<table>
<thead>
<tr>
<th>Title</th>
<th>On Decomposition of Lattice Ideals of a Lattice-Ordered Semigroup (半群論セミナー)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MURATA, KENTARO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1977), 292: 168-175</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1977-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/106174">http://hdl.handle.net/2433/106174</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
ON DECOMPOSITION OF LATTICE IDEALS OF A LATTICE-ORDERED SEMIGROUP

KENTARO MURATA

Our purpose of the present note is to obtain a unique decomposition theorem of lattice ideals of \( l \)-semigroups treated in [2]. The decomposition theorem is a generalization of the unique factorization of elements in the arithmetical \( l \)-groups [7]. Applying our theorem to submodules over a maximal bounded order of a ring, we obtain a decomposition of the modules [5].

1. PRELIMINARIES. Let \( L = (L, \cdot, \leq) \) be a (conditionally) complete \( l \)-semigroup with multiplicative unity \( e \). We assume the following two conditions:

1. \( L \) has a map \( a \mapsto a^{-1} \) into itself with two properties (i) \( aa^{-1}a \leq a \) and (ii) \( axa \leq a \) implies \( a \leq a^{-1} \).

2. \( e \) is maximally integral: \( c^2 \leq c \) and \( e \leq c \) imply \( c = e \).

For any \( a \) of \( L \) we define \( a^* = (a^{-1})^{-1} \), and define \( a^* \circ b^* = (a^*b^*)^* = (ab)^* \) [2]. Then the set \( L^* = \{a^*; a \in L\} \) is a complete \( l \)-group under \( \circ \) and \( \leq \) [3]. Hence the group \( (L^*, \circ) \) is commutative by the well known theorem of \( l \)-groups. If we classify \( L \) by the quasi-equal relation \( a \sim b \) defined by \( a^{-1} = b^{-1} \), then the set \( L/\sim \) of all cosets forms an \( l \)-group canonically and it is isomorphic to \( (L^*, \circ, \leq) \). We now put
the ascending chain condition in the sense of quasi-equality for integral elements of \( L \). Then we can prove that \( p^* = p \) for any prime \( p \) which is not quasi-equal to \( e \) \([2]\). In the following \( \mathcal{P} \) will denote the set of all primes not quasi-equal to \( e \). Then any element \( a \) of \( L \) is factored into a finite number of primes:

\[
a \sim \prod_{p \in \mathcal{P}} y(p, a)
\]

where \( y(p, a) \) is the \( p \)-exponent of \( a \). We have then (1°) \( y(p, a) = 0 \) for all but finite many \( p \in \mathcal{P} \), (2°) \( a \sim b \) if and only if \( y(p, a) = y(p, b) \) for all \( p \in \mathcal{P} \), (3°) \( y(p, a) = y(p, a^*) \), (4°) \( y(p, ab) = y(p, a) + y(p, b) \), (5°) \( y(p, a \cup b) = \min \{ y(p, a), y(p, b) \} \), (6°) \( a \leq b^* \) (i.e. \( a^* \leq b^* \)) if and only if \( y(p, a) \geq y(p, b) \) for all \( p \in \mathcal{P} \).

A lattice ideal (abbr. 1-ideal) \( J \) is called closed if \( a \in J \) implies \( a^* \in J \). Let \( A \) be any non-empty subset of \( L \), and let \( A' \) be the join semi-lattice generated by \( A \). Then the set-theoretical union of all principal closed 1-ideals \( J(a^*)' \)’s generated by \( a \in A' \) is the closed 1-ideal generated by \( A \). Let \( P \) be any subset of \( \mathcal{P} \). If \( P \) is non-void, the closed 1-ideal generated by \( \{ p_1^{-1} \cdots p_n^{-1} ; p_i \in P \} \) is called a \( P \)-component of the cone \( I \) and denoted by \( I_p \). If \( P \) is void, \( I_p \) means \( I \) itself. A \( P \)-component \( J_p \) of the closed 1-ideal \( J \) will be defined to be the closed 1-ideal generated by \( J \cdot I_p = \{ xy ; x \in J, y \in I_p \} \). For convenience the closed 1-ideal generated by the 1-ideal \( J \) will be denoted by \( J^* \). For two 1-ideals \( J_1 \) and \( J_2 \) we define quasi-equal relation by \( J_1 \sim J_2 \iff J_1^* = J_2^* \). \( J_1 \circ J_2 \) means the closed 1-ideal

(2)
generated by \( \{ xy; x \in J_1, y \in J_2 \} \) for any two \( l \)-ideals \( J_1 \) and \( J_2 \). Then the set of all closed \( l \)-ideals \( J = (J, o, \subseteq) \) forms a complete \( l \)-semigroup which contains the \( cl \)-semigroup \( (L^*, o, \subseteq) \) isomorphically. It can be seen that \((J, o)\) is a commutative semigroup.

The set-theoretical union \( Z_{-\infty} \) of the rational integers \( Z \) and the symbol \( -\infty \) is a totally ordered additive semigroup. For any \( l \)-ideal \( J \) of \( L \) we define

\[
\nu(p, J) = \inf \{ \nu(p, a); a \in J \}.
\]

Fixing \( J \) and moving \( p \) over \( \mathbb{P} \), \( \nu(p, J) \) is considered as a map from \( \mathbb{P} \) into \( Z_{-\infty} \). The map is written by \( \nu_J \), that is \( \nu_J(p) = \nu(p, J) \).

Let now \( \sigma \) be a map from \( \mathbb{P} \) into \( Z_{-\infty} \) such that \( \sigma(p) \leq 0 \) for almost all \( p \in \mathbb{P} \), and let \( S \) be the set of all such maps. Then the set \( G \) of all vectors \( [\sigma(p)] \) forms a complete \( l \)-semigroup under the usual addition and the order \( \preceq \) defined by \( [\sigma(p)] \preceq [\sigma'(p)] \iff \sigma(p) \geq \sigma'(p) \) for all \( p \in \mathbb{P} \). In symbol: \( G = (G, +, \preceq) \).

2. LEMMAS AND MAIN RESULTS.

**Lemma 1.** For each \( \sigma \in S \), the set \( K[\sigma] \) of all \( x \in L \) such that \( \nu(p, x) \geq \sigma(p) \) for all \( p \in \mathbb{P} \) forms a closed \( l \)-ideal of \( L \).

**Proof.** This is immediate by (2°), (5°) and (6°) in Section 1.

**Lemma 2.** For each closed \( l \)-ideal \( J \) we have \( K[\nu_J] = J \).

**Proof.** Similarly obtained as the proof of Lemma 3 in [7].

**Lemma 3.** For each \( \sigma \in S \) we have \( \nu_{K[\sigma]} = \sigma \).

**Proof.** Similarly obtained as the proof of Lemma 4 in [7].
By using LEMMAS 2 and 3 we obtain the following

THEOREM 1. The map \( f: J \to f(J) = [\mathcal{V}_J(p)] \) gives an 1-semigroup isomorphism from \((\mathcal{J}, \circ, \subseteq)\) onto \((G, +, \subseteq)\).

Let \( P_+(J), P_0(J), P_-(J) \) and \( P_\infty(J) \) be the sets of primes \( p \) in \( P \) such that \( \mathcal{V}_J(p) \) is positive, zero, negative and \(-\infty\), respectively.

LEMMA 4. Let \( J \) be a closed 1-ideal such that both \( P_+(J) \) and \( P_-(J) \) are void. If \( P_0(J) \) is contained in the set-theoretical union of \( P_0(J(a)) \) and \( P_+(J(a)) \), then \( a \) is contained in \( J \) and conversely.

By using Corollary to Theorem 2.3 in [2] we get the following

LEMMA 5. Let \( J \) be a closed 1-ideal. If \( J \) is multiplicatively closed, the vector \( f(J) \) has no integral coordinate except zero, and vice versa.

LEMMA 6. Let \( J \) be a closed 1-ideal containing the cone \( I \). If \( J \) is closed under multiplication, \( J \) is the \( P_\infty(J) \)-component of \( I \).

THEOREM 2. Any 1-ideal \( J \) of \( L \) is decomposed as follows:

\[
(*) \quad J \sim \prod_{p \in P_+} J(p \mathcal{V}_p) \cdot (\bigvee_{p \in P_0} \mathcal{V}_p) \cdot J_{I_p}.
\]

where \( \mathcal{V}_p = \mathcal{V}_J(p) \), \( P_+ = P_+(J*) \), \( P_- = P_-(J*) \), \( \bigvee' \) denotes a finite join and \( \bigvee \) denotes the set-theoretical union of all \( J(\bigcup' \mathcal{V}_p) \). Conversely, let \( A, B, C \) be any three subsets of \( P \) such that they are disjoint and one of them is finite, e. g. so is \( A \), and let \( \alpha_q, -\beta_q \) be positive and negative integers respectively such that \( \alpha_q \) corresponds \( q \in A \) and \( -\beta_q \) corresponds to \( q \in B \). Then

\[
(**) \quad \prod_{q \in A} J(q^{\alpha_q}) \cdot (\bigvee_{q \in B} J(-q^{\beta_q})) \cdot I_C
\]

(4)
is an 1-ideal of $L$. Moreover if $J$ of (*) is quasi-equal to (**), then $P_+ = A$, $P_- = B$, $P_{-\infty} = C$, $\forall p \in P_+$ $\forall p \in P_-$ by suitable enumeration of $p$; that is, the decomposition (*) is unique within quasi-equality.

Proof. Let $J$ be any 1-ideal of $L$. Firstly we suppose that $J$ is closed. $f(J)$ is represented as $f(J) = u_+(J) + u_-(J) + u_{-\infty}(J)$, where $u_+(J)$, $u_-(J)$, $u_{-\infty}(J)$ are the vectors whose $p$-coordinates are $\forall J(p)$ if $p$ is positive-, negative-, $-\infty$-spots (zero otherwise), respectively. It is clear that $f^{-1}(u_+(J)) = \prod_{p \in P} J(p)^{\lambda^+_p}$. Take any element $a$ of $f^{-1}(u_-(J))$, and let $a^* = p_1^{\lambda_1} \cdots p_n^{\lambda_n}$, $p_i \in P$. If $\lambda_i > 0$ for all $i$, then $a^*$ is integral, hence so is the element $a$. Therefore $a$ is contained in $\bigvee J(\bigcup_{p \in P} J(p))$. If $\lambda_1 < 0$, ..., $\lambda_r < 0$, $\lambda_{r+1} > 0$, ..., $\lambda_n > 0$ for $r$ with $0 < r \leq n$, then we obtain $a \leq a^* \leq p_1^{\lambda_1} \cdots p_r^{\lambda_r} \leq (p_1^{\lambda_1} \cup \cdots \cup p_r^{\lambda_r})^* = p_1^{\lambda_1} \cup \cdots \cup p_r^{\lambda_r}$. This implies $a \in J(p_1^{\lambda_1} \cup \cdots \cup p_r^{\lambda_r})$. Hence $f^{-1}(u_-(J)) \subseteq \bigvee J(\bigcup_{p \in P} J(p))$. The converse inclusion is easy to see. Next, by using LEMMAS 5 and 6 we obtain $f^{-1}(u_{-\infty}(J)) = I_{P_{-\infty}}$. The last part of the theorem can be proved easily.

3. APPLICATION.

1. Let $R$ be a noncommutative ring with a bounded maximal order $O$, and let $L$ be all the non-zero fractional two-sided $O$-ideals (abbr. ideals) in $R$ [4]. $L$ is then a conditionally complete 1-semigroup under module-product and set-inclusion. We assume throughout this paragraph that the ascending chain condition in the sense of
quasi-equality holds for integral ideals [1]. The term submodule
means a two-sided \( \mathcal{O} \)-submodule of \( R \) which contains a regular element
of \( R \). A submodule \( M \) of \( R \) is said to be closed if \( \mathfrak{a} \subseteq M \) implies \( \mathfrak{a}^* = (\mathfrak{a}^{-1})^{-1} \subseteq M \), where \( \mathfrak{a} \) is an ideal and \( \mathfrak{a}^{-1} \) is the inverse of \( \mathfrak{a} \).

The set-theoretical union \( M^* \) of \( \mathfrak{a}^* \) for the ideals \( \mathfrak{a} \) contained in \( M \)
is the closed submodule generated by \( M \). Two submodules \( M_1 \) and \( M_2 \) are
said to be quasi-equal iff \( M_1^* = M_2^* \). In symbol: \( M_1 \sim M_2 \). If we
define \( M_1 M_2 \) of any two closed submodules \( M_1 \) and \( M_2 \) to be the set-
theoretical union of all ideals \( \bigoplus_{i=1}^{n} \mathfrak{a}_i \cup \mathfrak{b}_i \) where \( \mathfrak{a}_i \subseteq M_1 \), \( \mathfrak{b}_i \subseteq M_2 \), then the set \( \mathfrak{m}^* = (\mathfrak{m}^*, \subseteq) \) of all closed submodules of \( R \)
forms a commutative cl-semigroup. If we classify the cl-semigroup
\( \mathfrak{m} \) consisting of all submodules of \( R \) by the quasi-equal relation \( \sim \),
then \( \mathfrak{m}/\sim \), the set of all cosets \( [M_1], [M_2], \ldots \), is a commutative
cl-semigroup which is isomorphic to \( (\mathfrak{m}^*, \subseteq) \), where the product of
two cosets is the coset containing \( (M_1 M_2)^* \) and the order \( \subseteq \) is defined
by \( [M_1] \subseteq [M_2] \iff M_1^* \subseteq M_2^* \). Let \( J \) be any closed 1-ideal of \( \mathcal{L} \).

Then the set-theoretical union \( M(J) \) of all ideals in \( J \) is a closed
submodule of \( R \). Conversely the correction \( J(M) \) of all ideals in the
closed submodule \( M \) is an 1-ideal of \( \mathcal{L} \). Then we have \( J \mapsto M(J) \mapsto
J(M(J)) = J \) and \( M \mapsto J(M) \mapsto M(J(M)) = M \). Let \( (\mathcal{L}^*, \circ, \subseteq) \) be the cl-
semigroup consisting of all closed 1-ideals in \( \mathcal{L} \), where "\( \circ \)" is
defined as in the former section. Then the map \( M \mapsto J(M) \) gives an 1-
semigroup isomorphism from \( (\mathfrak{m}^*, \circ, \subseteq) \) onto \( (\mathcal{L}^*, \circ, \subseteq) \). Under that
isomorphism the cl-group consisting of all ideals corresponds to the
principal lattice ideals. By using THEOREM 2 we obtain:

\[ M \sim \prod_{i=1}^{n-1} \mathfrak{p}_i^{\alpha_i} \times \prod_{\mathfrak{q} \in \mathcal{P}} (\sum_{\mathfrak{x} \in \mathfrak{q}} - \mathfrak{p}) \cdot \mathcal{P}_\mathfrak{p} \]

where \( \mathcal{P}(\mathcal{J}(M)) = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \} \), \( \mathcal{P}_\mathfrak{p} = \mathfrak{p} \setminus \mathfrak{p}_0 \), \( \mathcal{P}_\mathfrak{p} = - \mathfrak{p}_0 \), \( \mathfrak{p}_0 \) is the complement of \( \mathcal{P}_\mathfrak{p}(\mathcal{J}(M)) \) in the set of all prime ideals not quasi-equal to \( \mathcal{J} \), \( \sum' \) denotes the restricted sum, and \( \mathcal{P}_\mathfrak{p} \) is the \( \mathcal{P} \)-component of \( \mathcal{J} \). Moreover the above decomposition is unique within quasi-equality. If in particular \( \mathcal{J} \) is Asano, each (non-zero) submodule of \( R \) is uniquely decomposed (within commutativity) as follows:

\[ M = \prod_{i=1}^{n-1} \mathfrak{p}_i^{\alpha_i} \times \prod_{\mathfrak{q} \in \mathcal{P}} (\sum_{\mathfrak{x} \in \mathfrak{q}} - \mathfrak{p}) \cdot \mathcal{P}_\mathfrak{p} \]

Furthermore the \( \mathcal{P}_1 \)-component \( \mathcal{M}_{\mathcal{P}_1} \) of \( M \) is represented as follows:

\[ \mathcal{M}_{\mathcal{P}_1} = \prod_{i=1}^{n-1} \mathfrak{p}_i^{\alpha_i} \times \prod_{\mathfrak{q} \in \mathcal{P}} (\sum_{\mathfrak{x} \in \mathfrak{q}} - \mathfrak{p}) \cdot \mathcal{P}_\mathfrak{p} \]

where \( \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \} \setminus \mathfrak{p}_1 \) and \( \{ \mathfrak{q}_1 \} = \{ \mathfrak{q}_1 \} \setminus \mathfrak{p}_1 \) (Cf. [4] and [5]).

2. Let \( \mathcal{J} \) be a Dedekind domain with its quotient field \( K \). Then any non-zero \( \mathcal{J} \)-submodule \( M \) of \( K \) can be decomposed as in the case of the former paragraph. By using the decomposition we can prove the following statements.

The map \( \gamma : x \mapsto \gamma(x) \) from a non-zero \( \mathcal{J} \)-submodule \( M_1 \) to a non-zero \( \mathcal{J} \)-submodule \( M_2 \) is an \( \mathcal{J} \)-isomorphism if and only if there exists a non-zero element \( t \) of \( K \) such that \( \gamma(x) = tx \) for all \( x \in M_1 \).

Two non-zero \( \mathcal{J} \)-submodules \( M_1 \) and \( M_2 \) are said to have the same -type iff \( \mathcal{J}_{\mathcal{P}_{\mathcal{J}}(M_1)} = \mathcal{J}_{\mathcal{P}_{\mathcal{J}}(M_2)} \). Then in order that \( M_1 \) and \( M_2 \) have

(7)
the same $-\infty$-type, it is necessary and sufficient that there is an
ideal $\mathfrak{a}$ such that $M_2 = M_1 \mathfrak{a}$. Let $\mathfrak{m}$ be the ideal generated by
all prime ideals in $P_{-\infty}(M)$, and let $\mathfrak{a}$ be an ideal. Then $M$ is $\mathcal{O}$-
isomorphic to $M \mathfrak{a}$ if and only if $\mathfrak{a}$ is represented as $\mathfrak{a} = \mathfrak{m}(a)$
for a non-zero element $a$ of $K$. Any intermediate ring $T$ of $\mathcal{O}$ and $K$
is $P$-component of $\mathcal{O}$, and it is a Dedekind ring. An integral $T$-
ideal $\mathfrak{p}$ of $T$ is prime if and only if $\mathfrak{p} = \mathfrak{p}T$, where $\mathfrak{p}$ is a prime
ideal in $P_0(T)$.

REFERENCES

(1949) 98-134.

Journ. Institute of Polytec., Osaka City Univ. 4 (1953) 9-33.


Acad. 50 (1974) 584-588.

[6] K. Murata, On lattice ideals in a conditionally complete lattice-
oordered semigroup (Forthcoming).

[7] K. Murata, On lattice ideals in arithmetical lattice-ordered groups,