# On (sub)-holonomicity of
# Some Modules and b-functions

### By

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本記録においては、$\mathcal{D}$-Module の section $u$ に対して、
$\pi = \mathcal{D}[s] f^s u$, $\mathcal{D} f^\alpha u$ ($\alpha \in \mathbb{C}$) を定義し、その性質をのべる。
この $\pi$ は $\mathcal{D}[t,s]$-Module の構造をもち、$t$ は injective になっている。$\pi$ の $b$ 函数は常に存在し、その根によって、$\mathcal{D} u$ が holonomic の場合が重要であり、その場合 $\mathcal{D} f^\alpha u$ と $\pi/(s-\alpha)\pi$ の同型性も判列される。定理 1, 17, 20, 21, 24 が重要である。又、reduced b-function についての事項もおさめた。詳細は下記の二論文、又それにつづくものに発表される。

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Chapter I   Genelarities

In this chapter, we study the basic features of general $\mathcal{D}$[t,s]-Modules and b-functions associated with them, which are indispensable to later chapters.   The author develop the general theory of such b-functions and Modules in [32].

§ 1.   $\mathcal{D}$ [t,s] - Modules and b-functions.

Let   $\mathbb{C}$[t,s]   be the associative algebra over   $\mathbb{C}$   with generators   s   and   t   and defining relation

$$ts - st = t. \tag{1}$$

Set   $\mathcal{D}$[t,s] = $\mathcal{D} \underset{\mathbb{C}}{\otimes} \mathbb{C}$[t,s].

A $\mathcal{D}$- Module $\mathfrak{M}$ is called a $\mathcal{D}$[s] - Module (restectively $\mathcal{D}$[t,s] - Module), if $\mathfrak{M} \supset s\mathfrak{M}$ (respectively $\mathfrak{M} \supset s\mathfrak{M}$, $\mathfrak{M} \supset t\mathfrak{M}$) holds.   In this chapter, all Modules are $\mathcal{D}$[t,s] - Modules unless otherwise stated.   Since   $t^{\nu}s = (s + \nu)t^{\nu}$   in view of (1), Ker $t^{\nu}$, Coker $t^{\nu}$ and   Im $t^{\nu}$   are $\mathcal{D}$ [t,s]-Modules along with given $\mathcal{D}$[t,s]-Module.

<u>Definition 1.1</u>    Let $\mathcal{L}$ be a $\mathcal{D}$[s]-Module.  If  $s \in \mathcal{E}nd_{\mathcal{D}}'(\mathcal{L})$

has the non-zero minimal polynomial, we denote it by $d_{\mathcal{L}}(s)$,

and say  " $d_{\mathcal{L}}(s)$ exists." "b-functions" for a $\mathcal{D}$[t,s]-Module $\mathcal{M}$

are defined by       $b_{\mathcal{M},\nu}(s) = d_{\mathcal{M}/t^\nu \mathcal{M}}(s), \quad \nu = 1,2\cdots.$

Usually, $b_{\mathcal{M},1}$ is abbribiated as $b_{\mathcal{M}}$. As is easily seen, $b_{\mathcal{M},\nu}$

exist if and only if $b_{\mathcal{M}}$ exists.

It should be remarked that if $\mathcal{L}$ is a holonomic $\mathcal{D}$[t,s]-

Module $d_{\mathcal{L}}(s)$ exists, since $\mathcal{E}nd_{\mathcal{D}}(\mathcal{L})_x$ ($x \in X$) is finite

dimensional and $\mathcal{E}nd_{\mathcal{D}}(\mathcal{L})$ is coherent [13].

Standard example of $\mathcal{D}$[t,s] - Module is constructed as

follows.  Let f be a holomorphic function on $U \subset X$, let $\mathcal{L}$ be a

coherent $\mathcal{D}$-Module and let  u  be its section over  U.  We

denote the annihilator of u by $\mathcal{A}$ , that is; $\mathcal{A} = \{Q \in \mathcal{D} \mid Qu = 0\}$.

Define the ideal $\mathcal{J}(s) \subset \mathcal{D}[s]$  by the condition that


$$P(s,x,D) \in \mathcal{J}(s) \qquad \text{if and only if}$$
$$f^m P(s,x,D + \frac{s}{f} \text{grad } f) \in \mathbb{C}[s] \otimes \mathcal{A} \qquad , \text{ for some m.}$$


We denote by $\mathcal{M}$ the Module $\mathcal{D}[s]/\mathcal{J}(s)$ and by $f^s u$ the

class (1 mod $\mathcal{J}(s)$ ). $\mathcal{M} = \mathcal{D}[s]f^s u$ is a $\mathcal{D}$[t,s]-Module

with actions of t and s given by ,


$$t: P(s) \longmapsto P(s+1)f \quad , \qquad s: P(s) \longmapsto P(s)s .$$

The map  t is injective  in  $\mathcal{N}$ .  In fact, if  $P(s+1)f \in \mathcal{J}(s)$

then

$$f^m P(s+1,x,D+\tfrac{s}{f} \text{ grad } f)f = \sum Q_j s^j$$

for some m and  $Q_j \in \mathcal{O}$ .  The left-hand side equals to

$$f^{m+1} P(s+1,x,D+\tfrac{s+1}{f} \text{ grad } f),$$

and the right-hand side can be rewritten in the form

$$\sum R_j (s+1)^j$$

for some  $R_j \in \mathcal{O}$ .  Therefore,

$$f^{m+1} P(s,x,D+\tfrac{s}{f} \text{ grad } f) = \sum R_j s^j ,$$

which implies  $P(s) \in \mathcal{J}(s)$.

The  $\mathcal{L}$- Module  $\mathcal{D} f^s u$  is coherent,  and if u is a holonomic section,  $\mathcal{D} f^s u$  is subholonomic  $(\text{See } [32])$.

Definition 1.2    With a non-zero polynomial  p(s), we associate a number  $w(p) \in \mathbb{N}_0$  in the following manner (w(p) is called the width of p.)

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then

i) If $p(s) \in \mathbb{C}^*$  $w(p) = 0$,

ii) If $p(s) = \prod_{i=0}^{k}(s+\alpha+i)^{\varepsilon_i}$, $\alpha \in \mathbb{C}$, $\varepsilon_0 \varepsilon_k \neq 0$  then  $w(p) = k+1$,

iii) If $p(s)$ has the form

$p(s) = \prod_{j=1}^{k} p_j(s)$, where each $p_j(s)$ is of the form

in ii), $p_j(s) = \prod (s + \alpha_j + i)^{\varepsilon_i^{(j)}}$ , and $\alpha_j \not\equiv \alpha_{j'}$

mod $\mathbb{Z}$ $(j \neq j')$;   then $w(p) = \max_j w(p_j)$.

**Theorem 1.3**    If $d_{\mathcal{L}}(s)$ exists, then $t^{w(d_{\mathcal{L}})}\mathcal{L} = 0$.   Furthermore if we assume that $t$ is injective or surjective, then $\mathcal{L} = 0$.

Proof)  we have

$$0 = d_{\mathcal{L}}(s)\mathcal{L} \supset d_{\mathcal{L}}(s)t^{w(d_{\mathcal{L}})}\mathcal{L} ,$$

and by virtue of (1),

$$0 = t^{w(d_{\mathcal{L}})}d_{\mathcal{L}}(s)\mathcal{L} = d_{\mathcal{L}}(s+w(d_{\mathcal{L}}))t^{w(d_{\mathcal{L}})}\mathcal{L} .$$

It follows from the definition of $w(d_{\mathcal{L}})$ that

$$\text{g.c.d.}(d_{\mathcal{L}}(s), d_{\mathcal{L}}(s+w(d_{\mathcal{L}}))) = 1.$$

Hence the assertion follows.  When $t$ is injecti ve or surjective, it is obvious that $\mathcal{L} = 0$.          Q.E.D.

A coherent $\mathcal{D}$-Module $\mathcal{L}$ is called holonomic (resp. sub-holonomic) if $\mathcal{E}xt_{\mathcal{D}}^i(\mathcal{L}, \mathcal{D}) = 0$ for $i < n$ (resp. $i < n-1$).

This condition is equivalent to codim $\check{SS}(\mathcal{X}) \geqslant n$ (resp.

codim $\check{SS}(\mathcal{X}) \geqslant n-1$). $\mathcal{X}$ is called purely subholonomic

if $\mathcal{E}xt_{\mathcal{Y}}^{i}(\mathcal{X},\mathcal{Y}) = 0$ for $i \neq n-1$. It is known that for any

coherent $\mathcal{Y}$-Module , $\mathcal{E}xt_{\mathcal{Y}}^{n}(\mathcal{X},\mathcal{Y})$ (resp. $\mathcal{E}xt_{\mathcal{Y}}^{n-1}(\mathcal{X},\mathcal{Y})$) is holonomic

(resp. sub-holonomic) and $\mathcal{E}xt_{\mathcal{Y}}^{i}(\mathcal{X},\mathcal{Y}) = 0, i > n$. Let $W$ be an

irreducible component of $\check{SS}(\mathcal{X})$. Then the multiplicity of $\mathcal{X}$

at a generic point $x_0$ of an irreducible component of $\check{SS}(\mathcal{X})$

can be defined (which is denoted by $m_{x_0}(\mathcal{X})$), and has the

additivity, that is, if

$$0 \leftarrow \mathcal{X}_1 \leftarrow \mathcal{X}_2 \leftarrow \mathcal{X}_3 \leftarrow 0,$$

is an exact sequence of coherent $\mathcal{Y}$-Modules, $m_{x_0}(\mathcal{X}_2) =$

$m_{x_0}(\mathcal{X}_1) + m_{x_0}(\mathcal{X}_3)$.


Corollary 1.4    Let $\mathcal{M}$ be a sub-holonomic $\mathcal{Y}[t,s]$-Module

such that $t : \mathcal{M} \to \mathcal{M}$ is injective. Then, $\mathcal{M}$ is

purely sub-holonomic.


Proof)    Consider the exact sequence

$$0 \leftarrow \mathcal{M}/t\mathcal{M} \leftarrow \mathcal{M} \overset{t}{\leftarrow} \mathcal{M} \leftarrow 0.$$

Set $\mathcal{X} = \mathcal{E}xt_{\mathcal{Y}}^{n}(\mathcal{M},\mathcal{Y})$. Then $\mathcal{X}$ is holonomic and the long

exact sequence of $\mathcal{E}xt$ gives us the surjection $\mathcal{X} \overset{t}{\to} \mathcal{X} \to 0$ .

Therefore $\mathcal{X} = 0$ by virtue of Theorem 1.3.    Q.E.D.


Proposition 1.5 Upon the conditions in Corollary 1.4, $b_{\mathcal{M}}$ exists.

Proof)      Consider an irreducible component W of $\overset{\vee}{SS}(\mathcal{M})$.

Since  t  is injective, the multiplicity of  $\mathcal{M}/t\mathcal{M}$  at

a generic point of W vanishes.  Therefore  codim $\overset{\vee}{SS}(\mathcal{M}/t\mathcal{M}) \geqslant n$

which implies that  $\mathcal{M}/t\mathcal{M}$  is holonomic.  Thus  $\ell_{\mathcal{M}}$ exists

$\left(\text{and so does} \ell_{\mathcal{M},\nu},\right.$ by the argument after Definition 1.1.). Q.E.D.


The conditions in Corrollary 1.4  are satified for

$\mathcal{M} = \mathcal{D}[s]f^s u$, if one of the  following two conditions

holds.


i)     f is arbitrary holomorphic function, u = 1.


ii)     f is quasi-homogeneous,  $\mathcal{D}u$ is holonomic.


In the present paper, we restrict ourselves to case i).

We investigate case ii) in [32], where the detailed structure

of $b_{\mathcal{M},\nu}(s)$  and the relation between  $\mathcal{M}_\alpha$ and  $\mathcal{D}f^\alpha u$ ($\alpha \in \mathbb{C}$)

are also discussed.  The existence of $b_{\mathcal{M}}(s)$ for  $\mathcal{M} =$

$\mathcal{D}[s]f^s u$  with general f  and $\mathcal{D}u$ being holonomic can

be derived from that of case ii), following the

tecnique in      of [14].(See [32] §.   ).


以下は 別の原稿よりとったので, 以上と直接は つながらない.

A. General structure of $\mathcal{B}$ [t,s]-Modules

In §1 we study the structure of $b_{\pi,\nu}(s)$ and define reduced b-functions. The relation between reduced b-function of $\pi_1$ and that of a sub-Module $\pi_2$ is studied in §2.

The key theorem is the following.

Theorem 1.1 (Theorem 1.3 in [Y])

Let $\mathcal{L}$ be a $\mathcal{B}$ [t,s]-Module such that $d_{\mathcal{L}}(s)$ exist. Then $t^{w(d_{\mathcal{L}})}\mathcal{L} = 0$.

Here, $w(d_{\mathcal{L}})$ is the width of $d_{\mathcal{L}}$. We recall its meaning and add some more definitions.

Definition 1.2 For a non-zero polynomial $p(s)$, we associate a number $w(p)$, called the width of $p$, and polynomials $\hat{p}(s)$ and $\check{p}(s)$ in the following manner.

i) $p(s) \in \mathbb{C}^*$; $w(p) = 0$, $\hat{p}(s) = \check{p}(s) = 1$

ii) $p(s) = \prod_{i=0}^{k}(s+\alpha+i)^{\varepsilon_i}$, $\alpha \in \mathbb{C}$, $\varepsilon_0 \varepsilon_k = 0$; $w(p) = k+1$,
$\hat{p}(s) = (s+\alpha)^{\varepsilon_0}$, $\check{p}(s) = (s+\alpha+k)^{\varepsilon_k}$.

iii) $p(s) = \prod_{j=1}^{k} p_j(s)$, where each $p_j(s)$ is of the form
in ii), $p_j(s) = \prod(s+\alpha_j+i)^{\varepsilon_i^{(j)}}$ , and $\alpha_j \neq \alpha_{j'}$
mod $\mathbb{Z}$ ($j \neq j'$); $w(p) = \max_j w(p_j)$, $\hat{p}(s) = \prod_j \hat{p}_j(s)$,
$\check{p}(s) = \prod_j \check{p}_j(s)$.

§1.    Structure of $b_{\pi,\nu}(s)$.

We first note that $b_{\pi,\nu}(s)$ is of the significant structure.  Given a rational function  p(s), we use the notation

$$[p(s)]_\nu = \prod_{i=0}^{\nu-1} p(s+i) \quad \nu > 0, \quad [p(s)]_0 = 1.$$

<u>Theorem 1.3</u>    i)  There are a rational function $\overline{b}_\pi(s)$, polynomials $\overline{b}'_\pi(s)$ and $c_\pi(s)$, unique up to a constant multiple, and $\nu_0 \in \mathbb{N}_0$, such that for $\nu \geq \nu_0$,

$$b_{\pi,\nu}(s) = [\overline{b}_\pi(s)]_\nu c_\pi(s+\nu) \tag{2}$$

$$= c_\pi(s)[\overline{b}'_\pi(s)]_\nu , \tag{3}$$

$$\overline{b}_\pi(s)c_\pi(s+1) = c_\pi(s)\overline{b}'_\pi(s). \tag{4}$$

ii)   If $t : \pi \to \pi$ is injective, $\overline{b}_\pi(s)$ is also a polynomial, and for $\nu \leq \nu_0$  there are polynomials $c_{\pi,\nu}(s)$ and $c'_{\pi,\nu}(s)$ such that

$$b_{\pi,\nu}(s) = [\overline{b}_\pi(s)]_\nu c_{\pi,\nu}(s+\nu)$$

$$b_{\pi,\nu}(s) = [\bar{b}_\pi(s)]_\nu c_{\pi,\nu}(s+\nu)$$

$$= c'_{\pi,\nu}(s)[\bar{b}'_\pi(s)]_\nu,$$

$$c_{\pi,\nu}(s) \mid c_{\pi,\nu'}(s), \quad c'_{\pi,\nu}(s) \mid c'_{\pi,\nu'}(s) \quad \text{for } \nu \leqslant \nu'.$$

$$\hat{b}_\pi(s) \mid \bar{b}_\pi(s), \quad \check{b}_\pi(s) \mid \bar{b}'_\pi(s). \tag{5}$$

Moreover, it is possible to take $\nu_0 = w(b_\pi) - 1$, and the following relations hold.

$$c_\pi(s) \mid [\bar{b}_\pi(s)]_{\nu_0}, [\bar{b}'_\pi(s-\nu_0)]_{\nu_0}, \tag{6}$$

$$b_\pi(s) \mid [\bar{b}_\pi(s)]_{\nu_0+1}, [\bar{b}'_\pi(s-\nu_0)]_{\nu_0+1}, \tag{7}$$

$$w(c_\pi) \leqslant \nu_0. \tag{8}$$

<u>Corollary 1.4</u>  As easily seen, $\bar{b}_\pi(s)$ and $\bar{b}'_\pi(s)$ can be so determined that

$$\bar{b}_\pi(s) = b_{\pi,\nu+1}(s)/b_{\pi,\nu}(s+1),$$

$$\bar{b}'_\pi(s) = b_{\pi,\nu+1}(s-\nu)/b_{\pi,\nu}(s-\nu), \quad \nu \geqslant \nu_0.$$

$\bar{b}_\pi(s)$ is called the reduced b-function of $\pi$. The special case of the part of this theorem is substantially due to M.Sato [2].

この証明は省略する.

Here after $R$, $\overline{R}$, $\overline{R}'$, $R_\nu$ and $C$ denote the set of the roots of equations $b_\pi(s) = 0$, $\overline{b}_\pi(s) = 0$, $\overline{b}'_\pi(s) = 0$ $b_{\pi,\nu}(s) = 0$ and $c_\pi(s) = 0$ respectively, when t is injective.

__Proposition 1.5__    $R \supset \overline{R}$ , $\overline{R}'$ , $C$;    $R \cap (R+1) \supset C$

$$R_k + k = \bigcup_{i=1}^{k}(R+i) = \bigcup_{i=1}^{k}(\overline{R}+i) \cup C, \quad R_k = \bigcup_{i=0}^{k-1}(R-i) = \bigcup_{i=0}^{k-1}(\overline{R}-i) \cup C.$$

Proof is straightforward.

We end this section by adding the following remarks when $t \in \mathcal{End}(\pi)$ is not necessarily injective.

__Definition 1.6__    We define the $\mathcal{B}[t,s]$-Module $\overset{\sim}{\pi}$ by $\pi / \bigcup_{\nu \geq 1} \mathrm{Ker}\, t^\nu$ . (Hence t is injective in $\overset{\sim}{\pi}$ .)

We can prove $\overline{b}_\pi(s) = \dfrac{\tilde{c}(s)}{\tilde{c}(s+1)}\overline{b}_{\overset{\sim}{\pi}}(s)$, where $\tilde{c}(s) = c_\pi(s)/c_{\overset{\sim}{\pi}}(s)$ is a polynomial, and $\overline{b}'_\pi(s) = \overline{b}'_{\overset{\sim}{\pi}}(s)$. The proof is omitted.

Proposition 1.7    Let $0 \to \mathcal{X} \hookrightarrow \pi \to \pi' \to 0$ be an exact sequence of $\mathcal{B}[t,s]$-Modules, let $t \in \mathcal{End}_{\mathcal{B}}(\pi')$ be injective, and let $d_{\mathcal{X}}(s)$ exist. Then $\pi' \cong \overset{\sim}{\pi}$.

For, since $\pi \twoheadrightarrow \pi'$ and $t|_{\pi'}$ is injective, $\bigcup_\nu \mathrm{Ker}(t)^\nu_\pi \hookrightarrow \mathcal{X}$. On the other hand, $t^{w(d_{\mathcal{X}})}\mathcal{X} = 0$ by Theorem 1.1. Therefore,

$$\mathcal{X} = \mathrm{Ker}\, t^{w(d_{\mathcal{X}})} = \bigcup \mathrm{Ker}\, t^\nu, \text{ and } \overset{\sim}{\pi} \cong \pi'.$$

## §2. b-functions for a sub-Module

In terms of $b_{\pi}(s)$, we can estimate the b-function of a submodule of $\pi$ .

Theorem 1. 8    Let $\pi_1$ be $\mathcal{D}[t,s]$-Module and let $\pi_2$ be its submodule. Further assume  1° $t \in \mathcal{E}nd(\pi_1)$ is injective, 2° $d_{\pi_1/\pi_2}(s)$ exists and 3° $b_{\pi_1}(s)$ or $b_{\pi_2}(s)$ exists. Then, $\deg \bar{b}_{\pi_1} = \deg \bar{b}_{\pi_2}$ $(= d)$ and there are polynomials $c(s)$ and $c'(s)$, unique up to a constant multiple, such that

$$c(s) \ , \ c'(s) \ \Big| \ d_{\pi_1/\pi_2}(s), \tag{15}$$

$$c_{\pi_1}(s)c'(s) = c_{\pi_2}(s)c(s), \tag{16}$$

$$\bar{b}_{\pi_1}(s) = \frac{c(s)}{c(s+1)} \bar{b}_{\pi_2}(s), \quad \bar{b}'_{\pi_1}(s) = \frac{c'(s)}{c'(s+1)} \bar{b}'_{\pi_2}$$

Corollary 1. 9

$$b_{\pi_2}(s) \ \Big| \ [b_{\pi_1}(s)]_{\nu_0+1}, \quad b_{\pi_1}(s) \ \Big| \ [b_{\pi_2}(s-\nu_0)]_{\nu_0+1}, \tag{18}$$

$$\bar{b}_{\pi_2}(s) \ \Big| \ [\bar{b}_{\pi_1}(s)]_{\nu_0+1}, \quad \bar{b}_{\pi_1}(s) \ \Big| \ [\bar{b}_{\pi_2}(s-\nu_0)]_{\nu_0+1}, \tag{19}$$

$$b_{\pi_2}(s) \ \Big| \ [\bar{b}_{\pi_1}(s)]_{\nu_0+\nu'}, \quad b_{\pi_1}(s) \ \Big| \ [\bar{b}_{\pi_2}(s-\nu_0)]_{\nu_0+\nu'}, \tag{20}$$

$$\Big| \deg \, c_{\pi_1} - \deg \, c_{\pi_2} \Big| \le \nu_0 d, \tag{21}$$

where $\nu_0 = w(d_{\pi_1/\pi_2})$, $\nu' = \min \, ( \, w(b_{\pi_1}), \, w(b_{\pi_2}))$.

Proof of Theorem 1. 7)    It       follows from Thm.1.( and condition 2  that  $\pi_1 \supset t^{\nu_0}\pi_2$ .  Consider the following diagram for  $\nu \geq \nu_0$ ,

$$\pi_1 \supset \pi_2 \begin{array}{c} \supset t^{\nu_0}\pi_1 \supset \\ \\ \supset t^{\nu-\nu_0}\pi_2 \supset \end{array} t^{\nu}\pi_1 \supset t^{\nu}\pi_2$$

This immediately reads

i) $b_{2,\nu-\nu_0}(s) \mid b_{1,\nu}(s)$ $\qquad\qquad$ (22)

$b_{1,\nu-\nu_0}(s+\nu_0) \mid b_{2,\nu}(s)$ $\qquad\qquad$ (23)

ii) $b_{2,\nu}(s) \mid b_{1,\nu}(s)d(s+\nu)$, $\qquad\qquad$ (24)

$b_{1,\nu}(s) \mid d(s)b_{2,\nu}(s)$. $\qquad\qquad$ (25)

Here, we have used the notations, $b_{i,\nu}(s) = b_{\pi_i,\nu}(s)$, $c_i(s) = c_{\pi_i}(s)$, $d(s) = d_{\pi_1/\pi_2}(s)$. (22) and (23) tell us that the existence of $b_1$ and that of $b_2$ are equivalent. In particular, setting $\nu = \nu_0+1$, we have (18).

i) gives also,

$(\nu - \nu_0)\deg b_2 + \deg c_2 \leq \deg b_1 + \deg c_1$,

$(\nu - \nu_0)\deg b_1 + \deg c_1 \leq \deg b_2 + \deg c_2$,

and letting $\nu$ tend to infinity, we have $\deg b_1 = \deg b_2$.
This implies (21).

Because of (22) and (23), we can assume, $\bar{b}_i(s) = \prod_{j=1}^{d}(s+n_j^{(i)})$,
and $n_j^{(i)} \leqslant n_{j+1}^{(i)}$ for $n_j^{(i)} \in \mathbb{Z}$. Setting $\nu \gg 0$ in formula
(22), we have $s+n_1^{(2)}|[\bar{b}_1(s)]$, hence $n_1^{(2)} \geqslant n_1^{(1)}$. Similarly
by (23), $n_1^{(1)}+\nu_0 \geqslant n_1^{(2)}$. Therefore $r_{1,\nu}(s) = [s+n_1^{(1)}]_\nu/[s+n_1^{(2)}]_{\nu-\nu_0}$
is a polynomial. Then the relation

$$[\prod_{j=2}^{d}(s+n_j^{(2)})]_{\nu-\nu_0}c_2(s+\nu-\nu_0) \mid r_{1,\nu}(s)[\prod_{j=2}^{d}(s+n_j^{(1)})]_\nu c_1(s+\nu),$$

for $\nu \gg 0$ yields $n_2^{(2)} \geqslant n_2^{(1)}$, and similarly, $n_2^{(1)}+\nu_0 \geqslant n_2^{(2)}$.
Proceeding in this way, we have

$$n_j^{(1)} + \nu_0 \geqslant n_j^{(2)} \geqslant n_j^{(1)} \qquad j = 1,..,d. \qquad (26)$$

Set $c(s) = \prod_{j=1}^{d}[s+n_j^{(1)}]_{n_j^{(2)}-n_j^{(1)}}$. Then clearly $c(s)$ is a poly-
nomial and the first of (17) holds. Uniqueness of $c(s)$ is
obvious. We apply (17) to (25) and have, after cancellation,

$$c(s)\,c_1(s+\nu) \mid d(s)c_2(s+\nu)c(s+\nu),$$

taking $\nu \gg 0$, $c(s) \mid d(s)$.

Statements about $c'(s)$ can be proved analogously, and
equation (17) applied to (2) and (3) gives (16). From
equation (17), we have

$$c(s)[\bar{b}_2(s)]_\nu = [\bar{b}_1(s)]_\nu c(s+\nu). \qquad (27)$$

(with $\nu = \nu_0 + 1$)

The definition of $c(s)$, together with (26) and (27) gives

(19). Analogously, (18), (26) and (27) (with $\nu = \nu_0 + \nu'$) prove (20).

$$Q.E.D.$$

Remark   1.   $b_{\pi_2}(s) \mid [b_{\pi_1}(s)]_{\nu_0 + 1}$        (28)

holds even when t is not injective.

2.   (27), (16) and (2) give

$$c(s)b_{2,\nu}(s) = b_{1,\nu}(s)c'(s+\nu).$$ (29)

3.   Let h and k be the minimum and the maximum of the indices which satisfy $n_i^{(1)} < n_i^{(2)}$ respectively.  Then by (15) $n_k^{(2)} - n_h^{(1)} \leqslant \nu_0$.  It should be noted that this inequality improves (26).

Theorem 1.10   Let $X'$ and $X$ be complex analytic and let $\pi : X' \to X$ be projective holomorphic map. For an $f(x) \in \mathcal{O}_X$, we set $f' = f \cdot \pi$.  We assume $X' - \pi^{-1}(f^{-1}(0)) \cong X - f^{-1}(0)$.  Then, $\pi = \mathcal{D}_X[s]f^s$ is a sub-Module of $\pi'' = \int \pi'$, $\pi' = \mathcal{D}_{X'}[s]f'^s$, and

$$b_{f,x}(s) \mid [b_{f', \pi^{-1}(x)}(s)]_{\nu_0 + 1}, \quad [\overline{b_{f', \pi^{-1}(x)}}(s)]_{\nu_0 + w(b_{f'})}.$$ (30)

Here $\nu_0 = w(d \; \pi''/\pi)$.

この 証明は 省略する.

B. Structure of $\mathcal{L}[s](f^s u)$

In the following sections, we investigate the structure of special $\mathcal{D}[t,s]$-Module $\mathcal{M} = \mathcal{L}[s](f^s u)$. It is to be proved that if $u$ is a holonomic section, $\mathcal{D} f^s u$ is subholonomic and $\mathcal{D} f^\lambda u$ is holonomic. The characterization of reduced b-function is also given.

In the sequel, $\mathcal{M}$ always denote a $\mathcal{D}[t,s]$-Module $\mathcal{L}[s](f^s u)$ which is defined in $\S 1$ [Y]. Recall that the operation $t: P(s)(f^s u) \longmapsto P(s+1)f(f^s u)$ is $\underline{\text{injective}}$ in $\mathcal{M}$.

We denote by $\mathcal{J}$ the annihilator of $u$. Basic concept and notations are same with S-K-K and [3]. Especially, a coherent $\mathcal{D}$-Module is called a System.

$\S 3.$    Preliminary results on Systems

We define some general concepts and collect propositions which we shall need.

$\underline{\text{Definition 1.11}}$    For a system $\mathcal{L}$, we define

$$\text{hol}(\mathcal{L}) = \begin{cases} \dim X - \text{codim } \overset{\vee}{SS}(\mathcal{L}) & , \mathcal{L} \neq 0, \\ -\infty & , \mathcal{L} = 0. \end{cases}$$

Note that $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{L}, \mathcal{D}) = 0$ for $i < \dim X - \text{hol}(\mathcal{L})$.

Definition $1.12$    1.    Let  $\varphi : Y \to X$  be a holomorphic

map and let  $\mathcal{L}$  be a system on  X.  We define the induced

Module of  $\mathcal{L}$  on Y  by

$$\varphi^* \mathcal{L} = \mathcal{D}_{Y \to X} \underset{\mathcal{D}_X}{\otimes} \mathcal{L} \ .$$

2.   Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be systems on  $X_1$  and  $X_2$ , respectively.
The product Module of  $\mathcal{L}_1$  and  $\mathcal{L}_2$   on  $X_1 \times X_2$  is defined by:

$$\mathcal{L}_1 \overset{\wedge}{\otimes} \mathcal{L}_2 = \mathcal{D}_{X_1 \times X_2} \underset{\mathcal{D}_{X_1} \otimes \mathcal{D}_{X_2}}{\otimes} (\mathcal{L}_1 \underset{\mathbb{C}}{\otimes} \mathcal{L}_2).$$

3.   Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be systems on X.  The product Module
of them on X is defined to be

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 = \Delta^* (\mathcal{L}_1 \overset{\wedge}{\otimes} \mathcal{L}_2),$$

where   $\Delta : X \to X \times X$  is a diagonal embedding.

    For the Definition $1.12$, 1. and 2.  and the following

Theorem, we refer the reader to S-K-K  and M.Kashiwara [ 3 ],[  ].

Theorem $1.13$    1. Assume that  for  $V \subset P^*Y$, the map

induced from the canonical projection is proper.

$$\rho^{-1}(V) \cap \varpi^{-1}(SS(\mathcal{L})) \longrightarrow V.$$

Then,  $\varphi^* \mathcal{L}$  is a system on Y and the following isomorphism holds

$$\varphi^* \mathbb{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{Z}, \mathcal{S}_X)[\dim X] \xleftarrow{\sim} \mathbb{R}\, \mathcal{H}om_{\mathcal{D}_Y}(\varphi^*\mathcal{Z}, \mathcal{S}_Y)[\dim Y].$$

2. $\mathcal{Z}_1 \widehat{\otimes} \mathcal{Z}_2$ is always a system and $(\mathcal{Z}_1, \mathcal{Z}_2) \mapsto \mathcal{Z}_1 \widehat{\otimes} \mathcal{Z}_2$ is an exact functor.

3. If $\overset{\vee}{SS}(\mathcal{Z}_1) \cap \overset{\vee}{SS}(\mathcal{Z}_2) \subset X$, $\mathcal{Z}_1 \boxtimes \mathcal{Z}_2$ is a system.

Statement 3 is derived from 1 and 2 easily.

<u>Proposition 1.14</u>  Upon the conditions in Theorem 1.13 ,

1.  $\mathrm{hol}\,(\varphi^*\mathcal{Z}) \leqslant \mathrm{hol}\,(\mathcal{Z})$

2.  $\mathrm{hol}\,(\mathcal{Z}_1 \widehat{\otimes} \mathcal{Z}_2) = \mathrm{hol}\,(\mathcal{Z}_1) + \mathrm{hol}\,(\mathcal{Z}_2),$

3.  $\mathrm{hol}\,(\mathcal{Z}_1 \boxtimes \mathcal{Z}_2) \leqslant \mathrm{hol}\,(\mathcal{Z}_1) + \mathrm{hol}\,(\mathcal{Z}_2).$

Since this is an easy Corrollary of Theorem 1.13, we omit the proof.

We note that Prof. Bernstein considered above theorems under a little different situation in     of [+]. The notation $\boxtimes$ is barrowed from it.

4.  Holonomicity and subholonomicity of some Modules

In this section, we study the structure of $\mathcal{S}[s]f^s u$ and $\mathcal{S}f^\alpha u$ when $\mathcal{Z}u$ is holonomic.

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We define the Modules $\Pi_{\alpha}$ and $\mathcal{D}f^{\alpha}u$ for $\alpha \in \mathbb{C}$ as follows.

Definition 1.15.

$$\Pi_{\alpha} = \Pi /(s-\alpha)\Pi .$$

We sue the notation

$$\mathcal{J}(\alpha) = \left\{ P \in \mathcal{S} \mid P=Q(\alpha) \text{ for some } Q(s) \in \mathcal{J}(s) \right\} .$$

Then $\Pi_{\alpha}$ is isomorphic to $\mathcal{S}/\mathcal{J}(\alpha)$. Let $v \in \Pi$. Then, $v \bmod (s-\alpha)\Pi$ is denoted by $(v)_{\alpha}$. Especially, $(f^{s}u)_{\alpha}$ is the class $1 \bmod \mathcal{J}(\alpha)$.

We define

$$\mathcal{J}_{\alpha} = \left\{ P \in \mathcal{S} \mid f^{m}P(x,D+\frac{\alpha}{f} df) \in \mathcal{J} \text{ for some } m. \right\}$$

Consider the Module $\mathcal{S}/\mathcal{J}_{\alpha}$ and denote $1 \bmod \mathcal{J}_{\alpha}$ by $f^{\alpha}u$. Thus $\mathcal{D} f^{\alpha}u = \mathcal{S}/\mathcal{J}_{\alpha}$.

We also define

$$\mathcal{J}^{(0)} = \mathcal{J}(s) \cap \mathcal{D} .$$

The following inclusions hold.

$$\mathcal{J}^{(0)} \subset \mathcal{J}^{(\alpha)} \subset \mathcal{J}_{\alpha} .$$

Proposition 1.16    Ideals $\mathcal{J}^{(0)}$,    and $\mathcal{J}_\alpha$ are coherent.

Proof.  The proof relies on the following theoremof M.Kashi-wara.

"Let $\mathcal{l}$ be an ideal of $\mathcal{D}$ with filtration: $\mathcal{l} = \bigcup_m \mathcal{l}_m$, $\mathcal{D}^{(h)}\mathcal{l}_m \subset \mathcal{l}_{m+h}$. In order to be coherent for $\mathcal{l}$ over $\mathcal{D}$ , it is necessary and sufficient that each $\mathcal{l}_m$ is coherent over $\mathcal{O}$ ."

From this, the coherency of $\mathcal{J}^{(0)}$ and $\mathcal{J}_\alpha$ follows.

Q.E.D.

Thus we have three systems with canonical surjections.

$$\mathcal{D}f^s u \rightarrow \mathcal{N}_\alpha \rightarrow 0, \qquad \mathcal{N}_\alpha \rightarrow \mathcal{D}f^\alpha u \rightarrow 0.$$

We study (sub-)holonomicity of these Modules in the following.

Theorem 1.17   $\cdots$ is        subholonomic., when $\mathcal{D}u$ is holonomic.

Proof.  Since $\mathcal{D}u$ is holonomic, M.Kashiwara's theorem in [5]says that $\overset{\vee}{SS}(\mathcal{D}u) \subset \bigcup_{X_j} T^*_{X_j} X$ for some Whitney stratification $X = \bigcup X_j$.        We first prove

Lemma 1.13   $\mathcal{D} f^s \boxtimes \mathcal{D}u$   is subholonomic outside $f^{-1}(0)$.

Proof.  It is sufficient to show $\overset{\vee}{SS}(\mathcal{D} f^s) \cap \overset{\vee}{SS}(\mathcal{D}u) \subset X$ outside $f^{-1}(0)$ by 3. of Theorem 1.14.  We refine the stratification, if necessary, such that each $X_j$ is contained in $f^{-1}(0)$ or disjoint to it.  Assume that there exists $(x_0, \xi_0)$ which has the following properties:[*] $x_0 \in f^{-1}(0)$, and there is an analytic path $x(t)$ in some $X_j$ such that $x(0) = x_0$, $(x(t), \xi(t)) \in W$ for $t > 0$ and $\lim_{t \to 0} \xi(t) = \xi_0$.  Since the tangent of the curve $x = x(t)$ is $(\dot{x}_1(t),..,\dot{x}_n(t))$, we have

$$0 = \sum_{i=1}^{n} \dot{x}_i(t) \frac{f_i}{f} = \frac{d}{dt} f(x(t))$$

from the definition of $W$.  Therefore, the path $x = x(t)$ is included in $f^{-1}(0)$, and so is $X_j$.        q.e.d.


Owing to the canonical surjection and injection

$$\mathcal{D} f^s \boxtimes \mathcal{D}u \hookleftarrow \mathcal{D}(f^s \boxtimes u) \to \mathcal{D} f^s u \to 0,$$

$\mathcal{D} f^s u$ is subholonomic outside $f^{-1}(0)$.  We use the

argument in ___ of [ 3 ]. Take the subholonomic part of $\mathcal{D} f^s u$ and denote it by $\mathcal{L}$ . Lemma 1.18 shows that the support of the Module $\mathcal{D} f^s u / \mathcal{L} = \mathcal{D} \overline{f^s u}$ is contained in $f^{-1}(0)$ . Therefore, considering the coherent $\mathcal{O}$-Module $\mathcal{O} \overline{f^s u}$ , we have a natural number $k$ such that $f^k \cdot f^s u \in \mathcal{L}$ . Since $\mathcal{D} f^k \cdot f^s u$ and $\mathcal{D} f^s u$ are isomorphic, the subholonomicity of $\mathcal{D} f^s u$ is derived from that of $\mathcal{D} f^k \cdot f^s u$ .     Q.E.D.


We note that the holonomicity of $\mathcal{D} f^s u / \mathcal{D} f^k \cdot f^s u$ is an easy consequence of the above theorem and injectivity of $t$, considering multiplicity of each Module in the following exact sequence along irreducible components of $\overset{\vee}{SS}(\mathcal{D} f^s u)$.


$$ 0 \;\rightarrow\; \mathcal{D} f^s u \;\xrightarrow{\;t^k\;}\; \mathcal{D} f^s u \;\rightarrow\; \mathcal{D} f^s u / \mathcal{D} f^k \cdot f^s u \;\rightarrow\; 0. $$

When f is quasi-homogeneous, $\mathcal{N} \cong \mathcal{D}/\mathcal{J}^{(0)}$ and hence subholonomic. Thus $b_{\mathcal{N}}(s)$ exists, by Proposition 1.5 in [Y]      . In the general cases, we use the tecnique of adding   a parameter. Define f'(t,x) = tf(x). Then $\mathcal{N}' \cong \mathcal{D}_{\mathbb{C} \times X}/\mathcal{J}_{f'}^{(0)}$ and hence there exists b'(s) and $Q(t,x,D_t,D_x)$ such that

$$Q(t,x,D_t,D_x)f'^{s+1}u = b'(s)f'^s u.$$

Let $Q_0(tD_t,x,D_x)D_t = \sum_j a_j(x,D_x)(tD_t)^j \cdot D_t$ be the homogeneous part of degree -1 in t of Q. Then, defining P by

$$P(s,x,D) = Q_0(s,x,D),$$

we have

$$P(s,x,D)f^{s+1}u = \frac{b'(s)}{s+1} f^s u$$

Thus <u>b-function always exists</u>. We denote by R the set of roots of the equation b(s)=0.

<u>Theorem 1.20</u>    $\mathcal{D} f^{\alpha}u$ is holonomic, when $\mathcal{D}u$ is holonomic.

Proof) As in the proof of Theorem 1.17, one can see that $\mathcal{D} f^k(f^{\alpha}u)$ is holonomic for sufficiently large k. Then the following diagram proves the holonomicity of $\mathcal{D} f^{\alpha}u$.

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{D} f^k \cdot f^s u & \hookrightarrow & \mathcal{D} f^s u & \to & \mathcal{D} f^s u/ \mathcal{D} f^k \cdot f^s u & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{D} f^k \cdot f^{\alpha} u & \hookrightarrow & \mathcal{D} f^{\alpha} u & \to & \mathcal{D} f^{\alpha} u/ \mathcal{D} f^k \cdot f^{\alpha} u & \to & 0. \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Theorem 1.21 $\mathcal{M}_\alpha \rightarrow \mathcal{E}f^\alpha u$ if and only if $\alpha \notin R + \mathbb{N}$.

Proof. Let $P \in \mathcal{J}_\alpha$, ord $P = m$. Then, there is $Q(s) \in \mathcal{B}[s]$ such that

$$Pf^m \equiv (s+m-\alpha)Q(s,x,D) \quad \mod \mathcal{J}(s). \tag{40}$$

To prove (40) , we prepare

Lemma 1.22 For any $R \in \mathcal{L}$, ord $R = m$, the following equality holds for some $S(s,x,D) \in \mathcal{L}[s]$ with $\text{ord}^T S < m$.

$$\left\{ R(x,D+\tfrac{s}{f}\,df) - R(x,D+\tfrac{\gamma}{f}\,df) \right\} f^m = (s-\gamma)S(s,x,D+\tfrac{s}{f}\,df).$$

Proof. The proof is carried out by induction on $m$. When $m=0$, the result is trivial. Let $m \geqslant 1$. For the simplicity of the notation, we explain the case of one variable. General case is similar.

By the hypothesis of induction,

$$\left\{ (D+s\tfrac{f'}{f})^{m-1} - (D+\gamma\tfrac{f'}{f})^{m-1} \right\} f^{m-1} = (s-\gamma)Q_{m-2}(s,x,D+s\tfrac{f'}{f}).$$

Then,

$$\left\{ (D+s\tfrac{f'}{f})^m - (D+\gamma\tfrac{f'}{f})^m \right\} f^m$$

$$= \left\{ (D+s\tfrac{f'}{f})^{m-1} - (D+\gamma\tfrac{f'}{f})^{m-1} \right\} (D+s\tfrac{f'}{f})f^m + (s-\gamma)(D+\gamma\tfrac{f'}{f})^{m-1}f'f^{m-1}$$

$$= (s-\gamma)Q_{m-2}(s,x,D+s\tfrac{f'}{f})(f(D+s\tfrac{f'}{f})+mf') + (s-\gamma)Q'_{m-1}(s,x,D+s\tfrac{f'}{f})f'$$

where

$$Q'_{m-1}(s,x,D) = (fD+(\gamma-s+1)f')(fD+(\gamma-s+2)f')..(fD+(\gamma-s+m-1)f').$$

This yields the case m.                          q.e.d.


We apply this lemma for $R = P$, $\gamma = \lambda-m$.  Then, we have


$$P(x,D+\frac{s}{f}\,df)f^m-f^m P(x,D+\frac{\alpha}{f}\,df) = (s+m-\alpha)Q(s,x,D+\frac{s}{f}\,df),$$


which proves (40).


<u>Lemma</u>  1.23      $t^m \mathfrak{n} \cap (s+m-\lambda)\mathfrak{n} \subset (s+m-\lambda)t^m \mathfrak{n}$      .


Because of the condition on $\langle$, $(s+m-\lambda)$ is not a factor of $b_{\mathfrak{n},\,m}(s)^{*}$.  Hence we have an isomorphism


$$\mathfrak{n}/t^m \mathfrak{n} \overset{s+m-\alpha}{\underset{\sim}{\longrightarrow}} \mathfrak{n}/t^m \mathfrak{n}.\tag{41}$$


Now take an element $v = (s+m-\alpha)w \in t^m \mathfrak{n} \cap (s+m-\lambda)\mathfrak{n}$.
If we consider $w \bmod t^m \mathfrak{n}$ in the left-hand side of (41),
it turns out to be 0 in the right-hand side. Hence $w \in t^m \mathfrak{n}$,
that is $v \in (s+m-\lambda)t^m \mathfrak{n}$.                    q.e.d.


Owing to this lemma and (40), we obtain


$$Pf^m \equiv (s+m-\lambda)Q'(s,x,D)f^m \qquad \bmod \mathfrak{z}(s),$$


Note that $R + \mathbb{N} = (\bar{R} + \mathbb{N})\cup C$.

and hence

$$P \equiv (s-\alpha)Q'(s-m,x,D) \qquad \mathrm{mod}\ \mathfrak{J}(s),$$

which proves the "if part".

("only if" part)  Suppose $\alpha \in R + \mathbb{N}$.  Then, $\exists \nu > 0$, such

that $b_{\pi,\nu}(\alpha - \nu) = 0$.  It follows from the definition of $b_{\pi,\nu}(s)$

that there exists $P_\nu(s) \in \mathcal{S}[s]$, such that

$$P_\nu(s+\nu)f^\nu \equiv b_{\pi,\nu}(s) \qquad \mathrm{mod}\ \mathfrak{J}(s).$$

Therfore $P_\nu(\alpha) \in \mathfrak{J}_\alpha$.  If $\pi_\alpha \simeq \mathcal{S}f^\alpha$ were valid, we should have

$$P_\nu(\alpha) = Q(s) + (s-\alpha)R(s), \quad {}^\exists Q(s) \in \mathfrak{J}(s).$$

Then, if we set $R_\nu(s) = (P_\nu(s) - P_\nu(\alpha))/(s-\alpha) + R(s)$,

$$R_\nu(s+\nu)f^\nu \equiv b_{\pi,\nu}(s)/(s+\nu-\alpha) \qquad . \quad \mathrm{mod}\ \mathfrak{J}(s)$$

This contradicts the minimality of $b_{\pi,\nu}(s)$.  Q.E.D.

There is a canonoical map

$$\tau: \ \pi_{\alpha+1} \to \pi_\alpha \ , \qquad (f^\alpha u)_{\alpha+1} \mapsto f \cdot (f^\alpha u)_\alpha .$$

As a map between $\mathcal{S}[s]$-Modules, this is $Q(s)\bar{1} \mapsto Q(s+1)f \cdot \bar{1}$.
Since $(\mathfrak{J}(s+1) + \mathcal{S}[s](s-\alpha))f \subset \mathfrak{J}(s) + \mathcal{S}[s](s-\alpha)$,
this map is well-defined.  There is also a map $\mathcal{S}f^{\alpha+1}u \to \mathcal{S}f^\alpha u$,

defined by $f^{\alpha+1}u \longrightarrow f \cdot f^{\alpha}u$. These maps are compatible with the surjection (35).

__Theorem 1.24__  i) $\pi_{\alpha+1} \backsimeq \pi_\alpha$  if and only if $\alpha \notin R$.

ii)  When $\alpha \notin R$, $\mathcal{G}f^{\alpha+1}u \backsimeq \mathcal{G}f^\alpha u$.

Proof)  We first prove "if part" of i) and ii).  If $\alpha \notin R_f$, we can define the map $\rho : \pi_\alpha \to \pi_{\alpha+1}$, by $\bar{1} \mapsto b(\alpha)^{-1} P(\alpha) \bar{1}$. As a map between $\mathcal{G}[s]$-Modules, $\rho$ is $R(s)\bar{1} \mapsto b(\alpha)^{-1} R(s-1) P(\alpha) \bar{1}$. Then this is a well-defined homomorphism, since if

$R(s) \in \mathfrak{J}(s) + \mathcal{G}[s](s-\alpha)$, $R(s-1)P(\alpha)f^s u \equiv 0 \mod (s-\alpha-1)\mathcal{G}[s] \cdot f^s u$. Similarly,

$$Q(s)fP(\alpha)f^s u = Q(s)f\{P(s-1) + (P(\alpha)-P(s-1))\} f^s u$$

$$\equiv Q(s)b(s-1)f^s u$$

$$\equiv b(\alpha)Q(s)f^s u \mod (s-\alpha-1)\mathcal{G}[s]f^s.$$

Therefore, $\rho\tau(Q(s)\bar{1}) = \rho(Q(s+1)f \cdot \bar{1}) = b(\alpha)^{-1}Q(s)fP(\alpha)\bar{1} = Q(s)\bar{1}$. Analogously, $\tau \rho(R(s)\bar{1}) = R(s)\bar{1}$. Thus, $\rho$ is the inverse of $\tau$.  The proof of ii) can be given in the same manner. ("only if" part) $\pi_{\alpha+1} \simeq \pi_\alpha$  implies

$$(\mathfrak{J}(s+1) + \mathcal{G}[s](s-\alpha))f = \mathfrak{J}(s) + \mathcal{G}[s](s-\alpha).$$

Hence, if $R(s)f^s u = (s-\alpha)Q(s)f^s u$, then there is $Q'(s)$ such that $R(s)f^s u = (s-\alpha)Q'(s)f \cdot f^s u$.  Therefore, if $\alpha \in R$ were valid, the relation

$$P(s)f \cdot f^s u = (s-\varkappa)\frac{b(s)}{s-\varkappa}f^s u = (s-\varkappa)Q'(s)f \cdot f^s u$$

shows $Q'(s)f \cdot f^s u = \frac{b(s)}{s-\varkappa}f^s u$. This contradicts the minimality

of $b(s)$.                                                    Q.E.D.


<u>Corollary</u> 1.25      i) When $\varkappa \notin R + \mathbb{N}$, the following

commutative diagram exists for any $k \in \mathbb{N}_0$.

When $\varkappa \notin R + \mathbb{Z}$, it holds for $k = \mathbb{Z}$.

$$\begin{array}{ccc} \mathfrak{N}_\varkappa & \xrightarrow{\sim} & \mathfrak{N}_{\varkappa-k} \\ \downarrow & & \downarrow \\ \mathcal{S}f^\varkappa u & \xrightarrow{\sim} & \mathcal{S}f^{\varkappa-k}u \end{array}$$

ii) $\varinjlim \mathfrak{N}_{\varkappa-k} \cong \varinjlim \mathcal{S}f^{\varkappa-k}u$

is holonomic for $\forall_\varkappa \in \mathbb{C}$.


Proof)    i) is direct consequence of Theorems 1.24 and 1.21.

ii) follows from ).

## §5. Reduced b-functions

We can realize a reduced b-function as a b-function of some $\mathcal{G}[t,s]$-Modules. The characterization of these Modules are also given. We are indebted to M.Sato[2] for basic ideas in this section.

Definition 1.26

$$\mathcal{M}_{\#} = \bigcup_{\nu \geq 0} [\bar{b}(s-\nu)]_{\nu} \, t^{-\nu} \mathcal{M} \, ,$$

$$\mathcal{M}^{\#} = \left\{ v(s) \in \bigcup_{\nu \geq 0} \mathcal{J}[s] t^{-\nu} \mathcal{M} \, \Big| \, \exists \, m, \; [\bar{b}(s)]_m v(s) \in t^m \mathcal{M}_{\#} \right\}.$$

Proposition 1.27

i) $\mathcal{M}_{\#}$ and $\mathcal{M}^{\#}$ are $\mathcal{G}[t,s]$-Modules. If $\mathcal{M}$ is coherent, $\mathcal{M}_{\#}$ is also coherent.

ii) $b_{\mathcal{M}_{\#}} = b_{\mathcal{M}^{\#}} = \bar{b}$ .

Proof) i) $\mathcal{M}_{\#}$ and $\mathcal{M}^{\#}$ are easily seen to be $\mathcal{G}[t,s]$-Modules. To see the coherency of $\mathcal{M}_{\#}$, we use the operators $P_{\nu}(s)$ which satisfy

$$P_{\nu}(s+\nu) f^{\lambda} \equiv b_{\mathcal{M}, \nu}(s) \qquad \text{mod} \; \mathcal{J}(s).$$

Since $\mathcal{G}[s]$ is a noetherian ring, there is $m \in \mathbb{N}$ such that

$$P_m(s) + A_1(s) P_{m-1}(s) + \cdots + A_m(s) P_0(s) = 0.$$

for some $A_\nu(s) \in \mathcal{L}[s]$. Since

$$P_h(s)P_n(s+n) \equiv c(s)P_{h+n}(s+n) \qquad \text{mod } \mathcal{J}(s+n),$$

multiplying $P_n(s+n)$ from the right, cancelling $c(s)$ and rewriting s to s-n, we have

$$P_{m+n}(s) + A_1(s-n)P_{m+n-1}(s) + \dots + A_m(s-n)P_n(s) \equiv 0 \text{ mod } \mathcal{J}(s).$$

Therefore, $\mathcal{N}_\# = \bigcup_{\nu=0}^{m-1} [\bar{b}(s-\nu)]_\nu \, t^{-\nu}\mathcal{N} \subset t^{-m+1}\mathcal{N}$.

ii) Obviously, $[\bar{b}(s)]_\nu \mathcal{N}_\# \subset t^\nu \mathcal{N}_\#$. Set $\bar{b}_\nu(s) = b_{\mathcal{N}_\#,\nu}(s)$. If $\bar{b}_\nu(s) \neq [\bar{b}(s)]_\nu$ for some $\nu$, there is $k' < k = \deg \bar{b}(s)$, such that $\deg \bar{b}_\nu(s) < \nu k'$ for $\nu \gg 0$. But the following diagram shows $\nu k \leq (\nu+m-1)k'$. This is a contradiction. Thus we proved $b_{\mathcal{N}_\#} = \bar{b}$.

$$
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{N}/t^{\nu+m-1}\mathcal{N}_\# \longrightarrow \mathcal{N}/t^\nu\mathcal{N} \longrightarrow 0 \\
\downarrow \\
\mathcal{N}_\# \cdot /t^{\nu+m-1}\mathcal{N}_\# \cdot
\end{array}
$$

It follows from the definition of $\mathcal{N}^\#$ that $\mathcal{N}^\# \supset \mathcal{N}_\#$ and $\bar{b}(s)\mathcal{N}^\# \subset t\mathcal{N}^\#$. Set $\bar{\bar{b}}(s) = b_{\mathcal{N}^\#}(s)$ and assume $\neq \bar{b}(s)$. Then or $v(s) \in \mathcal{N}_\#$, $\bar{\bar{b}}(s)v(s) \in t\mathcal{N}^\#$ yields $[\bar{b}(s)]_m \bar{\bar{b}}(s-1)v(s-1) \in t^m\mathcal{N}_\#$. This relation is equivalent to $\bar{\bar{b}}(s)[b(s+1)]_m v(s) \in t^{m+1}\mathcal{N}_\#$. Since $\mathcal{N}_\#$ is finitely generated over $\mathcal{L}[s]$, we see that $b_{\mathcal{N}_\#,m}(s)$ is a strict divisor of $[\bar{b}(s)]_m$ for sufficiently large m. That is a contradiction. Hence $b_{\mathcal{N}^\#}(s) = \bar{b}(s)$. Q.E.D.

**40**

It is not for certain whether $\mathcal{M}^{\#}$ is coherent or not when $\mathcal{M}$ is coherent. We have, however, the following characterization.

Theorem 1.28    Let $\mathcal{M}'$ be a $\mathcal{L}[t,s]$-Module satisfying,

$$t^{-k}\mathcal{M} \supset \mathcal{M}' \supset \mathcal{M} \qquad \text{for some k. Then } b_{\mathcal{M}'}(s) = \bar{b}(s)$$

if and only if

$$\mathcal{M}^{\#} \supset \mathcal{M}' \supset \mathcal{M}_{\#} \ .$$

Proof)    ("only if" part)    Since $b_{\mathcal{M}'}(s) = \bar{b}(s)$, we have relations

$$\mathcal{M}' \supset \bar{b}(s-1)t^{-1}\mathcal{M}' \supset \bar{b}(s-1)\bar{b}(s-2)t^{-2}\mathcal{M}' \supset [\bar{b}(s-h)]_h t^{-h}\mathcal{M}' \ .$$

Therefore,

$$\mathcal{M}' \supset \bigcup_{h \geq 0} [\bar{b}(s-h)]_h t^{-h}\mathcal{M}' \supset \bigcup_{h \geq 0} [\bar{b}(s-h)]_h t^{-h}\mathcal{M} = \mathcal{M}_{\#} \ .$$

Then the following diagram

$$\mathcal{M}' \supset \mathcal{M}_{\#} \supset t^m \mathcal{M}' \supset t^m \mathcal{M}_{\#} \qquad\qquad (46)$$

shows that $d_{\mathcal{M}'/t^m\mathcal{M}_{\#}}(s)$    divides    both $d'(s)[\bar{b}(s)]_m$ and $[\bar{b}(s)]_m d'(s+m)$ (where we set $d'(s) = d_{\mathcal{M}'/\mathcal{M}_{\#}}$), and hence one of $[\bar{b}(s)]_m$ for $m \gg 0$. But $[\bar{b}(s)]_m$ is the best possible for the pair $\mathcal{M}_{\#} \supset t^m \mathcal{M}_{\#}$ . Therefore, $d_{\mathcal{M}'/t^m\mathcal{M}_{\#}}(s) = [\bar{b}(s)]_m$. Thus the definition of $\mathcal{M}$ proves $\mathcal{M}^{\#} \supset \mathcal{M}'$ .

("if" part)      Consider the following diagram for $m \gg 0$.

$$\pi^{\#} \supset \pi' \supset \pi_{\#} \supset t^m \pi' \supset t^m \pi_{\#} \quad .$$

Then the definition of $\pi^{\#}$ implies $[\bar{b}_{\pi'}(s)]_m c_{\pi'}(s+m) \mid [\bar{b}(s)]_m$.

On the other hand, equation (17) of Theorem 1.8 shows

$\bar{b}_{\pi'}(s) = (c'(s)/c'(s+1))\bar{b}(s)$. From these formulae,

we have $c' = c_{\pi'} = 1$, and then $b_{\pi'}(s) = \bar{b}(s)$.      Q.E.D.


Corollary 1.29   Assume that $w(\bar{b}) = 1$ in addition to the

condition on $\pi'$ in Theorem 1.28. Then,

$b_{\pi'} = \bar{b}$, if and only if $\pi' = \pi_{\#}$


Proof)   The "if" part is trivial. Consider the diagram

(46) when $b_{\pi'} = \bar{b}$. $d_{\pi'/t^m \pi_{\#}}(s) = [\bar{b}(s)]_m$ is shown in

the proof of Theorem 1.27. Therefore, $d'(s) = d_{\pi'/\pi_{\#}}(s)$

and $d'(s+m)$ are divisors of $[\bar{b}(s)]_m$ for large m. Since

$w(\bar{b}) = 1$, this is actually possible only when $d'(s) = 1$,

that is, $\pi' = \pi_{\#}$ .                        Q.E.D.