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ON IMPLICATIONAL CLASSES OF STRUCTURES

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In the previous paper [1], we have studied that the least universal Horn class containing a given class K is constructed by taking all isomorphic copies of direct limits of substructures of direct products of structures in K. A universal Horn class may be also called a generalized implicational class. However, this generalized implicational class is restricted to being defined by a set of generalized implicational sentences of finite length.

In this paper, a (generalized) $L(\textcircled{m}, \textcircled{n})$-implicational class will be defined by a set of (generalized) $L(\textcircled{m}, \textcircled{n})$-implicational sentences each of which contains a conjunction of length $< \omega_{\textcircled{n}}$ and a universal quantification over a string of variables of length $< \omega_{\textcircled{m}}$ where $\textcircled{m}, \textcircled{n}$ are infinite cardinals, and $\omega_{\textcircled{m}}, \omega_{\textcircled{n}}$ are the initial ordinals of powers $\textcircled{m}, \textcircled{n}$ respectively. We shall make the similar investigation for a (generalized) $L(\textcircled{m}, \textcircled{n})$-implicational class as in the above for a universal Horn class. The method of this study is analogous to that of the paper [1], but the results are not mere generalizations of the results in [1]. It can be seen from our results, especially from Theorem 1, that the lengths of conjunctions and quantifications are closely connected with direct limits and unions respectively. Theorem 2 is a direct generalization of the above result in [1]. The characterization (iii) of an $L(\textcircled{m}, \textcircled{n})$-implicational class in the final remark appears to make the substance of the first main theorem of the paper [3] clear, with the help of Theorem 2 in [2].

§ 1. Preliminaries.

Let L be a first order language with equality which has a set $\{v_\eta \mid \eta < \omega_{\text{Ⓜ}}\}$ of individual variables, where $\omega_{\text{Ⓜ}}$ is the initial ordinal of an infinite power Ⓜ. All operation and relation symbols are assumed to be finitary. We use $x_0, x_1, \ldots x_\xi, \ldots$ as syntactical variables which vary through the variables v_η, $\eta < \omega_{\text{Ⓜ}}$, and denote by $(x_\xi \mid \xi < \rho)$ a subsequence of the sequence $(v_\eta \mid \eta < \omega_{\text{Ⓜ}})$. A formula Φ of L which contains at most some of x_ξ, $\xi < \rho$, as free variables is denoted by $\Phi(x_\xi \mid \xi < \rho)$, if the variables x_ξ, $\xi < \rho$, need to be indicated. If ρ is finite, $\Phi(x_\xi \mid \xi < \rho)$ may be simply denoted by $\Phi(x_0, \ldots, x_{\rho-1})$. An atomic formula of L means a formula of the form $t_1 = t_2$ or of the form $rt_1 \ldots t_n$, where r is an n-ary relation symbol of L and $t_1, \ldots, t_n$ are terms of L. A structure Ⓐ of the similarity type corresponding to the language L is simply called a structure for L. The domain of Ⓐ is denoted by D[Ⓐ]. Let $\Phi(x_\xi \mid \xi < \rho)$ be a formula of L, and let $(a_\xi \mid \xi < \rho)$ be a ρ-sequence of elements in D[Ⓐ]. Then we write (Ⓐ; $(a_\xi \mid \xi < \rho)$) $\models \Phi(x_\xi \mid \xi < \rho)$, if $(a_\xi \mid \xi < \rho)$ satisfies $\Phi(x_\xi \mid \xi < \rho)$ in Ⓐ when the free variables x_ξ, $\xi < \rho$, are assigned the values a_ξ, $\xi < \rho$, respectively. If ρ is finite, (Ⓐ; $(a_\xi \mid \xi < \rho)$) $\models \Phi(x_\xi \mid \xi < \rho)$ may be denoted by (Ⓐ; $a_0, \ldots, a_{\rho-1}$) $\models \Phi(x_0, \ldots, x_{\rho-1})$.

Let Ⓐ and Ⓑ be structures for a language L. A mapping h of D[Ⓐ] into (or onto) D[Ⓑ] is called an L-homomorphism of Ⓐ into (or onto) Ⓑ, if for any atomic formula $\Theta(x_\xi \mid \xi < \rho)$ of

L and for any ρ-sequence $(a_\xi \mid \xi < \rho)$ of elements in $D[Ⓐ]$, $(Ⓐ; (a_\xi \mid \xi < \rho)) \models \Theta(x_\xi \mid \xi < \rho)$ implies $(Ⓑ; (h(a_\xi) \mid \xi < \rho)) \models \Theta(x_\xi \mid \xi < \rho)$. An L-homomorphism h of Ⓐ onto Ⓑ is called an L-isomorphism of Ⓐ onto Ⓑ, if the mapping h is one-to-one and the inverse mapping h^{-1} is also an L-homomorphism.

Let $(Ⓐ_i \mid i \in I)$ be a family of structures for L. A structure Ⓐ for L is called the direct product of the $Ⓐ_i$, $i \in I$, if the following two conditions hold:

(1) $D[Ⓐ]$ is the Cartesian product $\Pi(D[Ⓐ_i] \mid i \in I)$;

(2) For any atomic formula $\Theta(x_\xi \mid \xi < \rho)$ and any ρ-sequence $(a_\xi \mid \xi < \rho)$ of elements in $D[Ⓐ]$, $(Ⓐ; (a_\xi \mid \xi < \rho)) \models \Theta(x_\xi \mid \xi < \rho)$ holds if and only if $(Ⓐ_i; (a_\xi(i) \mid \xi < \rho)) \models \Theta(x_\xi \mid \xi < \rho)$ holds for all $i \in I$, where $a_\xi(i)$ denotes the i-th component of a_ξ.

The above definition of a direct product is equivalent to the usual definition of a direct product. Hence for any family of structures for L, the direct product of this family exists. From the above definition, the direct product of the empty family of structures for L is a one-element structure for L in which every atomic formula is valid. Such a structure is called an L-trivial structure.

X is called an operator if for every class K of structures for L, $X(K)$ is also a class of structures for L. If X and Y are operators, the operator XY is defined by $XY(K) = X(Y(K))$. The operators I, S, P, and P^* are defined as follows:

$I(K)$: all L-isomorphic copies of structures in K;

$S(K)$: all substructures of structures in K;

$P(K)$: all direct products of non-empty families of structures in K;

P*(K): all direct products of empty or non-empty families of structures in K.

Let E be a set of constant symbols (i.e. nullary operation symbols) not belonging to the language L. Then, a new first order language can be obtained from L by adjoining all the constant symbols e in E, which is denoted by L(E). If L(E) contains at least one constant symbol, then E is said to be L-generative. Now let Ⓐ be a structure for L, and ψ a mapping of E into D[Ⓐ]. Then Ⓐ can be expanded to a structure for L(E) by considering $\psi(e)$ as realizations of e to Ⓐ. Such an expanded structure is denoted by Ⓐ(ψ). An ordered pair (E, Ω) is called an L-defining pair, if E is an L-generative set of constant symbols not belonging to L and Ω is a set of atomic sentences of L(E). For any infinite cardinals ⓟ and ⓠ, an L-defining pair (E, Ω) is called an L(ⓟ, ⓠ)-defining pair if $\overline{\overline{E}}$ < ⓟ and $\overline{\overline{\Omega}}$ < ⓠ, where $\overline{\overline{E}}$ and $\overline{\overline{\Omega}}$ denote the cardinals of E and Ω respectively.

Let K be a class of structures for a language L, and let (E, Ω) be an L-defining pair. Now let Ⓐ be a structure for L, and ψ a mapping of E into D[Ⓐ]. The pair (Ⓐ, ψ) is called a K-model of (E, Ω), if Ⓐ is in K and every atomic sentence in Ω is valid in Ⓐ(ψ). We denote by (E, Ω; K) the class of all K-models of (E, Ω). A K-model of (E, Ω), say (Ⓕ, ϕ), is said to be free (in (E, Ω; K)), if Ⓕ is generated by $\{\phi(e) \mid e \in E\}$ and for any (Ⓐ, ψ) $\in$ (E, Ω; K), there exists an L(E)-homomorphism of Ⓕ(ϕ) into Ⓐ(ψ), i.e. there exists an L-homomorphism of Ⓕ into Ⓐ that maps $\phi(e)$ to $\psi(e)$ for each $e \in E$. We denote by $F(E, \Omega; K)$ the class of all free K-models of (E, Ω). Note that if $(\textcircled{F}, \phi)$ and $(\textcircled{F}', \phi')$ are in $F(E, \Omega; K)$, then $\textcircled{F}(\phi)$ and $\textcircled{F}'(\phi')$ are $L(E)$-isomorphic.

The following criterion for a class K to possess free K-models can be immediately obtained from Theorem 2 in [1]:

CRITERION. *Let K be a class of structures for a language L. Then, in order that for any L-defining pair (E, Ω), $(E, \Omega; K) \neq \emptyset$ implies $F(E, \Omega; K) \neq \emptyset$, it is necessary and sufficient that $S(K) \subseteq I(K)$ and $P(K) \subseteq I(K)$.*

§ 2. The definition of a (generalized) $L(\textcircled{m}, \textcircled{n})$-implicational class and its simple properties.

Let $\textcircled{m}$, $\textcircled{n}$ be any infinite cardinals, and let $\omega_{\textcircled{m}}$, $\omega_{\textcircled{n}}$ be the initial ordinals of powers $\textcircled{m}$, $\textcircled{n}$ respectively. Let L be a first order language with equality which has a set $\{v_\xi \mid \xi < \omega_{\textcircled{m}}\}$ of variables. A new expression ----- which contains a conjunction of length $< \omega_{\textcircled{n}}$ and a quantification over a string of variables of length $< \omega_{\textcircled{m}}$ ----- of the form

(*) $$\forall (x_\xi \mid \xi < \alpha)[\bigwedge(\Theta_\eta \mid \eta < \beta) \rightarrow \Theta]$$

is called a (generalized) $L(\textcircled{m}, \textcircled{n})$-implicational sentence, if $\alpha < \omega_{\textcircled{m}}$, $\beta < \omega_{\textcircled{n}}$, and all Θ_η and Θ are (identically false or) atomic formulas of L which contain at most some of the variables x_ξ, $\xi < \alpha$. Note that every $L(\textcircled{m}, \textcircled{n})$-implicational sentence is a generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentence.

Let $\textcircled{A}$ be a structure for L. The sentence (*) is said to be valid in $\textcircled{A}$, if for any α-sequence $(a_\xi \mid \xi < \alpha)$ of elements in $D[\textcircled{A}]$,

(#) $(\textcircled{A}; (a_\xi \mid \xi < \alpha)) \models \Theta_\eta(x_\xi \mid \xi < \alpha)$ for all $\eta < \beta$ implies $(\textcircled{A}; (a_\xi \mid \xi < \alpha)) \models \Theta(x_\xi \mid \xi < \alpha)$.

Hence, if Θ is an identically false formula, the condition (#) can be replaced by

(##) $(\textcircled{A}; (a_\xi \mid \xi < \alpha)) \models \neg\Theta_\eta(x_\xi \mid \xi < \alpha)$ for some $\eta < \beta$.

Therefore the sentence (*) in this special case may be denoted by

$$\forall(x_\xi \mid \xi < \alpha)[\bigvee(\neg\Theta_\eta \mid \eta < \beta)]$$

which contains a disjunction of length $< \omega_{\textcircled{n}}$. Let Φ be a usual or generalized L(ⓜ, ⓝ)-implicational sentence of L. If Φ is valid in a structure Ⓐ for L, then we write $\textcircled{A} \models \Phi$.

Let Σ be a set of generalized L(ⓜ, ⓝ)-implicational sentences. A structure Ⓐ for L is called a model of Σ, if every sentence in Σ is valid in Ⓐ. The class of all models of Σ is denoted by Σ^*. A class K of structures for L is called a (generalized) L(ⓜ, ⓝ)-implicational class, if $K = \Sigma^*$ for some set Σ of (generalized) L(ⓜ, ⓝ)-implicational sentences. Note that every L(ⓜ, ⓝ)-implicational class is a generalized L(ⓜ, ⓝ)-implicational class.

The following lemmas can be easily obtained from the above definitions:

LEMMA 1. *Let K be a generalized L(ⓜ, ⓝ)-implicational class. Then K is closed under the formation of substructures, i.e. $S(K) \subseteq K$.*

LEMMA 2. *Let K be a (generalized) L(ⓜ, ⓝ)-implicational class. Then K is closed under the formation of direct products of (non-empty) families of structures. That is, $P(K) \subseteq K$ for every generalized L(ⓜ, ⓝ)-implicational class K, especially $P^*(K) \subseteq K$ for every L(ⓜ, ⓝ)-implicational class K.*

Let M be a partially ordered set, and let ⓟ be any infinite cardinal. M is said to be ⓟ-directed if for any subset N of M which satisfies $\overline{\overline{N}} < \textcircled{p}$, there exists a element $\mu \in M$ such that $\nu \leqq \mu$ for all $\nu \in N$. A family $(\textcircled{A}_\mu \mid \mu \in M)$ of structures for L indexed by a set M is said to be $\textcircled{p}$-directed if M is an $\textcircled{p}$-directed partially ordered set and $\textcircled{A}_\mu \subseteq \textcircled{A}_\nu$ whenever $\mu \leqq \nu$. Let $(\textcircled{A}_\mu \mid \mu \in M)$ be a $\textcircled{p}$-directed family of structures for L. A structure $\textcircled{A}$ for L is called a union of $(\textcircled{A}_\mu \mid \mu \in M)$ and denoted by $\bigcup(\textcircled{A}_\mu \mid \mu \in M)$, if $D[\textcircled{A}] = \bigcup(D[\textcircled{A}_\mu] \mid \mu \in M)$ and each $\textcircled{A}_\mu$ is a substructure of $\textcircled{A}$. Let K be a class of structures for L. We denote by $U_{\textcircled{p}}(K)$ the class of all structures that are unions of $\textcircled{p}$-directed families of structures in K.

Now we shall prove the following:

LEMMA 3. *Let K be a generalized $L(\textcircled{m}, \textcircled{n})$-implicational class. Then K is closed under the formation of unions of $\textcircled{m}$-directed families of structures in K, i.e. $U_{\textcircled{m}}(K) \subseteq K$.*

Proof. Let Σ be a set of generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences such that $\Sigma^* = K$. Let $F = (\textcircled{A}_\mu \mid \mu \in M)$ be any $\textcircled{m}$-directed family of structures in Σ^*, and let $\textcircled{A}$ be the union of F. Now let

$$\Phi = \forall(x_\xi \mid \xi < \alpha)[\bigwedge(\Theta_\eta \mid \eta < \beta) \rightarrow \Theta]$$

be any generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentence in Σ, and let $(a_\xi \mid \xi < \alpha)$ be any α-sequence of elements in $D[\textcircled{A}]$.

Now assume that

$$(\textcircled{A};\ (a_\xi \mid \xi < \alpha)) \models \Theta_\eta(x_\xi \mid \xi < \alpha) \quad \text{for all } \eta < \beta.$$

We shall prove that

$$(\textcircled{A};\ (a_\xi \mid \xi < \alpha)) \models \Theta(x_\xi \mid \xi < \alpha).$$

By the definition of a union, there exists a subfamily $(\textcircled{A}_{\mu_\xi} \mid \xi < \alpha)$ of F such that $a_\xi \in D[\textcircled{A}_{\mu_\xi}]$ for each $\xi < \alpha$.

Hence there exists a structure $\textcircled{A}_\mu \in F$ such that $\textcircled{A}_{\mu_\xi} \subseteq \textcircled{A}_\mu$ for all $\xi < \alpha$, because $\alpha < \omega_{\textcircled{m}}$ and F is an $\textcircled{m}$-directed family. Hence

$$(\textcircled{A}_\mu;\ (a_\xi \mid \xi < \alpha)) \models \theta_\eta(x_\xi \mid \xi < \alpha) \quad \text{for all } \eta < \beta,$$

because $\textcircled{A}_\mu \subseteq \textcircled{A}$ and $a_\xi \in D[\textcircled{A}_\mu]$ for all $\xi < \alpha$. Since $\textcircled{A}_\mu \in \Sigma^*$, we have $\textcircled{A}_\mu \models \Phi$. Hence

$$(\textcircled{A}_\mu;\ (a_\xi \mid \xi < \alpha)) \models \theta(x_\xi \mid \xi < \alpha),$$

and hence

$$(\textcircled{A};\ (a_\xi \mid \xi < \alpha)) \models \theta(x_\xi \mid \xi < \alpha).$$

Therefore every generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentence in Σ is valid in $\textcircled{A}$, i.e. $\textcircled{A} \in \Sigma^*$. This completes the proof.

Let $(\textcircled{A}_\mu \mid \mu \in M)$ be a family of structures for L indexed by a directed partially ordered set M, and let $(f_\mu^\nu \mid \mu, \nu \in M$ and $\mu \leqq \nu)$ be a family of L-homomorphisms f_μ^ν of $\textcircled{A}_\mu$ into $\textcircled{A}_\nu$ such that f_μ^μ is the identity mapping for each $\mu \in M$ and $f_\mu^\nu f_\lambda^\mu = f_\lambda^\nu$ whenever $\lambda \leqq \mu \leqq \nu$. Then the system $S = \langle(\textcircled{A}_\mu \mid \mu \in M),\ (f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leqq \nu)\rangle$ is called a direct system. Let $A = \bigcup(D[\textcircled{A}_\mu] \times \{\mu\} \mid \mu \in M)$, and let $\sim$ be the equivalence relation on A defined by

$\langle a, \mu\rangle \sim \langle b, \nu\rangle$ if and only if for some $\lambda \in M$, $f_\mu^\lambda(a) = f_\nu^\lambda(b)$.

Now let $\hat{A}$ be the set of all equivalence classes of A defined by the relation $\sim$. Then a structure $\hat{\textcircled{A}}$ for L is called a direct limit of the direct system S if the following two conditions hold:

(1) $D[\hat{\textcircled{A}}] = \hat{A}$;

(2) For any atomic formula $\theta(x_1, \ldots, x_n)$ of L and for any elements $\hat{a}_1, \ldots, \hat{a}_n$ in $D[\hat{\textcircled{A}}]$, $(\hat{\textcircled{A}};\ \hat{a}_1, \ldots, \hat{a}_n) \models \theta(x_1, \ldots, x_n)$ if and only if there exist

some $\mu \in M$ and some elements $a_1, \ldots, a_n$ in $D[Ⓐ_\mu]$ such that $(Ⓐ_\mu; a_1, \ldots, a_n) \models \Theta(x_1, \ldots, x_n)$ and $\langle a_i, \mu\rangle \in \hat{a}_i$ for each $i = 1, \ldots, n$.

Note that the above definition of a direct limit is equivalent to the usual definition of a direct limit. Hence for any direct system S, there exists the direct limit of S.

Let ⓟ be any infinite cardinal. A direct system $\langle (Ⓐ_\mu \mid \mu \in M), (f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leqq \nu)\rangle$ is called a ⓟ-direct system if the index set M is ⓟ-directed. Let K be a class of structures for L. We denote by $L_{ⓟ}(K)$ the class of all structures that are direct limits of ⓟ-direct systems of structures in K.

LEMMA 4. <u>Let K be a generalized L(ⓜ, ⓝ)-implicational class. Then K is closed under the formation of direct limits of ⓝ-direct systems of structures in K, i.e. $L_{ⓝ}(K) \subseteq K$.</u>

Proof. Let Σ be a set of generalized L(ⓜ, ⓝ)-implicational sentences such that $\Sigma^* = K$. Let

$$S = \langle (Ⓐ_\mu \mid \mu \in M), (f_\mu^\nu \mid \mu, \nu \in M \text{ and } \mu \leqq \nu)\rangle$$

be any ⓝ-direct system of structures in Σ^*, and let $\hat{Ⓐ}$ be the direct limit of S. Now let

$$\Phi = \forall(x_\xi \mid \xi < \alpha)[\bigwedge(\Theta_\eta \mid \eta < \beta) \rightarrow \Theta_\beta]$$

be any generalized L(ⓜ, ⓝ)-implicational sentence in Σ, and let $(\hat{a}_\xi \mid \xi < \alpha)$ be any α-sequence of elements in $D[\hat{Ⓐ}]$.

Now assume that

$$(\hat{Ⓐ}; (\hat{a}_\xi \mid \xi < \alpha)) \models \Theta_\eta(x_\xi \mid \xi < \alpha) \quad \text{for all} \quad \eta < \beta.$$

We shall prove that

$$(\hat{Ⓐ}; (\hat{a}_\xi \mid \xi < \alpha)) \models \Theta_\beta(x_\xi \mid \xi < \alpha).$$

For each $\eta \leqq \beta$, we define X_η as the sequence of ordinals such that $\{x_\xi \mid \xi \in X_\eta\}$ is the set of all variables appearing in Θ_η. Since $\hat{Ⓐ}$ is the direct limit of S, for each $\eta < \beta$, there exist an element $\mu_\eta \in M$ and a sequence $(a_\xi^{\mu_\eta} \mid \xi \in X_\eta)$ of elements in $D[Ⓐ_{\mu_\eta}]$ such that

$$(Ⓐ_{\mu_\eta}; (a_\xi^{\mu_\eta} \mid \xi \in X_\eta)) \models \Theta_\eta(x_\xi \mid \xi \in X_\eta), \text{ and}$$
$$\langle a_\xi^{\mu_\eta}, \mu_\eta \rangle \in \hat{a}_\xi \quad \text{for each} \quad \xi \in X_\eta.$$

Moreover, there exist an element $\mu_\beta \in M$ and a sequence $(a_\xi^{\mu_\beta} \mid \xi \in X_\beta)$ of elements in $D[Ⓐ_{\mu_\beta}]$ such that

$$\langle a_\xi^{\mu_\beta}, \mu_\beta \rangle \in \hat{a}_\xi \quad \text{for each} \quad \xi \in X_\beta.$$

For each $\xi \in \bigcup(X_\eta \mid \eta \leqq \beta)$, we now define Y_ξ as the set of all η such that $X_\eta \ni \xi$. Then for all $\eta \in Y_\xi$, $\langle a_\xi^{\mu_\eta}, \mu_\eta \rangle$ are in $\hat{a}_\xi$. Hence for any pair $\langle \eta, \eta' \rangle \in Y_\xi \times Y_\xi$, there exists an element $\nu_{\eta,\eta'} \in M$ such that

$$f_{\mu_\eta}^{\nu_{\eta,\eta'}}(a_\xi^{\mu_\eta}) = f_{\mu_{\eta'}}^{\nu_{\eta,\eta'}}(a_\xi^{\mu_{\eta'}}).$$

Since $\overline{\overline{Y_\xi \times Y_\xi}} < ⓝ$ and M is ⓝ-directed, there exists an element $\nu_\xi \in M$ such that $\nu_{\eta,\eta'} \leqq \nu_\xi$ for all $\langle \eta, \eta' \rangle \in Y_\xi \times Y_\xi$. Hence all $f_{\mu_\eta}^{\nu_\xi}(a_\xi^{\mu_\eta})$, $\eta \in Y_\xi$, are the same element in $D[Ⓐ_{\nu_\xi}]$. Since each X_η is finite, $\overline{\overline{\bigcup(X_\eta \mid \eta \leqq \beta)}} < ⓝ$. Hence there exists an element $\nu \in M$ such that $\nu_\xi \leqq \nu$ for all $\xi \in \bigcup(X_\eta \mid \eta \leqq \beta)$. And hence for each element $\xi \in \bigcup(X_\eta \mid \eta \leqq \beta)$,

all $f_{\mu_\eta}^{\nu}(a_\xi^{\mu_\eta})$, $\eta \in Y_\xi$, are the same element in $D[Ⓐ_\nu]$.

Therefore for each $\xi \in \bigcup(X_\eta \mid \eta \leqq \beta)$, we can define an element a_ξ in $D[Ⓐ_\nu]$ by

$$a_\xi = f_{\mu_\eta}^{\nu}(a_\xi^{\mu_\eta}) \quad \text{for some} \quad \eta \in Y_\xi.$$

Then we can immediately obtain the following:

$(Ⓐ_\nu; (a_\xi \mid \xi \in \bigcup(X_\eta \mid \eta \leqq \beta))) \models \Theta_\eta(x_\xi \mid \xi \in \bigcup(X_\eta \mid \eta \leqq \beta))^{1)}$ for all $\eta < \beta$, and $\langle a_\xi, \nu \rangle \in \hat{a}_\xi$ for each $\xi \in \bigcup(X_\eta \mid \eta \leqq \beta)$.

Since $Ⓐ_\nu \models \Phi$, we have

$$(\textcircled{A}_\nu;\ (a_\xi \mid \xi \in \bigcup(X_\eta \mid \eta \leq \beta)) \models \theta_\beta(x_\xi \mid \xi \in \bigcup(X_\eta \mid \eta \leq \beta))^{2)}.$$

Hence by the definition of a direct limit, we have

$$(\hat{\textcircled{A}};\ (\hat{a}_\xi \mid \xi < \alpha)) \models \theta_\beta(x_\xi \mid \xi < \alpha),$$

as desired. Hence every generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentence in Σ is valid in $\hat{\textcircled{A}}$, i.e. $\hat{\textcircled{A}} \in \Sigma^*$. This completes the proof.

§ 3. Some lemmas concerning free structures and natural limit structures.

Let K be a class of structures for L, and let (E, Ω) be any L-defining pair. We denote by $M_{\textcircled{p},\textcircled{q}}(E, \Omega)$ the set of all $L(\textcircled{p}, \textcircled{q})$-defining pairs (X, Γ) which satisfy $X \subseteq E$ and $\Gamma \subseteq \Omega$, where $\textcircled{p}$ and $\textcircled{q}$ are infinite cardinals. For (X, Γ), $(Y, \Delta) \in M_{\textcircled{p},\textcircled{q}}(E, \Omega)$, we define $(X, \Gamma) \leq (Y, \Delta)$ as both $X \subseteq Y$ and $\Gamma \subseteq \Delta$. Then $M_{\textcircled{p},\textcircled{q}}(E, \Omega)$ forms a directed partially ordered set. Now assume that for each $(X, \Gamma) \in M_{\textcircled{p},\textcircled{q}}(E, \Omega)$, $F(X, \Gamma; K) \neq \emptyset$, i.e. there exists $(\textcircled{A}_{(X,\Gamma)}, \phi_{(X,\Gamma)})$ in $F(X, \Gamma; K)$. Then, for all (X, Γ), $(Y, \Delta) \in M_{\textcircled{p},\textcircled{q}}(E, \Omega)$ satisfying $(X, \Gamma) \leq (Y, \Delta)$, there exists an $L(X)$-homomorphism $f^{(Y,\Delta)}_{(X,\Gamma)}$ of $\textcircled{A}_{(X,\Gamma)}(\phi_{(X,\Gamma)})$ into $\textcircled{A}_{(Y,\Delta)}(\phi_{(Y,\Delta)})$, i.e. L-homomorphism $f^{(Y,\Delta)}_{(X,\Gamma)}$ of $\textcircled{A}_{(X,\Gamma)}$ into $\textcircled{A}_{(Y,\Delta)}$ which maps $\phi_{(X,\Gamma)}(e)$ to $\phi_{(Y,\Delta)}(e)$ for each $e \in X$. These homomorphisms have the properties that $f^{(X,\Gamma)}_{(X,\Gamma)}$ is the identity mapping and that $f^{(Z,\Lambda)}_{(Y,\Delta)} f^{(Y,\Delta)}_{(X,\Gamma)} = f^{(Z,\Lambda)}_{(X,\Gamma)}$ if $(X, \Gamma) \leq (Y, \Delta) \leq (Z, \Lambda)$. Hence the pair of families $(\textcircled{A}_{(X,\Gamma)} \mid (X, \Gamma) \in M_{\textcircled{p},\textcircled{q}}(E, \Omega))$ and $(f^{(Y,\Delta)}_{(X,\Gamma)} \mid (X, \Gamma), (Y, \Delta) \in M_{\textcircled{p},\textcircled{q}}(E, \Omega)$ and $(X, \Gamma) \leq (Y, \Delta))$ forms a direct system, which

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1) 2) In this expression, $\bigcup(X_\eta \mid \eta \leq \beta)$ denotes the subsequence of $(\xi \mid \xi < \alpha)$ which consists of all ordinals belonging to the set-union $\bigcup(X_\eta \mid \eta \leq \beta)$.

is called a direct system Ⓟ, Ⓠ)-naturally defined by (E, Ω; K). The direct limit of a direct system (Ⓟ, Ⓠ)-naturally defined by (E, Ω; K) is called a (Ⓟ, Ⓠ)-natural limit structure with respect to (E, Ω; K) and denoted by $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)$. Note that $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)$ is unique up to L-isomorphism if it exists. Now we define a mapping ϕ of E into $D[Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)]$ as $\phi(e) = \overline{\langle \phi_{(X,\Gamma)}(e), (X, \Gamma)\rangle}$ for some $(X, \Gamma) \in M_{Ⓟ,Ⓠ}(E, \Omega)$ satisfying $X \ni e$, where $\overline{\langle \phi_{(X,\Gamma)}(e), (X, \Gamma)\rangle}$ denotes the member of $D[Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)]$ that contains $\langle \phi_{(X,\Gamma)}(e), (X, \Gamma)\rangle$. Of course, this is well defined, because if $(X, \Gamma) \leqq (Y, \Delta)$ then

$$\overline{\langle \phi_{(X,\Gamma)}(e), (X, \Gamma)\rangle} = \overline{\langle f^{(Y,\Delta)}_{(X,\Gamma)} \phi_{(X,\Gamma)}(e), (Y, \Delta)\rangle} = \overline{\langle \phi_{(Y,\Delta)}(e), (Y, \Delta)\rangle}.$$

The mapping ϕ defined as above is called a natural interpretation of E to $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)$.

Under the above definitions and notation, we shall prove the

LEMMA 5. <u>Suppose $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)$ is in K. Then $(Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K), \phi)$ is in $F(E, \Omega; K)$.</u>

Proof. It is easily seen that $(Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K), \phi)$ is in $(E, \Omega; K)$ and $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)$ is generated by $\{\phi(e) \mid e \in E\}$. Now let $(Ⓑ, \psi)$ be any member of $(E, \Omega; K)$. We shall prove that there exists an L(E)-homomorphism of $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)(\phi)$ into $Ⓑ(\psi)$.

Let Θ be any atomic sentence of $L(E)$ which is valid in $Ⓛ_{Ⓟ,Ⓠ}(E, \Omega; K)(\phi)$. Then there exists some (X, Γ) in $M_{Ⓟ,Ⓠ}(E, \Omega)$ such that $Ⓐ_{(X,\Gamma)}(\phi_{(X,\Gamma)}) \models \Theta$. Since $(Ⓑ, \psi)$ is in $(E, \Omega; K)$, $(Ⓑ, \psi\lceil X)$[3] is in $(X, \Gamma; K)$. Hence there exists an L(X)-homomorphism of $Ⓐ_{(X,\Gamma)}(\phi_{(X,\Gamma)})$ into $Ⓑ(\psi\lceil X)$, because $(Ⓐ_{(X,\Gamma)}, \phi_{(X,\Gamma)})$ is in $F(X, \Gamma; K)$. Hence we have $Ⓑ(\psi\lceil X) \models \Theta$,

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3) $\psi\lceil X$ denotes the mapping which is the restriction of ψ to X.

and hence $\textcircled{B}(\psi) \models \theta$. Therefore there exists an L(E)-homomorphism of $\textcircled{L}_{\textcircled{p},\textcircled{q}}(E, \Omega; K)(\phi)$ into $\textcircled{B}(\psi)$. Hence $(\textcircled{L}_{\textcircled{p},\textcircled{q}}(E, \Omega; K), \phi)$ is in $F(E, \Omega; K)$. This completes the proof.

LEMMA 6. Let K be a class of structures for L such that for any $L(\textcircled{m}, \textcircled{n})$-defining pair (X, Γ), $(X, \Gamma; K) \neq \emptyset$ implies $F(X, \Gamma; K) \neq \emptyset$. And let Σ be the set of all generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences that are valid in all structures in K. Then the following assertions hold for any $L(\textcircled{m}, \textcircled{n})$-defining pair (E, Ω):

(1) $F(E, \Omega; \Sigma^*) \neq \emptyset$ if and only if $F(E, \Omega; K) \neq \emptyset$.

(2) If $(\textcircled{F}, \phi) \in F(E, \Omega; K)$ and $(\textcircled{G}, \psi) \in F(E, \Omega; \Sigma^*)$, then $\textcircled{F}(\phi)$ and $\textcircled{G}(\psi)$ are L(E)-isomorphic.

Proof. Let $E = \{e_\xi \mid \xi < \alpha\}$ and let $\Omega = \{\theta_\eta(e_\xi \mid \xi < \alpha) \mid \eta < \beta\}$[4], where $\alpha < \omega_{\textcircled{m}}$ and $\beta < \omega_{\textcircled{n}}$.

First we shall prove the assertion (1). Assume that $F(E, \Omega; \Sigma^*) = \emptyset$. Then by Lemmas 1, 2 and the Criterion, $(E, \Omega; \Sigma^*) = \emptyset$. Hence $(E, \Omega; K) = \emptyset$, because $(E, \Omega; K) \subseteq (E, \Omega; \Sigma^*)$. Hence we have $F(E, \Omega; K) = \emptyset$. Conversely assume that $F(E, \Omega; K) = \emptyset$. Then by the assumption of this lemma, $(E, \Omega; K) = \emptyset$. Hence for any $(\textcircled{A}, \theta) \in (E, \emptyset; K)$,

$$\textcircled{A}(\theta) \models \bigvee(\neg\theta_\eta(e_\xi \mid \xi < \alpha) \mid \eta < \beta).$$

And hence for every structure $\textcircled{A}$ in K,

$$\textcircled{A} \models \forall(x_\xi \mid \xi < \alpha)[\bigvee(\neg\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta)].$$

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4) We denote by $\theta_\eta(e_\xi \mid \xi < \alpha)$ the atomic sentence of L(E) which is obtained from an atomic formula $\theta_\eta(x_\xi \mid \xi < \alpha)$ of L by replacing the variables x_ξ by the constant symbols e_ξ respectively. Note that any atomic sentence of L(E) can be written in such a form.

Therefore the generalized L(ⓜ, ⓝ)-implicational sentence

$$\forall(x_\xi \mid \xi < \alpha)[\bigvee(\neg\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta)]$$

belongs to Σ. Hence $(E, \Omega; \Sigma^*) = \emptyset$, and hence $F(E, \Omega; \Sigma^*) = \emptyset$.

Next we shall prove the assertion (2). Assume that (Ⓕ, ϕ) $\in F(E, \Omega; K)$ and (Ⓖ, ψ) $\in F(E, \Omega; \Sigma^*)$. Since (Ⓕ, ϕ) $\in (E, \Omega; K) \subseteq (E, \Omega; \Sigma^*)$, there exists an L(E)-homomorphism h of Ⓖ(ψ) onto Ⓕ(ϕ). Now let $\theta(e_\xi \mid \xi < \alpha)$ be any atomic sentence of L(E) such that Ⓕ(ϕ) $\models \theta(e_\xi \mid \xi < \alpha)$. Then, for any (Ⓐ, θ) $\in (E, \Omega; K)$, we have

$$Ⓐ(\theta) \models \theta(e_\xi \mid \xi < \alpha),$$

because there exists an L(E)-homomorphism of Ⓕ(ϕ) into Ⓐ(θ). Hence for any (Ⓑ, τ) $\in (E, \emptyset; K)$,

$$Ⓑ(\tau) \models \bigwedge(\theta_\eta(e_\xi \mid \xi < \alpha) \mid \eta < \beta) \rightarrow \theta(e_\xi \mid \xi < \alpha).$$

And hence for every Ⓑ $\in K$,

$$Ⓑ \models \forall(x_\xi \mid \xi < \alpha)[\bigwedge(\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta) \rightarrow \theta(x_\xi \mid \xi < \alpha)].$$

Therefore the L(ⓜ, ⓝ)-implicational sentence

$$\forall(x_\xi \mid \xi < \alpha)[\bigwedge(\theta_\eta(x_\xi \mid \xi < \alpha) \mid \eta < \beta) \rightarrow \theta(x_\xi \mid \xi < \alpha)]$$

belongs to Σ. Since Ⓖ $\in \Sigma^*$ and Ⓖ(ψ) $\models \theta_\eta(e_\xi \mid \xi < \alpha)$ for all $\eta < \beta$, we have

$$Ⓖ(\psi) \models \theta(e_\xi \mid \xi < \alpha).$$

Hence the L(E)-homomorphism h of Ⓖ(ψ) onto Ⓕ(ϕ) is an L(E)-isomorphism. This completes the proof.

The following lemma can be easily obtained from the above lemma and the definition of an (ⓜ, ⓝ)-natural limit structure.

LEMMA 7. <u>Let K and Σ be the same as in Lemma 6. Then the following assertions hold for any L-defining pair (E, Ω):</u>

(1) <u>$Ⓛ_{ⓜ,ⓝ}(E, \Omega; K)$ exists if and only if $Ⓛ_{ⓜ,ⓝ}(E, \Omega; \Sigma^*)$ exists.</u>

(2) $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E, \Omega; K)$ and $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E, \Omega; \Sigma^*)$ are L-isomorphic if both exist.

§ 4. Main theorems.

Throughout this section, we assume that L is a first order language with equality and with an infinite set $\{v_\xi \mid \xi < \omega_{\textcircled{m}}\}$ of $\textcircled{m}$ variables as in the preceding sections.

THEOREM 1. Assume that $\textcircled{m}$ and $\textcircled{n}$ are regular infinite cardinals, and let K be any class of structures for L. Then $U_{\textcircled{m}} IL_{\textcircled{n}} SP(K)$ is the least generalized $L(\textcircled{m}, \textcircled{n})$-implicational class containing K. That is, if Σ is the set of all generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences that are valid in all structures in K, then

$$\Sigma^* = U_{\textcircled{m}} IL_{\textcircled{n}} SP(K).$$

Proof. By Lemmas 1, 2, 3, and 4, it is clear that

$$\Sigma^* \supseteq U_{\textcircled{m}} IL_{\textcircled{n}} SP(K).$$

We shall prove that

$$\Sigma^* \subseteq U_{\textcircled{m}} IL_{\textcircled{n}} SP(K).$$

Assume that $\textcircled{A}$ is any structure in Σ^*. Now let M be the set of all non-empty subsets of $D[\textcircled{A}]$ whose cardinals are less than $\textcircled{m}$. Since $\textcircled{m}$ is regular, M forms an $\textcircled{m}$-directed partially ordered set under the inclusion relation. For each $\mu \in M$, let $\textcircled{A}_\mu$ be the substructure of $\textcircled{A}$ generated by μ. Then $(\textcircled{A}_\mu \mid \mu \in M)$ forms an $\textcircled{m}$-directed family of structures, and clearly

$$\textcircled{A} = \bigcup(\textcircled{A}_\mu \mid \mu \in M).$$

Hence, in order to prove $\Sigma^* \subseteq U_{\textcircled{m}} IL_{\textcircled{n}} SP(K)$, it suffices to prove that each $\textcircled{A}_\mu$ is in $IL_{\textcircled{n}} SP(K)$.

By Lemma 1, each $\textcircled{A}_\mu$ is in Σ^*. Therefore we have

$$(\textcircled{A}_\mu, \psi_\mu) \in F(E_\mu, \Omega_\mu; \Sigma^*),$$

where $\overline{\overline{E}}_\mu = \overline{\overline{\mu}}$, ψ_μ is a one-to-one mapping of E_μ onto μ, and Ω_μ is the set of all atomic sentences of $L(E_\mu)$ which are valid in $\textcircled{A}_\mu(\psi_\mu)$. Hence for any $L(\textcircled{m}, \textcircled{n})$-defining pair $(X, \Gamma) \in M_{m,n}(E_\mu, \Omega_\mu)$, $(\textcircled{A}_\mu, \psi_\mu \lceil X)$ is in $(X, \Gamma; \Sigma^*)$, and hence $F(X, \Gamma; \Sigma^*) \neq \emptyset$ follows from Lemmas 1, 2 and the Criterion. Therefore there exists a direct system $(\textcircled{m}, \textcircled{n})$-naturally defined by $(E_\mu, \Omega_\mu; \Sigma^*)$, which is an $\textcircled{n}$-direct system consisting of structures in Σ^*, because $\overline{\overline{E}}_\mu < \textcircled{m}$ and $\textcircled{n}$ is regular. Hence the $(\textcircled{m}, \textcircled{n})$-natural limit structure $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; \Sigma^*)$ exists, and by Lemma 4, it is in Σ^*. Therefore by Lemma 5, we have

$$(\textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; \Sigma^*), \phi_\mu) \in F(E_\mu, \Omega_\mu; \Sigma^*),$$

where ϕ_μ is the natural interpretation of E_μ to $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; \Sigma^*)$. Hence we have that $\textcircled{A}_\mu$ and $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; \Sigma^*)$ are L-isomorphic, that is,

$$\textcircled{A}_\mu \cong_L \textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; \Sigma^*).$$

Since $SSP(K) \subseteq ISP(K)$ and $PSP(K) \subseteq ISP(K)$, it follows from the Criterion that for any $L(\textcircled{m}, \textcircled{n})$-defining pair (Y, Δ), $(Y, \Delta; SP(K)) \neq \emptyset$ implies $F(Y, \Delta; SP(K)) \neq \emptyset$. Moreover Σ can be considered as the set of all generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences that are valid in all structures in $SP(K)$, because $K \subseteq SP(K) \subseteq \Sigma^*$ follows from Lemmas 1 and 2. Hence by Lemma 7, $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; SP(K))$ exists, and it is L-isomorphic to $\textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; \Sigma^*)$. Therefore we have

$$\textcircled{A}_\mu \cong_L \textcircled{L}_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu; SP(K)).$$

Since $M_{\textcircled{m},\textcircled{n}}(E_\mu, \Omega_\mu)$ is $\textcircled{n}$-directed, we have that each $\textcircled{A}_\mu$ is L-isomorphic to a direct limit of an $\textcircled{n}$-direct system consisting of structures in $SP(K)$, i.e. $\textcircled{A}_\mu \in IL_{\textcircled{n}}SP(K)$, as desired. This completes the proof.

THEOREM 2. *Assume that $\textcircled{n}$ is a regular infinite cardinal not greater than the cardinal $\textcircled{m}$, and let K be any class of structures for L. Then $IL_{\textcircled{n}}SP(K)$ is the least generalized $L(\textcircled{m}, \textcircled{n})$-implicational class containing K. That is, if Σ is the set of all generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences that are valid in all structures in K, then*

$$\Sigma^* = IL_{\textcircled{n}}SP(K).$$

Note that if $\textcircled{m} \geqq \textcircled{n}$, then every generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentence is equivalent to a generalized $L(\textcircled{n}, \textcircled{n})$-implicational sentence.

Proof. By Lemmas 1, 2, and 4, it is clear that

$$\Sigma^* \supseteq IL_{\textcircled{n}}SP(K).$$

We shall prove that

$$\Sigma^* \subseteq IL_{\textcircled{n}}SP(K).$$

Assume that $\textcircled{A}$ is any structure in Σ^*. Then we have

$$(\textcircled{A}, \psi) \in F(E, \Omega; \Sigma^*),$$

where $\overline{\overline{E}} = \overline{\overline{D[\textcircled{A}]}}$, ψ is a one-to-one mapping of E onto $D[\textcircled{A}]$, and Ω is the set of all atomic sentences of $L(E)$ that are valid in $\textcircled{A}(\psi)$. Hence for any $L(\textcircled{n}, \textcircled{n})$-defining pair $(X, \Gamma) \in M_{\textcircled{n},\textcircled{n}}(E, \Omega)$, $(\textcircled{A}, \psi\lceil X)$ is in $(X, \Gamma; \Sigma^*)$, and hence $F(X, \Gamma; \Sigma^*) \neq \emptyset$ follows from Lemmas 1, 2, and the Criterion. Therefore there exists a direct system $(\textcircled{n}, \textcircled{n})$-naturally defined by $(E, \Omega; \Sigma^*)$, which is an $\textcircled{n}$-direct system consisting of structures in Σ^*, because $\textcircled{n}$ is regular. Hence the $(\textcircled{n}, \textcircled{n})$-natural limit structure $\textcircled{L}_{\textcircled{n},\textcircled{n}}(E, \Omega; \Sigma^*)$ exists, and by Lemma 4 it is in Σ^*. Therefore by Lemma 5, we have

$$(\textcircled{L}_{\textcircled{n},\textcircled{n}}(E, \Omega; \Sigma^*), \phi) \in F(E, \Omega; \Sigma^*),$$

where ϕ is the natural interpretation of E to $\textcircled{L}_{\textcircled{n},\textcircled{n}}(E, \Omega; \Sigma^*)$.

Hence we have

$$Ⓐ \cong_L Ⓛ_{ⓝ,ⓝ}(E, \Omega; \Sigma^*).$$

On the other hand, for any L(ⓝ, ⓝ)-defining pair (Y, Δ), $(Y, \Delta; SP(K)) \neq \emptyset$ implies $F(Y, \Delta; SP(K)) \neq \emptyset$. Moreover Σ^* can be considered as the class defined by the set of all generalized L(ⓝ, ⓝ)-implicational sentences that hold in SP(K). Hence by Lemma 7, $Ⓛ_{ⓝ,ⓝ}(E, \Omega; SP(K))$ exists and it is L-isomorphic to $Ⓛ_{ⓝ,ⓝ}(E, \Omega; \Sigma^*)$. Therefore we have

$$Ⓐ \cong_L Ⓛ_{ⓝ,ⓝ}(E, \Omega; SP(K)).$$

This implies that $Ⓐ \in IL_{ⓝ}SP(K)$, because $M_{ⓝ,ⓝ}(E, \Omega)$ is ⓝ-directed. Therefore we have $\Sigma^* \subseteq IL_{ⓝ}SP(K)$. This completes the proof.

We denote by $A(L)$ the set of all atomic formulas of the language L.

THEOREM 3. <u>Assume that the infinite cardinal ⓜ is regular and ⓝ is any cardinal $> \overline{\overline{A(L)}}$, and let K be any class of structures for L. Then $U_{ⓜ}ISP(K)$ is the least generalized L(ⓜ, ⓝ)-implicational class containing K. That is, if Σ is the set of all generalized L(ⓜ, ⓝ)-implicational sentences that are valid in all structures in K, then</u>

$$\underline{\Sigma^* = U_{ⓜ}ISP(K).}$$

Proof. By Lemmas 1, 2, and 3, it is clear that

$$\Sigma^* \supseteq U_{ⓜ}ISP(K).$$

We shall prove that

$$\Sigma^* \subseteq U_{ⓜ}ISP(K).$$

Assume that Ⓐ is any structure in Σ^*. Now let M be the set of all non-empty subsets of D[Ⓐ] whose cardinals are less than ⓜ. Then M forms an ⓜ-directed partially ordered set under the inclusion relation, because $\textcircled{m}$ is regular. For each $\mu \in M$, let $\textcircled{A}_\mu$ be the substructure of $\textcircled{A}$ generated by μ. Then $(\textcircled{A}_\mu \mid \mu \in M)$ forms an $\textcircled{m}$-directed family of structures, and clearly

$$\textcircled{A} = \bigcup(\textcircled{A}_\mu \mid \mu \in M).$$

By Lemma 1, each $\textcircled{A}_\mu$ is in Σ^*. Hence we have

$$(\textcircled{A}_\mu, \psi_\mu) \in F(E_\mu, \Omega_\mu; \Sigma^*),$$

where $\overline{\overline{E}}_\mu = \overline{\overline{\mu}}$, ψ_μ is a one-to-one mapping of E_μ onto μ, and Ω_μ is the set of all atomic sentences of $L(E_\mu)$ that are valid in $\textcircled{A}_\mu(\psi_\mu)$. On the other hand, for any $L(\textcircled{m}, \textcircled{n})$-defining pair (Y, Δ), $(Y, \Delta; SP(K)) \neq \emptyset$ implies $F(Y, \Delta; SP(K)) \neq \emptyset$. Moreover Σ can be considered as the set of all generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences that hold in $SP(K)$. Hence by (1) of Lemma 6, we have $F(E_\mu, \Omega_\mu; SP(K)) \neq \emptyset$, because $F(E_\mu, \Omega_\mu; \Sigma^*) \neq \emptyset$ and (E_μ, Ω_μ) is an $L(\textcircled{m}, \textcircled{n})$-defining pair. Now take

$$(\textcircled{B}_\mu, \phi_\mu) \in F(E_\mu, \Omega_\mu; SP(K)).$$

Then by (2) of Lemma 6, $\textcircled{A}_\mu(\psi_\mu)$ and $\textcircled{B}_\mu(\phi_\mu)$ are $L(E_\mu)$-isomorphic. Hence $\textcircled{A}_\mu \in ISP(K)$, and hence $\textcircled{A} \in U_{\textcircled{m}}ISP(K)$. Therefore we have $\Sigma^* \subseteq U_{\textcircled{m}}ISP(K)$. This completes the proof.

As immediate consequences of Theorems 1, 2, and 3, we have the following characterizations of generalized $L(\textcircled{m}, \textcircled{n})$-implicational classes respectively:

COROLLARY 1. *Assume that $\textcircled{m}$ and $\textcircled{n}$ are regular infinite cardinals. Then, a class K of structures for L is a generalized $L(\textcircled{m}, \textcircled{n})$-implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P(K) \subseteq K$, $U_{\textcircled{m}}(K) \subseteq K$, and $L_{\textcircled{n}}(K) \subseteq K$.*

COROLLARY 2. *Assume that $\textcircled{n}$ is a regular infinite cardinal* not greater than the cardinal $\textcircled{m}$. Then, a class K of structures for L is a generalized $L(\textcircled{m}, \textcircled{n})$-implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P(K) \subseteq K$, and $L_{\textcircled{n}}(K) \subseteq K$.

COROLLARY 3. Assume that the infinite cardinal $\textcircled{m}$ is regular and $\textcircled{n}$ is any cardinal $> \overline{\overline{A(L)}}$. Then, a class K of structures for L is a generalized $L(\textcircled{m}, \textcircled{n})$-implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P(K) \subseteq K$, and $U_{\textcircled{m}}(K) \subseteq K$.

Remarks on $L(\textcircled{m}, \textcircled{n})$-implicational classes. From Theorem 1, we can easily obtain the following analogous theorem for $L(\textcircled{m}, \textcircled{n})$-implicational classes:

(I) Assume that $\textcircled{m}$ and $\textcircled{n}$ are regular infinite cardinals, and let K be any class of structures for L. Then $U_{\textcircled{m}}IL_{\textcircled{n}}SP^*(K)$ is the least $L(\textcircled{m}, \textcircled{n})$-implicational class containing K.

We simply expain this fact. Let Σ be the set of all $L(\textcircled{m}, \textcircled{n})$-implicational sentences valid in all structures in K, and let Γ be the set of all generalized $L(\textcircled{m}, \textcircled{n})$-implicational sentences valid in all structures in $K \cup \{\textcircled{E}\}$, where $\textcircled{E}$ is a L-trivial structure. Then it is clear that $\Sigma^* = \Gamma^*$ and $IP^*(K) = IP(K \cup \{\textcircled{E}\})$. Hence by Theorem 1, we have

$$\begin{aligned} U_{\textcircled{m}}IL_{\textcircled{n}}SP^*(K) &= U_{\textcircled{m}}IL_{\textcircled{n}}SIP^*(K) \\ &= U_{\textcircled{m}}IL_{\textcircled{n}}SIP(K \cup \{\textcircled{E}\}) = U_{\textcircled{m}}IL_{\textcircled{n}}SP(K \cup \{\textcircled{E}\}) = \Gamma^* = \Sigma^*. \end{aligned}$$

Hence $U_{\textcircled{m}}IL_{\textcircled{n}}SP^*(K)$ is the least $L(\textcircled{m}, \textcircled{n})$-implicational class containing K.

By the similar method as in the above, we can obtain the following theorems (II) and (III) analogous to Theorems 2 and 3 respectively.

(II) Assume that $\textcircled{n}$ is a regular infinite cardinal not greater than the cardinal $\textcircled{m}$, and let K be any class of structures for L. Then $IL_{\textcircled{n}}SP^*(K)$ is the least $L(\textcircled{m}, \textcircled{n})$-implicational class containing K.

(III) Assume that the infinite cardinal $\textcircled{m}$ is regular and $\textcircled{n}$ is any cardinal $> \overline{\overline{A(L)}}$, and let K be any class of structures for L. Then $U_{\textcircled{m}}ISP^*(K)$ is the least $L(\textcircled{m}, \textcircled{n})$-implicational class containing K.

The following characterizations of $L(\textcircled{m}, \textcircled{n})$-implicational classes are immediately obtained from the theorems (I), (II), and (III) respectively.

(i) Assume that $\textcircled{m}$ and $\textcircled{n}$ are regular infinite cardinals. Then, a class K of structures for L is an $L(\textcircled{m}, \textcircled{n})$-implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P^*(K) \subseteq K$, $U_{\textcircled{m}}(K) \subseteq K$ and $L_{\textcircled{n}}(K) \subseteq K$.

(ii) Assume that $\textcircled{n}$ is a regular infinite cardinal not greater than the cardinal $\textcircled{m}$. Then, a class K of structures for L is an $L(\textcircled{m}, \textcircled{n})$-implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P^*(K) \subseteq K$, and $L_{\textcircled{n}}(K) \subseteq K$.

(iii) Assume that the infinite cardinal $\textcircled{m}$ is regular and $\textcircled{n}$ is any cardinal $> \overline{\overline{A(L)}}$. Then, a class K of structures for L is an $L(\textcircled{m}, \textcircled{n})$-implicational class if and only if $I(K) \subseteq K$, $S(K) \subseteq K$, $P^*(K) \subseteq K$, and $U_{\textcircled{m}}(K) \subseteq K$.

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