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Razumikhin type theorems for differential equations with infinite delay

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Our concern is on the stability problem for functional differential equations with infinite delay

$$\dot{x}(t) = f(t,x_t). \tag{1}$$

For functional differential equations with infinite delay, there are several ways to specify the phase space. A typical one is the Hale's space $\mathcal{B}$ (see [1]) consisting of functions defined on $(-\infty,0]$, which is provided a norm $|\cdot|_{\mathcal{B}}$ and the conditions;

(i) if $x(t)$ is defined on $(-\infty,a)$, $a > 0$, continuous on $[0,a)$ and $x_0 \in \mathcal{B}$, then for $t \in [0,a)$, $x_t \in \mathcal{B}$ and it is continuous in t, where

$$x_t(s) = x(t+s) \quad \text{for} \quad s \in (-\infty,0];$$

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(ii) there exist two positive constants c, d such that

$$|\phi|_{\mathcal{B}} \leqq c \sup_{-\beta \leqq s \leqq 0} |\phi(s)| + d|\phi|_{\beta}$$

for any $\beta \geqq 0$, where

$$|\phi|_{\beta} = \inf \{ |\psi|_{\mathcal{B}};\ \psi \in \mathcal{B}, \psi(s) = \phi(s) \text{ on } (-\infty, -\beta]\}$$

together with other conditions.

In our case, the space $\mathcal{B}$ is assumed to satisfy the properties

$$|\phi(0)| \leqq M|\phi|_{\mathcal{B}},\quad |\phi|_{\beta} \leqq M(\beta)|\phi_{-\beta}|_{\mathcal{B}} \text{ if } \phi_{-\beta} \in \mathcal{B},$$

in addition to the conditions (i) and (ii), though c and d in (ii) may continuously depend on β. In particular, if $x(t)$ is defined on $(-\infty, a)$ and continuous on $[\tau, a), \tau < a$, and if $x_{\tau} \in \mathcal{B}$ then we have

$$(2) \qquad |x_t|_{\mathcal{B}} \leqq c(t-\tau) \sup_{\tau \leqq s \leqq t} |x(s)| + d(t-\tau)M(t-\tau)|x_{\tau}|_{\mathcal{B}}.$$

It is assumed for the equations (1) to have the trivial solution, where $f(t, \phi)$ in (1) is defined and continuous on

$R \times \mathcal{B}$.

The following definition will be made:

Definition. The trivial solution of (1) is said to be (I) *stable* if for any $\varepsilon > 0$ and any $\tau \geq 0$ there exists a $\delta > 0$ such that

$$|x_\tau|_{\mathcal{B}} < \delta \quad \text{implies} \quad |x(t)| < \varepsilon \quad \text{for all} \quad t \geq \tau;$$

(II) *asymptotically stable* if in addition to the stability for any $\tau \geq 0$ there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$ there is a T such that

$$|x_\tau|_{\mathcal{B}} < \delta_0 \quad \text{and} \quad t \geq \tau + T \quad \text{imply} \quad |x(t)| < \varepsilon;$$

where $x(t)$ denotes any solution of (1). Here, δ, δ_0, T may depend on τ but not on each solution. If these numbers are independent of τ, then the stabilities are called *uniform*.

The following theorem is a simple version of the Liapunov-Krasovskii's theorem (see [2] and also [3]).

Theorem A. Suppose that there exists a continuous function $V(t,\phi)$ defined on $R \times \mathcal{B}$ such that $V(t,0) = 0$,

(3) $$a(|\phi(0)|) \leqq V(t,\phi)$$

for a continuous, increasing, positive-definite function $a(r)$ and that for a continuous function $c(t,r) \geqq 0$, which is non-decreasing in r,

(4) $$\dot{V}(t,x_t) \leqq -c(t,V(t,x_t))$$

along any solution $x(t)$ of (1), where

$$\dot{V}(t,x_t) = \overline{\lim_{h\to+0}} \frac{1}{h}\{V(t+h,x_{t+h}) - V(t,x_t)\}.$$

Then the trivial solution of (1) is asymptotically stable if for any $r > 0$

(5) $$\int_t^{t+T} c(s,r)ds \to \infty \quad \text{as} \quad T \to \infty;$$

and uniformly asymptotically stable if the divergence in (5) is uniformly in t and if we have

(6) $$V(t,\phi) \leqq b(|\phi|_{\mathcal{B}})$$

for a continuous function $b(r)$ with $b(0) = 0$.

Since the solutions may belong to the more restrictive class as the time elapses, the following theorem is expected to be more effective. Such a theorem has been given by

Barnea[4] for the uniform stability of an autonomous system with finite delay (also refer [5]).

Thorem B. In Theorem A, it is sufficient for $V(t,\phi)$ to satisfy (4) under the case (*) $x(s)$ is a solution of (1) at least on the interval $[p(t,V(t,x_t)), t]$, where the continuous function $p(t,r) \leqq t$ is increasing in $t \geqq 0$ and in $r > 0$ and satisfies $p(t,r) \to \infty$ as $t \to \infty$, $p(t,r) \to \infty$ as $r \to 0$. For the uniform stability we assume

$$(7) \qquad p(t,r) = t - q(r).$$

Here, also we assume that the trivial solution of (1) is unique for the stability and that $f(t,\phi)$ in (1) satisfies

$$(8) \qquad |f(t,\phi)| \leqq L|\phi|_{\mathcal{B}}$$

for the uniform stability.

Proof. Let $\varepsilon > 0$ be given. Suppose that $V(\tau,x_\tau) < \frac{a(\varepsilon)}{2}$ but $V(t,x_t) > a(\varepsilon)$ for a $t > \tau$. Then there exists

$$t_1 = \inf \{t > \tau;\ V(t,x_t) \geqq a(\varepsilon)\}.$$

Set $t_2 = \max \{t < t_1;\ V(t,x_t) \leqq \frac{a(\varepsilon)}{2}\}$.

Since we have

$$|x_t|_{\mathcal{B}} \leqq c(t-\tau) \sup_{\tau \leqq s \leqq t} |x(s)| + d(t-\tau)M(t-\tau)|x_\tau|_{\mathcal{B}}$$

for $t \geqq \tau$ by (2) and since the uniqueness of the trivial solution implies

$$\sup_{\tau \leqq s \leqq t} |x(s)| \leqq K(t,\tau, |x_\tau|_{\mathcal{B}}) \tag{9}$$

with $K(t,\tau,r) \to 0$ as $r \to 0$, we shall have

$$t \in [t_2,t_1] \text{ and } |x_\tau|_{\mathcal{B}} < \delta \text{ imply } \tau < p(t,V(t,x_t)).$$

For this purpose, it is enough to choose δ so that $\delta < \frac{a(\varepsilon)}{2}$ and

$$|\phi|_{\mathcal{B}} < A(p_t^{-1}(\tau,\frac{a(\varepsilon)}{2}),\tau,\delta) \text{ implies } V(t,\phi) < \frac{a(\varepsilon)}{2}$$

if $\tau \leqq t \leqq p_t^{-1}(\tau,\frac{a(\varepsilon)}{2})$, where $A(t,\tau,r) = c(t-\tau)K(t,\tau,r) + d(t-\tau)M(t-\tau)r$. Thus, by the assumptions $V(t,x_t)$ is non-increasing on $[t_2,t_1]$, which contradicts $V(t_1,x_{t_1}) = a(\varepsilon)$.

If f in (1) satisfies (8), we may choose K in (9) so that

$$K(t,\tau,r) = K(t-\tau)r$$

for a continuous function $K(t)$. Hence, in this case A is a function of $t - \tau$ and r, and under the condition $p(t,r) = t - q(r)$ δ can be chosen independent of τ so that

$$r \leqq A(\tau + q(\frac{a(\varepsilon)}{2}), \tau, \delta) \text{ implies } b(r) < \frac{a(\varepsilon)}{2} .$$

In the second step, we should note that

$$\text{(10)} \qquad \dot{V}(t,x_t) \leqq - c(t,V(t,x_t)) \text{ as long as } V(t,x_t) \geqq p_r^{-1}(t,\tau)$$

and that $p_r^{-1}(t,\tau)$ tends to 0 as $t \to \infty$.

Let δ_o and T_1 be such that $\delta_o(\tau) = \delta(\tau,1)$ and

$$\int_{\sigma}^{\sigma+T_1} c(s,\varepsilon)ds > \eta(\sigma,\tau) - \varepsilon,$$

where $\sigma = p_t^{-1}(\tau,\varepsilon)$ and

$$\eta(\sigma,\tau) \geqq \sup \{V(\sigma,\phi);\ |\phi|_{\mathcal{B}} \leqq b(\sigma-\tau) + c(\sigma-\tau)M(\sigma-\tau)\delta_o(\tau)\}.$$

Suppose that for a $t_1 > T + \tau$, $T = T_1 + \sigma - \tau$, we have $V(t_1,x_{t_1}) \geqq \varepsilon$. Clearly,

$$V(t_1,x_{t_1}) > p_r^{-1}(t_1,\tau).$$

Let $t_2 = \max \{\sup \{t < t_1;\ V(t,x_t) = p_r^{-1}(t,\tau)\},\tau\}$. Then, by

(10), $V(t,x_t)$ is non-increasing on $[t_2, t_1]$. Hence, we have

$$p_r^{-1}(t_2,\tau) \geqq V(t_2,x_{t_2}) \geqq V(t_1,x_{t_1}) \geqq \varepsilon,$$

which implies

$$\tau \geqq p(t_2,\varepsilon).$$

Therefore, $\sigma \overset{\text{def}}{\equiv} p_t^{-1}(\tau,\varepsilon) \geqq t_2$, that is,

$$\dot{V}(t,x_t) \leqq - c(t,V(t,x_t)) \quad \text{and} \quad V(t,x_t) \geqq \varepsilon \quad \text{for} \quad t \in [\sigma,t_1],$$

and hence we have

$$\begin{aligned} \varepsilon \leqq V(t_1,x_{t_1}) &\leqq V(\sigma,x_\sigma) - \int_\sigma^{t_1} c(s,V(s,x_s))ds \\ &\leqq V(\sigma,x_\sigma) - \int_\sigma^{t_1} c(s,\varepsilon)ds, \end{aligned}$$

which implies

$$\int_\sigma^{t_1} c(s,\varepsilon)ds \leqq \eta(\sigma,\tau) - \varepsilon.$$

This contradicts $t_1 > \tau + T(\tau,\varepsilon)$.

When $p(t,r) = t - q(r)$, $\sigma = \tau + q(\varepsilon)$. Therefore, if the divergence in (5) is uniformly in t, then we can choose T independent of τ.

<u>Remark 1</u>. It is sufficient that in the Theorem B for each τ there exists a Liapunov function $V(t,\phi;\tau)$ which is defined on $\{(t,x_t);\ t \geqq \tau,\ x(t)$ is continuous on $[\tau,\infty)$, $x_\tau \in \mathcal{B}\}$ and satisfies the conditions (3), (4) with a, c independent of τ, and corresponding to (6) we assume

$$V(t,x_t;\tau) \leqq b(\sup_{\tau \leqq s \leqq t} |x_s|_{\mathcal{B}}),$$

because to estimate solutions we can choose different Liapunov function for each solution.

Now, we try to construct a Razumikhin type theorem for the equations (1). Such theorems have been given in [3], [6], [7]. Here, we shall state the following theorem by extending the ideas in [5], [8].

<u>Theorem C</u>. In Theorem B, suppose that $p(t,r)$ is of the form (7).

Then, we can restrict $x(s)$ in (*) within a solution of (1) satisfying

$$(11) \qquad V(s,x_s) \leqq F(V(t,x_t)) \quad \text{for} \quad s \in [p(t,V(t,x_t)),t],$$

where $F(r)$ is a continuous function such that $F(r) > r$ and $F(r)/r$ is non-decreasing for $r > 0$.

To prove Theorem C, by Remark 1 it is sufficient to construct a Liapunov function for each τ, which satisfies the conditions in Theorem B on $[\tau,\infty)$. The existence of such a Liapunov function follows from the following lemma.

Lemma. Let F be as in Theorem C, and let p be as in Theorem B with $q(t,r) = t - p(t,r)$ which is non-decreasing in t.

If a Liapunov function $V(t,\phi)$ satisfies (3), (4) under the condition (11) and

$$V(t,\phi) \leqq b(t,|\phi|_{\mathcal{B}}),$$

then for each τ there exists a Liapunov function $W(t,x_t;\tau)$ which satisfies

$$a(|x(t)|) \leqq W(t,x_t;\tau) \leqq b^*(t,\tau, \sup_{\tau\leqq s\leqq t} |x_s|_{\mathcal{B}}) \tag{12}$$

and

$$\dot{W}(t,x_t;\tau) \leqq - c^*(t,W(t,x_t;\tau)), \tag{13}$$

if $x(s)$ is a solution of (1) on $[p(t,W(t,x_t;\tau)),t]$, where

$$b^*(t,\tau,r) = \sup_{\tau \leq s \leq t} b(s,r),$$

$$c^*(t,r) = \min \{c(t,r),\ r\alpha(t,r)\},$$

a, b, c, p, q for V, and

$$\alpha(t,r) = \frac{1}{q(p_t^{-1}(t,F^{-1}(\frac{r}{2})),F^{-1}(\frac{r}{2}))} \log \frac{r}{F^{-1}(r)}.$$

Proof. Define

$$W(t,x_t;\tau) = \sup_{\tau \leq s \leq t} V(s,x_s)e^{\alpha(s,V(s,x_s))(s - t)},$$

and for a fixed $x(s)$ set

$$W(t) = W(t,x_t;\tau),\ V(t) = V(t,x_t),$$

$$P(s,t) = V(s)e^{\alpha(s,V(s))(s - t)}.$$

Since $\alpha(t,r) > 0$ $(r > 0)$, obviously we have (12).

To prove (13), we choose $s(t) \in [\tau, t]$ so that

$$W(t) = P(s(t),t).$$

For small $h > 0$ we may assume that $s(t+h) \to s(t)$ as $h \to 0$.

Case 1. $s(t+h) \leq t$ for small $h > 0$. In this case, since $W(t) \geq P(s(t+h),t)$, we have

$$\frac{W(t+h) - W(t)}{h} \leq \frac{P(s(t+h),t+h) - P(s(t+h),t)}{h}$$

$$\leq W(t+h)\frac{1}{h}\{1 - e^{\alpha(s(t+h),V(s(t+h)))h}\}$$

$$\leq - W(t)\alpha(s(t),V(s(t))) + o(1)$$

$$\leq - W(t)\alpha(t,W(t)) + o(1).$$

Here, we note that $\alpha(t,r)$ is non-decreasing in r, non-increasing in t and that $V(s(t)) \geq W(t)$.

Case 2. $t \leq s(t+h) \leq t + h$ for some arbitrarily small $h > 0$. Then, clearly $s(t) = t$. Therefore,

$$V(t) = W(t) \geq P(s,t) \quad \text{for any} \quad s \leq t.$$

Hence,

(14) $$V(t) \geq V(s)e^{-\alpha(s,V(s))q(t,V(t))} \quad \text{for any} \quad s \in [p(t,V(t)),t].$$

Assume that $x(s)$ is a solution of (1) at least on $[p(t, W(t)),t]$ and, in particular, $\tau \leq p(t,W(t))$.

If we can prove that

$$V(t) \geqq F^{-1}(\frac{V(s)}{2}), \tag{15}$$

immediately we have

$$t \leqq p_t^{-1}(s, F^{-1}(\frac{V(s)}{2})) \quad \text{if} \quad s \geqq p(t, V(t)),$$

and hence by the definition of $\alpha(t,r)$

$$\alpha(s, V(s))q(t, V(t)) \leqq \log \frac{V(s)}{F^{-1}(V(s))},$$

which implies $V(t) \geqq F^{-1}(V(s))$, that is,

$$F(V(t)) \geqq V(s) \quad \text{for} \quad s \in [p(t,V(t)),t] \quad \text{with (15).}$$

This fact also proves (15) for all $s \in [p(t,V(t)),t]$, and hence we have

$$F(V(t)) \geqq V(s) \quad \text{for all} \quad s \in [p(t,V(t)),t]. \tag{16}$$

Since $s(t) = t$, we have

$$\frac{W(t+h) - W(t)}{h} = V(s(t+h))\frac{1}{h}\{e^{\alpha(s(t+h),V(s(t+h)))(s(t+h)-t-h)} - 1\}$$

$$+ \frac{V(s(t+h)) - V(t)}{h}$$

$$= V(t)\alpha(t,V(t))\left\{\frac{s(t+h) - t}{h} - 1\right\} + V(t)\frac{s(t+h) - t}{h}$$

$$+ o(1)$$

$$\leqq - W(t)\alpha(t,W(t))\left\{1 - \frac{s(t+h) - t}{h}\right\}$$

$$- c(t,W(t))\frac{s(t+h) - t}{h} + o(1)$$

$$\leqq \; - c^*(t,W(t)) + o(1),$$

because V satisfies (4) under (16) and $\frac{s(t+h) - t}{h} \in [0, 1]$.

To complete the proof of Theorem C, it is sufficient to note that if q is independent of t, then so is α and that the property (5) for $c(t,r)$ implies the same property for $c^*(t,r)$.

Remark 2. As is clear from the lemma, for the stability it is sufficient that the property (5) holds for $c^*(t,r)$. In addition to the case given in Theorem C, this is satisfied if c is independent of t and

$$\int_t^{t+T} \frac{ds}{q(p_t^{-1}(s,r),r)} \to \infty \quad \text{as} \quad T \to \infty.$$

The asmptotic stability of

$$\dot{x}(t) = -ax(t) + b(t)x(p(t)),$$

$$|b(t)| \leqq \beta < a, \; p(t) = \varepsilon t, \; 0 < \varepsilon < 1,$$

can be proved as the case.

However, unfortunately the case where

$$p(t) = \frac{\sqrt{1 + 4t} - 1}{2}$$

is not covered by our result, though the asymptotic stability can be proved by the method in [3].

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