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On the Oscillation of Fourth Order Nonlinear Differential Equations with Deviating Argument

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Introduction

The purpose of this note is to report some results which we have recently obtained concerning the oscillation of solutions of the nonlinear differential equation

$$(*) \qquad [r(t)y''(t)]'' + f(y(g(t)), t) = 0,$$

where the following conditions are assumed to hold:

(a) $r(t)$ is continuous and positive on $[0,\infty)$;

(b) $g(t)$ is continuous on $[0,\infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(c) $f(y,t)$ is continuous on $(-\infty,\infty)\times[0,\infty)$ and $yf(y,t) > 0$ for $y \neq 0$.

It should be noticed that the deviating argument $g(t)$ is allowed to be retarded ($g(t) \leqq t$) or advanced ($g(t) \geqq t$) or otherwise.

Equation (*) is called superlinear if $f(y,t)/y$ is nondecreasing in y for $y > 0$ and nonincreasing in y for $y < 0$; (*) is called strongly superlinear if there is a number $\alpha > 1$ such that $f(y,t)/|y|^{\alpha}\operatorname{sgn} y$ is nondecreasing in $y > 0$ and nonincreasing in $y < 0$. Dually, equation (*) is called sublinear if $f(y,t)/y$ is nonincreasing in y for $y > 0$ and nondecreasing in y for $y < 0$; (*) is called strongly sublinear if there exists a positive number $\beta < 1$ such that $f(y,t)/|y|^{\beta}\operatorname{sgn} y$ is nonincreasing in $y > 0$ and nondecreasing in $y < 0$.

Our attention will be restricted to solutions $y(t)$ of (*) which exist on some ray $[T_y,\infty)$ and satisfy

$$\sup\{|y(t)| : t \geqq T\} > 0 \quad \text{for any} \quad T \geqq T_y.$$

Such a solution is said to be _oscillatory_ if it has arbitrarily large zeros and _nonoscillatory_ otherwise.

We are interested in finding necessary and sufficient conditions for all solutions of (*) to be oscillatory. The results obtained extend considerably those of [1], [2] for ordinary differential equations of the form less general than (*). Some of the results of this note will appear in [3].

It is convenient to distinguish the two cases:

(A) $$\int_0^\infty \frac{t}{r(t)}\,dt = \infty,$$

(B) $$\int_0^\infty \frac{t}{r(t)}\,dt < \infty.$$

The following notation will be used throughout:

$$g^*(t) = \max\{g(t), t\}, \qquad g_*(t) = \min\{g(t), t\},$$

$$R(t) = \int_0^t \frac{(t-s)s}{r(s)}\,ds, \qquad \rho(t) = \int_t^\infty \frac{s-t}{r(s)}\,ds.$$

1. Nonoscillation Theorems

We first present sufficient conditions which guarantee the existence of nonoscillatory solutions of (*) with special asymptotic properties. From these results necessary conditions for the oscillation of all solutions of (*) will readily be derived.

THEOREM 1A. _Let_ (*) _be either superlinear or sublinear_. _Suppose that_ (A) _holds_.

(i) *A necessary and sufficient condition for* (*) *to have a solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)/R(t) = a \neq 0$ *is that*

$$\int^{\infty} |f(cR(g(t)),t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

(ii) *A necessary and sufficient condition for* (*) *to have a solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t) = b \neq 0$ *is that*

$$\int^{\infty} R(t)|f(c,t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

THEOREM 1B. *Let* (*) *be either superlinear or sublinear. Suppose that* (B) *holds.*

(i) *A necessary and sufficient condition for* (*) *to have a solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)/t = a \neq 0$ *is that*

$$\int^{\infty} \rho(t)|f(cg(t),t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

(ii) *A necessary and sufficient condition for* (*) *to have a solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)/\rho(t) = b \neq 0$ *is that*

$$\int^{\infty} t|f(c\rho(g(t)),t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

We give an outline of the proof of Theorem 1B-(ii).

(Necessity) Let (*) have a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/R(t) = a$. We may suppose that $a > 0$. There exist positive constants T, a_1, a_2 such that

$$a_1 g(t) \leqq y(g(t)) \leqq a_2 g(t) \quad \text{for} \quad t \geqq T.$$

This implies that

(1.1) $f(y(g(t)),t) \geqq f(a_1 g(t),t)$ if (*) is superlinear, (1.2) $f(y(g(t)),t) \geqq (a_1/a_2)f(a_2 g(t),t)$ if (*) is sublinear.

It turns out that the following three cases are possible:

(I) $y'(t) > 0$, $y''(t) > 0$ and $[r(t)y''(t)]' > 0$ for $t \geqq T$;

(II) $y'(t) > 0$, $y''(t) < 0$ and $[r(t)y''(t)]' > 0$ for $t \geqq T$;

(III) $y'(t) > 0$, $y''(t) < 0$ and $[r(t)y''(t)]' < 0$ for $t \geqq T$.

Let Case (I) or (II) hold. Then, integrating (*), using (1.1), (1.2) and noting that $[r(t)y''(t)]' > 0$, we readily obtain

$$\int^{\infty} f(a_i g(t),t)dt < \infty,$$

where $i = 1$ if (*) is superlinear and $i = 2$ is (*) is sublinear. Let Case (III) hold. We multiply (*) by $\rho(t)$, integrate it from T to t, add $y'(t)$ to both sides of the resulting equation and let $t \to \infty$. Using the inequalities

$$y'(t) \geqq -[r(t)y''(t)]'\rho(t),$$

$$y'(t) \geqq -r(t)y''(t)\int_t^{\infty} \frac{ds}{r(s)},$$

we arrive at

$$\int^{\infty} \rho(t)f(a_i g(t),t)dt < \infty,$$

where $i = 1$ or 2 according as (*) is superlinear or sublinear.

(Sufficiency) Suppose $c > 0$. Put $a = c/2$ or c according as (*) is superlinear or sublinear. Take $T > 0$ so large that

$$\int_T^{\infty} \rho(t)f(cg(t),t)dt < \frac{a}{6} \quad \text{and} \quad T_0 = \inf_{t \geqq T} g_*(t) > 0.$$

Let Y designate the linear space of all continuous functions $y(t)$ on $[T_0,\infty)$ such that

$$\|y\| = \sup_{t \geqq T_0} [|y(t)|/t^2] < \infty$$

and put

$$X = \{y \in Y:\ at \leqq y(t) \leqq 2at \quad \text{for} \quad t \geqq T_0\}.$$

Clearly, Y is a Banach space with norm $\|\cdot\|$ and X is a bounded, convex and closed subset of Y.

We define the operator Φ by

$$(\Phi y)(t) = at + \int_T^t \left(\int_s^t \frac{(\sigma - s)\sigma}{r(\sigma)} d\sigma \right) f(y(g(s)),s)ds$$

$$+ t\int_t^\infty \frac{ds}{r(s)} \cdot \int_T^t (t - s) f(y(g(s)),s)ds$$

$$+ t\int_t^\infty \frac{s - t}{r(s)} ds \cdot \int_T^t f(y(g(s)),s)ds$$

$$+ t\int_t^\infty \left(\int_s^\infty \frac{\sigma - s}{r(\sigma)} d\sigma \right) f(y(g(s)),s)ds \qquad \text{for } t \geqq T,$$

$$(\Phi y)(t) = at + T\int_T^\infty \left(\int_s^\infty \frac{\sigma - s}{r(\sigma)} d\sigma \right) f(y(g(s)),s)ds \qquad \text{for } T_0 \leqq t \leqq T.$$

It can be shown that Φ is a continuous operator mapping X into a compact subset of X. Thus the Schauder fixed point theorem is applicable, and Φ has a fixed point in X. This fixed point provides the required solution of (*).

2. Oscillation Theorems

Our next task is to give sufficient conditions for the oscillation of all solutions of (*) by limiting ourselves to the strongly superlinear and strongly sublinear cases.

THEOREM 2A. Let (*) be strongly superlinear. Suppose (A) holds. If there is a differentiable function h(t) on $[0,\infty)$ such that

$$h(t) \leqq g_*(t), \quad h'(t) \geqq 0, \quad h(t) \to \infty \text{ as } t \to \infty,$$

and

$$\int^\infty R(h(t))|f(c,t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

THEOREM 2B. Let (*) be strongly superlinear. Suppose (B) holds. If

$$\int^\infty g_*(t)|f(c\varphi(g^*(t)),t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

THEOREM 3A. Let (*) be strongly sublinear. Suppose (A) holds. If

$$\int^\infty \frac{R(g_*(t))}{R(g(t))}|f(cR(g(t)),t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

THEOREM 3B. Let (*) be strongly sublinear. Suppose (B) holds. If

$$\int^\infty \frac{g_*(t)}{g(t)}\rho(g^*(t))|f(cg(t),t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

Here we sketch the proof of Theorem 3A. Let there exist a nonoscillatory solution y(t). Without loss of generality we may suppose that y(t) is eventually positive. It is easy to show that $y'(t) > 0$, $[r(t)y''(t)]' > 0$ for all large t, say $t \geqq T$, that there are positive constants a_1, a_2 such

(2.1) $$a_1 \leqq y(t) \leqq a_2 R(t), \quad t \geqq T,$$

and that the following inequality holds:

(2.2) $$y(t) \geqq R_T(t)[r(t)y''(t)]', \quad t \geqq T,$$

where

$$R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)}\,ds.$$

Let $T' > T$ be such that $g_*(t) \geqq T$ for $t \geqq T'$. An integration of (*) yields

(2.3) $$[r(t)y''(t)]' \geqq \int_t^\infty f(y(g(s)),s)\,ds, \quad t \geqq T'.$$

Combining (2.3) with the inequality

$$y(g(t)) \geqq R_T(g_*(t))[r(t)y''(t)]', \quad t \geqq T',$$

which follows from (2.2), we have

(2.4) $$y(g(t)) \geqq R_T(g_*(t))\int_t^\infty f(y(g(s)),s)\,ds, \quad t \geqq T'.$$

If we apply the lemma stated below to (2.4), observing from (2.1) that $y(g(t)) \leqq aR_T(g(t))$, $t \geqq T'$, for some constant $a > 0$, then we conclude that

$$\int^\infty \frac{R_T(g_*(t))}{R_T(g(t))} f(aR_T(g(t)),t)\,dt < \infty.$$

But this contradicts the assumption of Theorem 3A.

LEMMA. *Let* (*) *be strongly sublinear. Let* $u(t)$, $v(t)$, $w(t)$, $\mu(t)$ *be positive continuous functions on* (T,∞) *such that* $\mu(t) \leqq 1$, $u(t) \leqq aw(t)$, *and*

$$u(t) \geqq \mu(t)w(t)\int_t^\infty v(s)f(u(s),s)\,ds \quad \text{for } t > T,$$

where a *is a positive constant. Then,* It is of interest to observe that from the foregoing results necessary and sufficient conditions can be obtained for the oscillation of all solutions of certain classes of differential equations of the form (*).

THEOREM 4A. *Let (*) be strongly superlinear and advanced. Suppose (A) holds. Then all solutions of (*) are oscillatory if and only if*

$$\int^{\infty} R(t)|f(c,t)|dt = \infty \quad \text{\underline{for all}} \quad c \neq 0.$$

THEOREM 4B. *Let (*) be strongly superlinear and advanced. Suppose (B) holds. Then all solutions of (*) are oscillatory if and only if*

$$\int^{\infty} t|f(c\rho(g(t)),t)|dt = \infty \quad \text{\underline{for all}} \quad c \neq 0.$$

EXAMPLE 1. Consider the strongly superlinear retarded equation

$$[t^3 y''(t)]'' + (3/16)t^{-1}|y(t^{1/2})|^2 \operatorname{sgn} y(t^{1/2}) = 0,$$

which has a nonoscillatory solution $y(t) = t^{-1/2}$. As easily verified, the integral condition of Theorem 4B is satisfied for this equation. This example shows that, as regards the oscillation of strongly superlinear equations of the form (*) with general deviating argument, there is a gap between the necessary condition and the sufficient condition that were given in the above theorems.

THEOREM 5A. *Let (*) be strongly sublinear and retarded. Suppose (A) holds. Then all solutions of (*) are oscillatory if and only if*

$$\int^{\infty} |f(cR(g(t)),t)|dt = \infty \quad \text{\underline{for all}} \quad c \neq 0.$$

THEOREM 5B. *Let* (*) *be strongly sublinear and retarded. Suppose* (B) *holds. Then all solutions of* (*) *are oscillatory if and only if*

$$\int^{\infty} \rho(t)\,|f(cg(t),t)|\,dt = \quad \underline{\text{for all}} \quad c \neq 0.$$

EXAMPLE 2. Consider the strongly sublinear advanced equation

$$[t^{1/2}y''(t)]'' + (1/2)t^{-7/2}|y(t^2)|^{1/2}\operatorname{sgn} y(t^2) = 0.$$

Although the integral condition of Theorem 5A is satisfied, this equation has a nonoscillatory solution $y(t) = t^2$. Thus it follows that there is a gap between the necessary condition and the sufficient condition for the oscillation of all solutions of the strongly sublinear equation (*) with general deviating argument.

REFERENCES

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