some examples.

Example 1. The first example is related to the existence of invariant positive measure. Let X be a separable metric space, and $\pi \in D(X)$. Let μ be a Borel measure on X. We say that μ is an invariant positive measure with respect to π iff μ satisfies the following two conditions:

(1) μ is positive, that is, for every open subset $U \subset X$ we have $\mu(U) > 0$ and for each compact subset $K \subset X$ we have $\mu(K) < \infty$.

(2) μ is invariant, that is, for each $B \subset X$ and $t \in R$ we have $\mu(\pi(B, t)) = \mu(B)$.

Assume that $\pi \in D(X)$ and $\rho \in D(Y)$ are topologically equivalent. Then we can prove the following theorem.

Theorem 3 ([1]). If π admits an invariant positive measure, then we can construct an invariant positive measure for $\rho|Y - S_\rho$ (the restriction of ρ to $Y - S_\rho$).

As an easy application of Theorem 3, we have the following corollary.

Corollary 3.1 ([1]). If $S_\pi = \phi$ and π is strictly ergodic, then ρ is also strictly ergodic, where π is said to be strictly ergodic iff π admits a unique invariant positive measure.

In Theorem 3, if $S_\pi \neq \phi$, then we can not assert, in general, that ρ admits an invariant positive measure. This is verified by the following simple example. Let π and ρ be flows on 2-dimensional torus T^2 defined by the following differential equations (E) and (F):

$$\text{(E)} \begin{cases} x' = M(x, y) \\ y' = \alpha M(x, y) \end{cases} \qquad \text{(F)} \begin{cases} x' = N(x, y) \\ y' = \alpha N(x, y) \end{cases}$$

where α is an irrational number, M and N are continuous periodic function on R^2 with period 1 and positive except at $(x, y) = (0, 0)$ (mod. 1). Assume that

$$\int_{T^2} \frac{1}{M(x, y)}\, dxdy < \infty \quad \text{and} \quad \int_{T^2} \frac{1}{N(x, y)}\, dxdy = \infty.$$

We can easily see that $S_\pi = S_\rho = \{(0, 0)\}$, and that π and ρ are topologically equivalent. Further, we can show that the positive measures μ and ν defined by

$$\mu(B) = \int_B \frac{1}{M(x, y)}\,dxdy \quad \text{and} \quad \nu(B) = \int_B \frac{1}{N(x, y)}\,dxdy$$

for $B \subset T^2$ are invariant positive measures with respect to $\pi|T^2 - S_\pi$ and $\rho|T^2 - S_\rho$, respectively.(We can show that $\pi|T^2 - S_\pi$ and $\rho|T^2 - S_\rho$ are strictly ergodic.) Since the total measure of $T^2 - S_\pi$ with respect to μ is finite by the assumption, we can easily extend μ to the invariant positive measure with respect to π. But ρ does not admit any invariant positive measure, because the total measure of $T^2 - S_\rho$ with respect to ν is not finite.

Example 2. The second example is related to minimal flows on compact metric spaces. Let X be a compact metric space, and $\pi \in D(X)$ be a minimal flow, that is, for each $x \in X$ we have $\overline{C_\pi(x)} = X$. We say that π is totally minimal iff for each $x \in X$ and $\lambda \in R$ $(\lambda \neq 0)$ we have $\overline{\{\pi(x, n\lambda)\}}_{n\in Z} = X$, where Z denotes the set of integers. Concerning with the total minimality, we can show the following Theorem.

Theorem 4([3]). If π is minimal but not periodic, then there exists a total minimal flow which is topological equivarent to π.

It is well known that equicontinuous or distal flows are not totally minimal, where π is said to be equicontinuous iff the family $\{\pi_t\}_{t \varepsilon R}$ of homeomorphisms of X onto X is equicontinuous, and π is said to be distal iff for each pair of distinct points $x, y \varepsilon X$ we have $\inf_{t \varepsilon R} \{d_X(\pi(x, t), \pi(y, t))\} > 0$. By Theorem 4, we conclude that equicontinuity, distality and total minimality are not invariant under NS-isomorphisms. Further, we can show,

Theorem 5 ([2]). Every minimal flow which is topologically equivalent to some equicontinuous flow is totally minimal, if it is not equicontinuous.

As an easy application of Theorem 5, we obtain

Corollary 5.1. If π is minimal, but neither totally minimal nor equicontinuous, then π is not topologically equivalent to any equicontinuous flows.

Corollary 5.2. If π is a distal minimal flow, but not equicontinuous, then π is not topologically equivalent to any equicontinuous flows.

References

[1] J.Egawa, Isomorphisms and Local Dynamical Systems

admitting invariant positive measures, Funkcial. Ekvac. 17(1974), 307-319.

[2] J.Egawa, Eigen values of harmonizable minimal flows (in preparation).

[3] J.Egawa, Isomorphisms and totally minimal flows (in preparation).

[4] T.Ura and J.Egawa, Isomorphism and parallelizability in dynamical systems theory, Math. Systems Theory, 7(1973), 250-264.