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NOTES ON THE THEORY OF DOUBLY STOCHASTIC OPERATORS AND REARRANGEMENTS

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1. Introduction.

The purpose of the present paper is to present basic properties and recent results on the theory of doubly stochastic operators and rearrangements. In section 2, we shall refer to doubly stochastic matrices and rearrangements of vectors, and in section 3, to the infinite version of doubly stochastic matrices. And then, in section 4, we shall refer to doubly stochastic operators and rearrangements of functions. For doubly stochastic matrices we can consult an important paper of Mirsky [6]. Also for doubly stochastic operators and rearrangements of functions, we can consult Luxemburg [5] or Chong and Rice [1].

2. Doubly stochastic matrices and rearrangements of vectors.

We shall denote the set of all (n,n) real matrices by $\mathfrak{M}$. V stands for the set of all n-dimensional vectors. For $x = (x_1, \dots, x_n)^t$ we shall denote by $x_1^*, \dots, x_n^*$ the numbers arranged in non-ascending order of magnitude, and let $x^* = (x_1^*, \dots, x_n^*)^t$. $\delta_i(x)$ is the i-th projection. We shall write $x \geqq y$ whenever $\delta_i(x) \geqq \delta_i(y)$ $(i = 1, \dots, n)$. For $A \in \mathfrak{M}$, A^* is its adjoint. We shall write $A = (a_{ij}) \geqq 0$ whenever $a_{ij} \geqq 0$ $(i,j = 1, \dots, n)$. $\mathcal{P}$ stands for the set of all permutation matrices, $\mathcal{Q}$ for the set of all sub-permutation matrices. $\mathcal{R}$ stands for all $(r_{ij}) \in \mathfrak{M}$ such that $r_{ij} = 0$ or 1 $(i,j = 1, \dots, n)$. We shall denote by $x.y$ the multiplication of $x,y \in V$. $1 \in V$ is the unit vector $(1,\ldots,1)^t$. $A \in \mathfrak{M}$ will be called multiplicative whenever $\delta_i(A(x.y)) = \delta_i(Ax).\ \delta_i(Ay)$ $(i = 1,\ldots,n)$. We shall denote by $CO(\mathcal{S})$ the convex closure of $\mathcal{S}$, also let denote $\mathcal{E}(\mathcal{S})$ the set of all extreme points of a convex set $\mathcal{S}$.

DEFINITION 2.1. We shall define the following sets of matrices.

1. $s\text{-}\mathcal{M} = \{A \in \mathfrak{M} : A \geqq 0,\ A1 \leqq 1\}$; sub Markov.
2. $\mathcal{M} = \{A \in \mathfrak{M} : A \geqq 0,\ A1 = 1\}$; Markov.
3. $\mathcal{D} = \{A \in \mathfrak{M} : A \geqq 0,\ A1 = 1,\ A^*1 = 1\}$; doubly stochastic.
4. $\mathcal{S} = \{A \in \mathfrak{M} : A \geqq 0,\ A1 \leqq 1,\ A^*1 \leqq 1\}$; doubly substochastic.

THEOREM 2.2 ([13]). $\mathcal{E}(s\text{-}\mathcal{M}) = \mathcal{R} \cap s\text{-}\mathcal{M}$, $\mathcal{E}(\mathcal{M}) = \mathcal{R} \cap \mathcal{M}$, $\mathcal{E}(\mathcal{D}) = \mathcal{P}$, and $\mathcal{E}(\mathcal{S}) = \mathcal{Q}$. That is, extreme points of the whole set in Definition 2.2 coincide with each multiplicative elements. Moreover, $s\text{-}\mathcal{M} = CO(\mathcal{R} \cap s\text{-}\mathcal{M})$, $\mathcal{M} = CO(\mathcal{R} \cap \mathcal{M})$, $\mathcal{D} = CO(\mathcal{P})$, and $\mathcal{S} = CO(\mathcal{Q})$.

DEFINITION 2.3. Suppose $x,\ y \in V$.

1. $y \prec\!\!\prec x$ whenever $\sum_{i=1}^{k} y_i^* \leqq \sum_{i=1}^{k} x_i^*$ $(k = 1,\ldots,n)$.
2. $y \prec x$ whenever $y \prec\!\!\prec x$ and $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$.

We shall denote by $\mathcal{D}(x)$ (resp. $\mathcal{S}(x)$) the orbit of x by $D \in \mathcal{D}$ (resp. $S \in \mathcal{S}$). The following two theorems are fundamental in the theory of d.s. matrices (See [6]).

THEOREM 2.4. $y \prec x$ iff $y \in CO(Px : P \in \mathcal{P})$ iff $y \in \mathcal{D}(x)$, when $x,y \geqq 0$, $y \prec\!\!\prec x$ iff $y \in CO(Qx : Q \in \mathcal{Q})$ iff $y \in \mathcal{S}(x)$.

THEOREM 2.5.

1. $y \prec x$ iff $\sum_{i=1}^{n} \phi(y_i) \leqq \sum_{i=1}^{n} \phi(x_i)$ for any convex function $\phi : R \longrightarrow R$.
2. $y \prec\!\!\prec x$ iff $\sum_{i=1}^{n} \phi(y_i) \leqq \sum_{i=1}^{n} \phi(x_i)$ for any non-decreasing convex function $\phi : R \rightarrow R$.

3. Infinite doubly stochastic matrices and rearrangements of vectors

In this section, let us use several notations and terminology which appeared in section 2 as their infinite analogies. For instance, $\mathcal{D}$ stands for the set of all infinite d.s. matrices (d_{ij}) $(i, j = 1, 2, \ldots)$. This means that $d_{ij} \geq 0$, $\sum_{j=1}^{\infty} d_{ij} = 1$ and $\sum_{i=1}^{\infty} d_{ij} = 1$ $(i, j = 1, 2, \ldots)$. If $A = (a_{ij}) \in \mathfrak{M}$, $x = (x_1, x_2, \ldots) \in V$, we shall denote by Ax the infinite dimensional vector $(\sum_{j=1}^{\infty} a_{1j}x_j, \sum_{j=1}^{\infty} a_{2j}x_j, \ldots)$ whenever $\sum_{j=1}^{\infty} a_{ij}x_j$ is convergent for $i = 1, 2, \ldots$. Given a topological space $(X, \mathcal{T})$. We shall denote by $\overline{S}^{\mathcal{T}}$ the $\mathcal{T}$-closure of $S \subset X$. $\mathfrak{X}$ stands for the vector space consisting of the boundedly line-summable infinite matrices $A = (a_{ij})$ characterized by $A \in \mathfrak{X}$ if and only if $\|A\| = \max\left\{\sup_i \sum_{j=1}^{\infty} |a_{ij}|, \sup_j \sum_{i=1}^{\infty} |a_{ij}|\right\} < \infty$ (see [3]). The following was established by Kendall [4] .

THEOREM 3.1. If $\mathfrak{X}$ is given the weakest locally convex Hausdoroff topology $\mathcal{T}_K$ which makes all components, row-sums, and column-sums continuous as linear functionals, then $\mathcal{D} = \overline{\mathrm{co}}^{\mathcal{T}_K}(\mathcal{P})$. If the cartesian product $\mathcal{Y}$ of countably infinite sets of real lines is given the topology of pointwise convergence $\mathcal{T}_o$, $\mathcal{S} = \overline{\mathrm{co}}^{\mathcal{T}_o}(\mathcal{P})$. Further more, $\mathcal{E}(\mathcal{D}) = \mathcal{P}$ and $\mathcal{E}(\mathcal{S}) = \mathcal{Q}$.

Let $\mathcal{Z}$ be a vector space of matrices such that $\|A\|_\infty = \sup_i \sum_{j=1}^{\infty} |a_{ij}|$ is finite, for which we give the w*-operator topology. Where, a subbasic neighbourhood of $0 \in \mathfrak{M}$ in this topology is given by $N(f, u, \varepsilon) = \left\{A \in \mathcal{Z} : |\sum_{i=1}^{\infty} f_i \sum_{j=1}^{\infty} a_{ij}u_j| < \varepsilon\right\}$ $f = (f_1, f_2, \ldots) \in \ell^1$, $u = (u_1, u_2, \ldots) \in \ell^\infty$. The following is fundamental (see [13]).

THEOREM 3.2.

1. $\mathcal{E}(s\text{-}\mathcal{M}) = \mathcal{R} \cap s\text{-}\mathcal{M}$, $\mathcal{E}(\mathcal{M}) = \mathcal{R} \cap \mathcal{M}$, $\mathcal{E}(\mathcal{D}) = \mathcal{P}$,

$\mathcal{E}(\mathcal{S}) = \mathcal{Q}$, $\mathcal{E}(\mathcal{S} \cap \mathcal{M}) = \mathcal{Q} \cap \mathcal{M}$. And each set of the above formulas coincides with the multiplicative elements of the set in each bracket.

2. The above set in each bracket is compact in the w*-operator topology, and it is the w*-closure of each convex closure of extrme points except $\mathcal{D}$.

3. $\bar{\mathcal{P}}^{w*} = \mathcal{Q} \cap \mathcal{M}$, $\bar{\mathcal{D}}^{w*} = \mathcal{S} \cap \mathcal{M}$.

For any $x = (x_1, x_2, \ldots) \in \ell^\infty$ we shall define $M_k(x) = \sup x_{i_1} + \ldots + x_{i_k}$ and $m_k(x) = \inf x_{i_1} + \ldots + x_{i_k}$ $(k = 1, 2, \ldots)$, where the upperbound or lower bound are taken with respect to all sets of k distinct positive integers $i_1, \ldots, i_k$. The following are analogies of Theorem 2.5 (see [13]).

THEOREM 3.3.

1. $Sx = y$, $S \in \mathcal{S}$, $x \in \ell^\infty$ implies $M_k(\phi(|x_1|), \phi(|x_2|), \ldots) \geqq M_k(\phi(|y_1|), \phi(|y_2|), \ldots)$ for any non-decreasing convex function $\phi : R \to R$.

2. $Dx = y$, $D \in \mathcal{D}$, $x \in \ell^\infty$ implies $M_k(\phi(x_1), \phi(x_2), \ldots) \geqq M_k(\phi(y_1), \phi(y_2), \ldots)$ for any convex function $\phi : R \to R$.

In particular, $M_k(x) \geqq M_k(y) \geqq m_k(y) \geqq m_k(x)$.

If $x \in \ell^1$, then $\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} y_i$.

As an infinite version of Theorem 2.4, we have the following (see [12]).

THEOREM 3.4. Suppose $0 < x, y \in \ell^1$. Then $y \prec x$ iff $y \in \mathcal{D}(x)$.

Suppose $0 \leqq x, y \in \ell^1$. Then $y \prec\!\!\prec x$ iff $y \in \mathcal{S}(x)$.

4. Doubly stochastic operators and rearrangements of functions.

Let (X,Λ,μ) be a measure space. By $M(X,\Lambda,\mu)$ we shall denote the set of all extended real valued μ-measurable functions on X. Also we shall denote the set of all functions $f \geq 0$, $f \in M(X,\mu)$ by $M^{+}(X,\mu)$. If E is a set, then χ_E will denote the characteristic function of E. Let R be the set of all real numbers and $\bar{R}$ be the set of all extended real numbers. We shall denote by $d_f(t) = \mu\{f > t\}$ $(t \in R)$ the distribution function of $f \in M(X,\mu)$. Suppose (X,Λ,μ) and (X',Λ',μ') are measure spaces such that $\mu(X) = \mu'(X')$ in the sense that both may be infinite. Let $f \in M(X,\mu)$, $f' \in M(X',\mu')$. Then we shall say that f and f' are equimeasurable and write $f \sim f'$ iff $\mu(f^{-1}[J]) = \mu'(f'^{-1}[J])$ for every bounded closed interval J of $\bar{R}$ where J may be the singleton set $\{-\infty\}$ or $\{\infty\}$. If $\mu(X) = \mu'(X') = a < \infty$ and $f \in M(X,\mu)$, $f' \in M(X',\mu')$, $f \sim f'$ is equivalent to $d_f = d_{f'}$. Let $f \in M(X,\mu)$, $\mu(X) = a < \infty$. Then the right inverse of its distribution function d_f will be denoted by f^* and will be called the <u>decreasing rearrangement</u> of f. That is, if $0 \leq s \leq a$, then $f^*(s) = \inf\{t : d_f(t) \leq s\}$. Then f^* is a decreasing right continuous function on $[0,a]$ such that $f^* \sim f$. The next theorem play the important role on the theory of d.s. operators and decreasing rearrangements (see [8, 1]).

THEOREM 4.1. If (X,Λ,μ) is a non-atomic finite measure space and if $f \in M(X,\mu)$ then there exists a measure preserving transformation $\sigma : X \longrightarrow [0, \mu(X)]$ such that $f = f^* \circ \sigma$ μ-a.e.

DEFINITION 4.2. Suppose (X,Λ,μ) and (X',Λ',μ') are finite measure space with $\mu(X) = \mu'(X') = a$, that $f \in L^1(X,\mu)$, $g \in L^1(X',\mu')$. Then, we write

1. $f \ll g$ whenever $\int_0^s f^* dt \leqq \int_0^s g^* dt$ for all $0 \leqq s \leqq a$.

2. $f \prec g$ whenever $f \ll g$ and $\int_0^a f d\mu = \int_0^a g d\mu$.

In this case f is said to be majorized by g.

Evidently $\ll$ and $\prec$ are partially orders for certain elements of $M(X,\mu)$ and $M(X',\mu')$. Here, we list up fundamental properties for these partially orders.

PROPOSITION 4.3.

1. $f \sim g \in L^1$ is equivalent to $f \ll g$ and $g \ll f$ or $f \prec g$ and $g \prec f$.
2. If $f \prec g$, then $rf + s \prec rg + s$ for all $r, s \in R$.
3. If $f \prec g$, then $\|f\|_1 \leqq \|g\|_1$ and $\|f\|_\infty \leqq \|g\|_\infty$.
4. Suppose $f, f_n \in L^1(X,\mu)$, $g, g_n \in L^1(X',\mu')$ and $f_n \prec g_n$ $(n = 1, 2, \ldots)$. If $f_n \longrightarrow f$ and $g_n \longrightarrow g$ in L^1 or L^∞ norms, then $f \prec g$.
5. $f \prec \chi_X$ implies $f = \chi_X$ μ-a.e.
6. $(f_1 + \ldots + f_n) \prec (f_1^* + \ldots + f_n^*)$ for $f_1, \ldots, f_n \in L^1$.
7. If $0 \leqq f_1, \ldots, f_n \in L^1(X,\mu)$, we have $f_1 \ldots f_n \ll f_1^* \ldots f_n^*$.

The following is a simple characterization of $\prec$ (see [14]).

THEOREM 4.4. Suppose $\mu(X) = \mu'(X') < \infty$, that $f \in L^\infty(X)$, $g \in L^\infty(X')$. Then $f \prec g$ iff $\int f^{(\alpha)} d\mu \geqq \int g^{(\alpha)} d\mu'$ and $\int f d\mu = \int g d\mu'$ for all $\alpha \in R$. Where $f^{(\alpha)}$ is the trancation of f at α.

We shall denote by M^+ the set of all $f \in M^+(R)$ such that $d_f(t) < \infty$ for any $t \in R^+ = (0,\infty)$. For any $f \in M^+$, let f^* be defined on R by $f^*(s) = \inf\{t : d_f(t) \leqq s\}$ if $s > 0$, $f^*(s) = 0$ if $s \leqq 0$. Then the symmetrically decreasing rearrangement $\hat{f}$ of $f \in M^+$ is $\hat{f}(s) = f^*(2|s|)$. It is easy to see that the function $\hat{f}$ decreases symmetrically on each side of the origin and satisfies $\hat{f} \sim f$. If $\int_0^s f^* dt \leqq \int_0^s g^* dt$ for any $s \in R^+$ and $\int_{-\infty}^{\infty} f dt = \int_{-\infty}^{\infty} g dt$ in the sense that both may be infinite, we write $f \prec g$. $f \otimes g(s) = \int_{-\infty}^{\infty} f(t)g(s-t)dt$ is the convolution of $f,\ g \in M^+$.

THEOREM 4.5 ([11]). If $f_i \in M^+$ $(i = 1,\dots, n)$ and $g_j \in M^+$ $(j = 1,\dots, m)$ then $f_1 \otimes \dots \otimes f_n \prec \hat{f}_1 \otimes \dots \otimes \hat{f}_n$ and

$$(f_1 + \dots + f_n) \otimes (g_1 + \dots + g_m) \prec (\hat{f}_1 + \dots + \hat{f}_n) \otimes (\hat{g}_1 + \dots + \hat{g}_m).$$

THEOREM 4.6 ([14]). If $f,\ g \in L^1(R) \cap M^+$ then $f \prec g$ iff $\hat{f} \otimes \hat{h} \prec \hat{g} \otimes \hat{h}$ for every $h \in M^+$.

The following is the extension of the notion of doubly stochastic matrices to operators defined on the L^1 space.

DEFINITION 4.7. A linear operator $T : L^1(X', \mu') \longrightarrow L^1(X, \mu)$ is called <u>doubly stochastic</u> (in short d.s.) whenever $Tf \prec f$ for all $f \in L^1(X', \mu')$, where $\mu'(X') = \mu(X) < \infty$.

From now on to Definition 4.20 let us assume that $\mu(X) = \mu'(X') < \infty$. The following is a fundamental theorem in the theory of d.s. operators. It was first established for the $L^1[0, a]$ space by Ryff [8].

THEOREM 4.8 ([2]). Let T be a linear map of the simple functions of $L^1(\mu')$ into $L^1(\mu)$. The following are equivalent :

1. T extends to a d.s. operator on $L^1(\mu')$.
2. $0 \leq T\chi_E \leq \chi_X$ and $\int T\chi_E d\mu = \mu'(E)$ for all $E \in \Lambda'$.
3. There is a linear extension of T to $L^1(\mu')$ such that $Tf \prec f$ for all $f \in L^1(\mu')$.

THEOREM 4.9. A linear operator $T : L^1(X', \mu') \longrightarrow L^1(X, \mu)$ is d.s. iff

1. $T \geq 0$, $T\chi_{X'} = \chi_X$, $T^*\chi_X = \chi_{X'}$.
2. $T\chi_{X'} = \chi_X$, $\| T \|_1 \leq 1$, $T \geq 0$.

We shall denote by $\mathcal{D}(X',X)$ the set of all d.s. operators $T: L^1(X',\mu') \longrightarrow L^1(X,\mu)$. For every $f \in L^1(X',\mu')$ we set $\mathcal{D}_f(X',X) = \{Tf : T \in \mathcal{D}(X',X)\}$. The following theorem is due to Ryff [8] who first established it for the $L^1[0,1]$ space.

THEOREM 4.10. $\mathcal{D}(X',X)$ is convex and compact in the w*-operator topology, when it is regarded as a set of operators acting on $L^\infty(X')$.

THEOREM 4.11 ([2]). If $T \in \mathcal{D}(X',X)$, T^* (acting on L^∞) always admits a unique extension to L^1 operator which belongs to $\mathcal{D}(X,X')$.

By the above theorem, $\mathcal{D}(X,X)$ is a selfadjoint compact convex semigroup. The following extension of Ryff's theorem [8] to the $L^1(X,\mu)$ is due to Day [2].

THEOREM 4.12. Let $f \in L^1(X',\mu')$. Then $\mathcal{D}_f(X',X)$ is w-compact. In addition, if $g \in M(X,\mu)$, then $g \prec f$ iff $g \in \mathcal{D}_f(X',X)$.

The following theorem was originally given for positive functions on (0,1) by Lorentz and Shimogaki [9].

THEOREM 4.13 ([2]). If f_1, $f_2 \in L^1(X',\mu')$ and $g \in M(X,\mu)$ and $g \prec f_1 + f_2$ then there are g_1, $g_2 \in L^1(X,\mu)$ such that $g = g_1 + g_2$ and $g_1 \prec f_1$ and $g_2 \prec f_2$.

For $L^1((0,a))$ space we have the following (see [10]).

THEOREM 4.14.

1. If $f \sim g$ and f and g are simple functions on (0,a), then there exists an invertible measure preserving transformation σ on (0,a) for which $T_\sigma f = f \circ \sigma = g$ holds.

2. Suppose f, $g \in L^1(0,a)$ and $f \sim g$. Then $Tf = g$ holds for a d.s. operator which is a w*-cluster point of a sequence of members

of $\{T_\sigma\}$ where σ is an invertible measure preserving map on $(0,a)$.

For equimeasurability of functions we have the following two theorems which were first established for the $L^1(0,a)$ space by Sakai and Shimogaki [10] . $e(f:\lambda)$ is the set $\{x : f(x) > \lambda\}$ and each function $f \in L^1(X,\mu)$ will be called smooth if $\mu\{x : f(x) = \lambda\} = 0$ for all $\lambda \in R$.

THEOREM 4.15. Suppose $f \in L^1(X',\mu')$ and $g \in L^1(X,\mu)$ and $Tf = g$ for $T \in \mathfrak{D}(X',X)$. Then the following statements are equivalent :

1. $f \sim g$.
2. $Tf^{(\alpha)} = g^{(\alpha)}$ for all $\alpha \in R$.
3. $T\chi_{e(f:\lambda)} = \chi_{e(g:\lambda)}$ for all $\lambda \in R$.
4. $T^*g = f$.

THEOREM 4.16. For every smooth function $f \in L^1(X,\mu)$ there is one and only one d.s. operator T such that $Tf^* = f$. This operator T is induced by some measure preserving transformation. Moreover, $f^* = Sf$, $S \in \mathfrak{D}$ implies $S = T^*$.

The following are characterizations of d.s. operators which are induced by measure preserving transformations (see [10]).

THEOREM 4.17. Let T be a d.s. operator on $L^1(0,a)$. The following statements are equivalent :

1. T is a permutator : $Tf \sim f$ for all $f \in L^1((0,a))$.
2. T is trancation invariant : $Tf^{(\alpha)} = (Tf)^{(\alpha)}$ for all $\alpha \in R$ and all $f \in L^1(0,a)$.
3. T is multiplicative : $T(f.g) = Tf.Tg$ for all $f, g \in L^\infty$.
4. T is an isometry in $L^1(0,a)$.
5. $T^*T = I$.

6. T is induced by a measure preserving transformation. In particular, a d.s. operator T is induced by an invertible measure preserving transformation if and only if $TT^* = T^*T = I$.

THEOREM 4.18 ([16]). If θ and σ are measure preserving transformations on (0,1) with θ invertible, then $T = T_\theta \cdot T_\sigma^* \in \mathcal{E}(\mathcal{D}((0,1),(0,1)))$.

We shall denote by $\mathcal{S}$ the set of all $L^\infty(R^+)$ operators such that (1) $Sf \geq 0$ for $0 \leqq f \in L^\infty(R^+)$, (2) $S1 \leq 1$, and (3) $\int_0^\infty Sfdt \leq \int_0^\infty fdt$ for $0 \leqq f \in L^1(R^+) \cap L^\infty(R^+)$. In particular, let denote by $\mathcal{S}^*$ the set of all $S \in \mathcal{S}$ such that $S1 = 1$. Recently, Sakamaki and Takahashi [15] established the following .

THEOREM 4.19. Suppose $0 \leqq f,\ g \in L^1(R^+)$ and g is decreasing on R^+. If $\int_0^s gdt \leqq \int_0^s fdt$ for all $s \in R^+$ and $\int_0^\infty gdt = \int_0^s fdt$, then there exists $T \in \mathcal{S}$ such that $g = Tf$. Moreover if $f,\ g \in L^\infty(R^+)$, there exists an operator $T \in \mathcal{S}^*$ such that $g = Tf$.

Recently, Takeuchi [17] introduced the notion of $\mathcal{F}$-rearrangement.

DEFINITION 4.20. If a subfamily $\mathcal{F} = \{X_k : k \in \Gamma\}$ of Λ satisfies the following conditions, we shall call $\mathcal{F}$ a stratus. (1) $\Gamma = \{\mu(E) : E \in \Lambda,\ \mu(E) < \infty\}$. (2) $X_0 = \phi$, $\mu(X_k) = k$ $(k \in \Gamma)$. (3) $X = \bigcup_{k \in \Gamma^*} X_k$ and $k \leqq k'$ implies $X_k \subset X_{k'}$. If there exists a measure preserving mapping $m : X \longrightarrow X$ such that $\mu(m^{-1}(E) \triangle E')$ $= 0$ for each pair $E, E' \in \Lambda$ with $\mu(E) = \mu(E')$, then we shall call that (X,Λ,μ) is homogenuous. We shall call (X,Λ,μ) a stratus system whenever (X,Λ,μ) is homogenuous and have a stratus. We shall define $\rho(X) = \sup\{k : x \notin X_k\}$ and $\delta_f(x) = \inf\{t : d_f(t) \leqq \rho(x)\}$

for all $x \in X$. $\Gamma(x)$ will be called the $\mathcal{F}$-distance and $\delta_f(x)$ will be called the $\mathcal{F}$-rearrangement.

DEFINITION 4.21. Suppose (X,Λ,μ) is a stratus system. An operator $T: L^1(X,\mu) \longrightarrow L^1(X,\mu)$ will be called a doubly substochastic Markov operator whenever T satisfies the following conditions.

1. $T \geqq 0$.
2. $\int_{X_k} Tf d\mu \leqq \int_{X_k} f d\mu$ for all $0 \leqq f \in L^1(X,\mu)$ $(k \in \Gamma)$.
3. $T^* \chi_X = \chi_X$.

We shall denote by $\mathcal{S}_m$ the set of all doubly substochastic Markov operators.

THEOREM 4.22 ([17]). Suppose (X,Λ,μ) be a stratus system and $f, g \in L^1(X,\mu)$ satisfies $\delta_f \geqq 0$, $\delta_g \geqq 0$ and $\int_X \delta_f d\mu = \int_X \delta_g d\mu$. Then the following statements are equivalent :

1. $\int_{X_k} \delta_f d\mu \leqq \int_{X_k} \delta_g d\mu$ $(k \in \Gamma)$.
2. $\delta_f = T\delta_g$ for some $T \in \mathcal{S}_m$.

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