数理解析研究所講究録
288巻 1977年 1-11

The Boundary Layer Equation
$x''' + 2xx'' + 2\lambda(1-x'^2) = 0$

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In the theory of viscous fluids the following non-linear boundary value problem for a function $x(t)$ of a real variable $t \geqq 0$ involving a constant λ plays an important part ;

(1) $x''' + 2xx'' + 2\lambda(1-x'^2) = 0$

(2) $x(0) = x'(0) = 0$, $x'(\infty) = 1$

(3) $0 < x'(t) < 1$ for $0 < t < \infty$.

For $\lambda \geqq 0$ H.Weyl(1942) first proved that there exists a continuous solution of the problem.

For $\lambda < 0$ ($|\lambda|$ small) S.P.Hastings(1971) first showed the existence of solutions as far as we know.

On the other hand, it is known that the separation phenomenon of boundary layer occurs for $\lambda = -0.1988..$, and M.Iwano(1974) tried to show the existence of solutions for negative λ as small as possible.

In this report we shall extend the value of such λ as closely as possible to the value $-0.1988..$.

Our method of proof, which is close to that of W.A.Coppel(1960), owes to Kneser's property,which was shown by M.Hukuhara(1967). Although we can solve this problem by using the continuity dependence property of solutions to initial data, because the equation (1) has the property that the solution for an initial value problem is unique, our proof was found by examining the paper of M.Hukuhara(1967).

1. An Existence Theorem of Solutions

Theorem 1. If $\lambda > -1/6$, then the equation (1) has a continuous solution satisfying (2), (3).

We choose x as a new independent variable and $y = x'^2$ as a new dependent variable. The equation (1) is transformed into

$$(4) \qquad \ddot{y} = -y^{-\frac{1}{2}}(2x\dot{y} + 4\lambda(1-y)) = f(x,y,\dot{y})$$

the boundary condition (2) into

$$(5) \qquad y(0) = 0 \ , \quad y(\infty) = 1$$

and the condition (3) into

$$(6) \qquad 0 < y(x) < 1 \quad \text{for} \quad 0 < x < \infty .$$

Consequently, Theorem 1 replaced by the following

Theorem 1'. If $\lambda > -1/6$, then (4) has a continuous solution satisfying (5), (6).

To prove the Theorem, we construct three functions

$$\overline{\Omega}(x,y) \ , \quad \underline{\Omega}(x,y) \ , \quad \omega(x) .$$

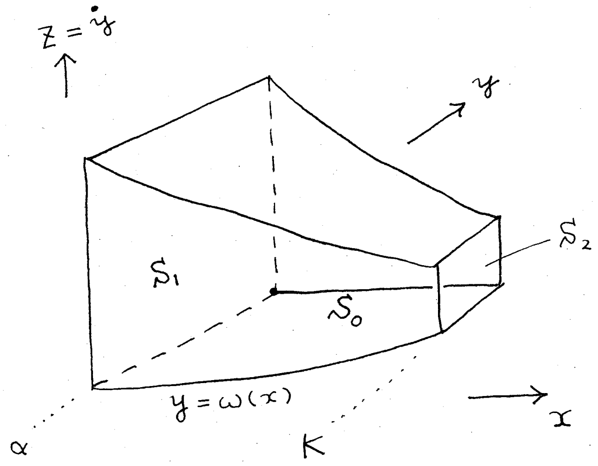
Here $y = \omega(x)$ is a continuous function for $0 \leq x < \infty$, satisfying the conditions : $\omega(0) = 0$, $0 < \omega(x) < 1$ for $0 < x < \infty$. And $\overline{\Omega}(x,y)$, $\underline{\Omega}(x,y)$ are continuous functions for $0 < x < \infty$, $\omega(x) \leq y < 1$, with $\underline{\Omega}(x,1) = 0$, $0 < \underline{\Omega}(x,y) < \overline{\Omega}(x,y)$.

Using these three functions, we define a compact subset D in the (x,y,z)-space as follows

$D: \alpha \leqq x \leqq K$

$\omega(x) \leqq y \leqq 1$

$\underline{\Omega}(x,y) \leqq z \leqq \overline{\Omega}(x,y)$



We divide the boundary ∂D into seven parts S_0 , S_1 ... S_6 .

S_0 is a segment : $\alpha \leqq x < K$, $y = 0$, $z = 1$.

We remark this segment is itself a solution curve of (4) .

S_1 : $x = \alpha$, $\omega(x) < y < 1$, $\underline{\Omega}(x,y) < z < \overline{\Omega}(x,y)$

S_2 : $x = K$, " , " .

Since the x-component of the velocity vector is 1 , any point of S_1 is a strictly ingress point and any point of S_2 is a strictly egress point. We call a point $(\bar{x},\bar{y},\bar{z})$ in ∂D an egress point if the solution curve $(x,y(x),z(x))$ passing through $(\bar{x},\bar{y},\bar{z})$ is in the interior of D for $\bar{x} - \varepsilon \leqq x < \bar{x}$ for some positive ε , if in addition there is a small $\varepsilon > 0$ such that for $\bar{x} < x \leqq \bar{x} + \varepsilon$ the solution curve $(x,y(x),z(x))$ is not in D , the point is called a strictly egress point. The ingress point and strictly ingress point are similarly defined.

And we define

S_3 : $\alpha \leqq x < K$,

$y = 1$,

$\underline{\Omega}(x,y) < z \leqq \overline{\Omega}(x,y)$.

It is easily verified that any point of S_3 is a strictly egress point.

S_4 : $\alpha \leqq x < K$, $\omega(x) \leqq y < 1$, $z = \underline{\Omega}(x,y)$,

S_5 : $\alpha \leqq x < K$, $y = \omega(x)$, $\underline{\Omega}(x,y) < z < \overline{\Omega}(x,y)$,

S_6 : $\alpha \leqq x < K$, $\omega(x) \leqq y < 1$, $z = \overline{\Omega}(x,y)$.

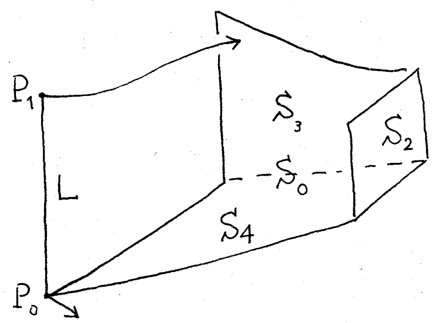
The boundary $\partial D = S_0 + S_1 + \dots + S_6$.

Now we impose S_4 , S_5 , S_6 the following conditions

(E) Any point of S_4 is a strictly egress point ,

(I) Any point of S_5 , S_6 is a strictly ingress point .

Then any point of ∂D is a strictly egress point or a strictly ingress point or in S_0 which is a solution curve contained in ∂D .



We consider solutions starting from a point of a segment

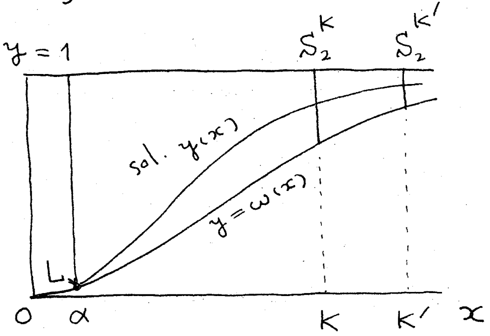
L : $x = \alpha$, $y = \omega(\alpha)$, $\underline{\Omega}(x,y) \leqq z \leqq \overline{\Omega}(x,y)$

contained in ∂D . And we define a map $p : L \to \partial D$ as follows ;

for $(\bar{x},\bar{y},\bar{z})$ in L , $p(\bar{x},\bar{y},\bar{z})$ is the first point (x,y,z) , $x \geqq \bar{x}$ where the solution starting from $(\bar{x},\bar{y},\bar{z})$ meets ∂D_e , the set of all egress points.

By the uniquness property for an initial value problem, the solution curve starting from a point in L cannot meet the set S_0 . In this case $\partial D_e = S_2 + S_3 + S_4$, and any egress point is a strictly egress point. Hence it is easy to prove that the map $p : L \to \partial D_e$ is continuous. If this condition is not satisfied, that is, there is a point which is an egress point but not a strictly egress point, then the map p is not always continuous.

Since L is a connected set, $p(L)$ is also a connected set contained in $S_2 + S_3 + S_4$. For the lowest point P_0 in L, $p(P_0) = P_0$ because P_0 in S_4 is an egress point. And if we construct these walls appropriately it is easily calculated that the solution starting from the highest point P_1 in L meets S_3 first. that is, $p(P_1) \in S_3$.



Consequently $p(L)$, which is contained in $S_2 + S_3 + S_4$, intersects S_3 and S_4. $p(L)$ is a connected set but $S_3 + S_4$ is not connected, hence $p(L)$ intersects S_2. That is, $p^{-1}(S_2)$ is a nonempty compact subset of L.

And we obtain a solution $y(x)$ of (4) for $\alpha \leqq x \leqq K$, with $y(\alpha) = \omega(\alpha)$ and $\Omega(\alpha, \omega(\alpha)) \leqq \dot{y}(\alpha)$, $\omega(x) < y(x) < 1$ for $\alpha < x \leqq K$.

If $K < K'$, a solution starting from a point of L reach to the set $S_2^{K'}$ corresponding to K' must pass through the set S_2^{K} corresponding to K. Therefore $p^{-1}(S_2^{K'}) \subset p^{-1}(S_2^{K})$. Since these sets are compact, there exists at least one point P

$$P \in \bigcap_{K \text{ large}} p^{-1}(S_2^{K})$$

And we obtain a solution $y(x)$ of (4) for $\alpha \leqq x < \infty$.

In this case $y(\alpha) > 0$, $\dot{y}(\alpha) > 0$. But this equation has a sort of monotone property as follows (P.Hartman(1964)).

If for the initial condition $y(\alpha) = \beta$, $\dot{y}(\alpha) = \gamma \geqq 0$, there is a solution of (4) satisfying (5), (6), then for the initial condition $0 \leqq y(\alpha) \leqq \beta$, $0 \leqq \dot{y}(\alpha) \leqq \gamma$, there is a solution of (4) satisfying (5), (6).

In particular there is a solution of (4) with initial conditions

$y(\alpha) = 0 \quad , \quad \dot{y}(\alpha) = 0 \quad$ for any $\alpha > 0$ small .

Using the continuity dependence property for the initial data, we obtain a solution $y(x)$ with initial conditions

$y(0) = 0 \quad , \quad \dot{y}(0) = 0$

for $0 \leqq x < \infty$.

This is the desired solution.

It is remained to construct three functions $\overline{\Omega}(x,y)$, $\underline{\Omega}(x,y)$ $\omega(x)$ satisfying the conditions (E) and (I) .

As sufficient conditions for these conditions, we have following

(E') $\quad \underline{\Omega}_x(x,y) + \underline{\Omega}_y(x,y)\,\underline{\Omega}(x,y) > f(x,y,\underline{\Omega}(x,y))$

for $\quad 0 < x < \infty \quad , \quad \omega(x) \leqq y < 1$

(I') $\quad \underline{\Omega}(x,\omega(x)) > \dot{\omega}(x)$

$\overline{\Omega}_x(x,y) + \overline{\Omega}_y(x,y)\,\overline{\Omega}(x,y) > f(x,y,\overline{\Omega}(x,y))$

for $\quad 0 < x < \infty$.

We can construct $\overline{\Omega}(x,y)$ to satisfy the condition (I') comparably easily. (From now on, we denote $\Omega \equiv \underline{\Omega}$.) Therefore it is essential to construct two functions $\Omega(x,y)$, $\omega(x)$ satisfying the following conditions

(E') $\quad k \equiv \Omega_x + \Omega_y\Omega - f > 0 \quad$ for $\quad 0 < x < \infty \quad , \quad \omega(x) \leqq y < 1$

(I') $\quad \Omega(x,\omega(x)) > \dot{\omega}(x) \quad$ for $\quad 0 < x < \infty$.

We put $\quad \Omega(x,y) = 2xy^{-\frac{1}{2}}(1-y)$.

In solving this problem, this function proposed by N.Kikuchi is essential.

Then we have

$$k = 2y^{-\frac{1}{2}}(1-y)(\ 1 + 2\lambda - x^2 y^{-\frac{3}{2}}(1-y))$$

If we define a continuous function $y = \omega(x)$ implicitly by

$$y = \omega(x) \quad \Longleftrightarrow \quad x^2 = y^{\frac{3}{2}}(1+2\lambda)/(1-y)$$

then $k \geqq 0$ for $0 < x < \infty$, $\omega(x) \leq y < 1$. Thus the condition (E') is satisfied (the equality in $k \geqq 0$ is not essential) .

By differentiating both sides of this relation w.r.t. y , we have

$$2x \frac{dx}{dy} = (1/2+ \lambda)y^{\frac{1}{2}}(3-y)/(1-y)^2 .$$

Then the condition (I') becomes

$$2xy^{-\frac{1}{2}}(1-y) > \frac{dy}{dx} ,$$

$$2x \frac{dx}{dy} > y^{\frac{1}{2}}/(1-y) ,$$

$$(1/2+ \lambda)y^{\frac{1}{2}}(3-y)/(1-y)^2 > y^{\frac{1}{2}}/(1-y) ,$$

and then

$$\lambda > -1/2 + (1-y)/(3-y) = h(y) .$$

Since $\sup_{0<y<1} h(y) = -1/6$, for $\lambda > -1/6$ this inequality holds, so the condition (I') is satisfied.

This completes the proof of Theorem 1 .

2. More precise estimate for λ

In order to have a more precise estimate for λ we shall construct $\Omega(x,y)$ by the following form.

$$\Omega(x,y) = 2xy^{-\frac{1}{2}}(1-y)u(y)$$

And we have obtained an estimate for λ of the following type.

Theprem 2. Let $u(y)$ be a continuous function on $0 \leqq y \leqq 1$ such that the following conditions are satisfied

(i) of class C^1 and piecewise C^2 on $0 < y < 1$

(ii) $1 \leqq u \leqq 2$, $u' \leqq 0$ on $0 < y < 1$

$u(1) = 1$, $\lim_{y \to 1} (1-y)u'(y) = 0$

(iii) $g(y) > 0$ on $0 < y < 1$.

Here $g(y)$ is defined as follows :

$$v(y) = 1 + (1+y)(u-1)/(1-y) - 2yu' \quad (\geqq 1 \text{ from (ii) })$$

$$g(y) = (3-y)/(1-y) - 2y(u'/u + v'/v) .$$

Then for $\lambda > \sup_{0<y<1} (-u/2 + (v-yu')/g)$ there exists a continuous solution for (1) satisfying (2) , (3) .

In this case

$$k = 2y^{-\frac{1}{2}}(1-y)[u + 2\lambda - x^2 y^{-\frac{3}{2}} u(1-y)v] .$$

As a implicit function of $k(x,y) = 0$ we take $y = \omega(x)$. That is , we define $y = \omega(x)$ implicitly by

$$y = \omega(x) \iff x^2 = y^{\frac{3}{2}} (1+2\lambda/u)/(1-y)v .$$

Differentiating this relation we have

$$2x \frac{dx}{dy} = y^{\frac{1}{2}}[(u/2+ \lambda)g + yu'] / (1-y)uv .$$

To satisfy the condition (I') $\Omega(x, \omega(x)) > \dot{\omega}(x)$

$$2xy^{-\frac{1}{2}}(1-y)u > \frac{dy}{dx} \quad (y = \omega(x))$$

$$2x \frac{dx}{dy} > y^{\frac{1}{2}} / (1-y)u$$

$$(u/2 + \lambda) g > v - yu' .$$

From the condition (iii) $g > o$ for $0 < y < 1$

$$\lambda > -u/2 + (v-yu')/g \equiv h(y)$$

By the similar way to the proof of Theorem 1 , we can obtaine a solution of (4) for $\lambda > \sup_{0<y<1} h(y)$

This completes the proof of Theorem 2 .

Using this Theorem, we can obtaine a more precise estimate for λ . If we construct a continuous function $u(y)$ on $0 \leqq y \leqq 1$ satisfying (i) , (ii) , fortunately the condition (iii) is satisfied in most case. Then for

$$\lambda > \sup_{0<y<1} h(y;u)$$

we have a solution of (1) satisfying (2) , (3) .

The function $u(y)$ has characters $u' \leqq 0$, $u(1) = 1$. If we take for example

$$u = 1 + 0.18(1-y)^{\frac{1}{3}}$$

then we have $\sup_{0<y<1} h = -o.1962..$.

And constructing the function $u(y)$ to make

$$\left| h(y;u) - (-0.1988) \right|$$

as small as possible, we can obtain the value $\lambda > -0.1988..$.
This function $u(y)$ was obtained almost by solving an ordinary differential equation

$$h(y;u,u',u'') = -0.1988$$

for an unknown function $u(y)$.

This equation is equivalent to the original one. In fact, if $h(y) \equiv \lambda$, we have $\Omega_x + \Omega_y \Omega(x, \omega(x)) = f(x, \omega(x), \Omega(x, \omega(x)))$

$$\Omega(x, \omega(x)) = \dot{\omega}(x)$$

$$\ddot{\omega} = \Omega_x + \Omega_y \dot{\omega}(x)$$
$$= \Omega_x + \Omega_y \Omega$$
$$= f(x, \omega(x), \Omega(x, \omega(x)))$$
$$= f(x, \omega(x), \dot{\omega}(x)) .$$

This relation shows that the function $\omega(x)$ is a solution of (4) .

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