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TOPICS ON OPEN PRIME IDEALS IN COMPACT SEMIGROUPS

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By a semigroup S we mean an abstract semigroup which is also a Hausdorff space, such that ab is a continuous function of a and b, a, b being elements of S. An ideal I of S is said to be

a completely prime ideal if $xy \in I$ implies $x \in I$ or $y \in I$,

a prime ideal if $xSy \subset I$ implies $x \in I$ or $y \in I$,

a completely semiprime ideal if $x^2 \in I$ implies $x \in I$,

and

a semiprime ideal if $xSx \subset I$ implies $x \in I$.

In the theory of compact semigroups, the ideals which are open completely prime, open prime, open completely semiprime or open semiprime, play an important role at various points, and these ideals have been treated by several authors ([2], [3], [4], [5], [6], [7] and [8]). In this note, we shall give two topics concerning these ideals in a compact semigroup.

Throughout the note we shall use the notation and terminology of A. B. Paalman-de Miranda [5].

§1. We list here some definitions and known results which will be used later.

Definition 1.1. Let I be an ideal of a semigroup S. An idempotent e of S is said to be I-primitive if $e \notin I$ and e is the only idempotent in $eSe \setminus I$.

Definition 1.2. An ideal Q of a semigroup S is said to be a q-ideal if Q is expressed as an intersection of open prime ideals of S.

Proposition 1.3 ([3; Theorem 2]). Let S be a compact semigroup. If P is a proper open prime ideal of S, then P has the form $P = J_0(S \setminus e)$ for some idempotent e. Conversely, for each idempotent e, $J_0(S \setminus e)$ is either an open prime ideal of S or the empty set.

Proposition 1.4 ([3; Lemma 9]). Let S be a semigroup and a, $b \in S$. Then $J_0(S \setminus a) \subset J_0(S \setminus b)$ iff $J(a) \subset J(b)$.

Proposition 1.5 ([4; Theorem 2.9]). Let S be a compact semigroup and let Q be a q-ideal of S. If e is an idempotent of S, then all of the following conditions are equivalent :

(i) e is a Q-primitive idempotent.

(ii) SeS is a minimal ideal not contained in Q.

(iii) Every idempotent in $SeS \setminus Q$ is a Q-primitive idempotent.

(Let A and B be ideals of a semigroup S. A is said to be a minimal ideal not contained in B, if A is a minimal member among the ideals of S which are not contained in B.)

Proposition 1.6 ([4; Theorem 2.7 and Lemma 3.10]). Let Q be an open semiprime ideal of a compact semigroup S. Then Q is a q-ideal. Moreover, if M is an ideal of S which is not contained in Q then M has a Q-primitive idempotent.

§2. In what follows, S will denote a compact semigroup.

An ideal of a semigroup which is completely prime is prime, but the converse is not true. In the case of normal semigroups (a semigroup S is said to be normal if $xS = Sx$ for every $x \in S$), however, these concepts coincide.

We are naturally led to the problem : <u>Is there any useful condition under which open prime ideals of</u> S <u>are completely prime</u> ? In this section we shall discuss about this problem.

<u>Lemma</u> 2.1. Let P be an open prime ideal of S, and let e be an idempotent of S. Then e is P-primitive iff $P = J_o(S \setminus e)$.

<u>Proof</u>. Assume that e is a P-primitive idempotent. Let f be an idempotent of S such that $P = J_o(S \setminus f)$. Since $e \notin P$, we have $J_o(S \setminus f) \subset J_o(S \setminus e)$. From this it follows that $SfS \subset SeS$ (see Proposition 1.4). As SeS is a minimal ideal not contained in P (see Proposition 1.5), $f \notin P$ and $SfS \subset SeS$ imply that $SfS = SeS$. Using Proposition 1.4 again, we can conclude that $J_o(S \setminus f) = J_o(S \setminus e)$.

Conversely, assume that $P = J_o(S \setminus e)$. It is clear that every ideal M of S which is properly contained in SeS does not contain e, and therefore $M \subset P$. Hence SeS is a minimal ideal not contained in P. Using Proposition 1.5, we obtain that e is a P-primitive idempotent.

<u>Lemma</u> 2.2. Let Q be a q-ideal of S and e a Q-primitive idempotent. Then $Q \cap SeS$ is a unique maximal ideal of the semigroup SeS.

<u>Proof</u>. Let us set $Q^* = Q \cap SeS$. It is obvious that Q^* is a proper ideal of the semigroup SeS. Assume that M^* is an ideal of the semigroup SeS such that $M^* \not\subset Q^*$. We shall show that $M^* = SeS$.

Let M denote the set $M^* \cup Q^*$; M is an ideal of the semigroup

SeS which properly contains Q^*. The set SMS is an ideal of the semigroup S which is contained in SeS. Since SeS is a minimal ideal not contained in Q, it follows that either $SMS \subset Q$ or $SMS = SeS$. In the former case, we have $SMS \subset Q^* \subset M$, and so M is an ideal of S. As Q is a semiprime ideal of S, $M^3 \subset SMS \subset Q$ implies that $M \subset Q$. This contradicts to the fact that Q^* is properly contained in M. In the latter case, we have

$$M \supset (SeS)M(SeS) = (Se)(SMS)(eS) = (Se)(SeS)(eS) \supset SeS,$$

and therefore $M = SeS$. From this it follows that $e \in M^*$, because $e \notin Q^*$. Hence we obtain

$$M^* \supset (SeS)e(SeS) \supset SeS.$$

Thus $M^* = SeS$.

<u>Lemma</u> 2.3. Let Q be a q-ideal of S, and let e be a Q-primitive idempotent. If P is the open prime ideal such that $P = J_o(S \setminus e)$, then $P \cap SeS = Q \cap SeS$.

<u>Proof</u>. By Lemma 2.1, e is a P-primitive idempotent. Therefore, by Lemma 2.2, $P \cap SeS$ is a unique maximal ideal of the semigroup SeS. On the other hand, $Q \cap SeS$ is also a unique maximal ideal of SeS. Hence we have $P \cap SeS = Q \cap SeS$.

<u>Theorem</u> 2.4. Let P be an open prime ideal of S, and let us suppose that P has the form $P = J_o(S \setminus e)$, where e is an idempotent. Then all of the following conditions are equivalent :

(1) P is a completely prime ideal.

(2) $SeS \setminus P$ is a subsemigroup of S.

(3) The product of two P-primitive idempotents does not lie in P.

(4) $a \in SeS \setminus P$ implies $a^2 \in SeS \setminus P$.

(5) $SeS \setminus P$ is the disjoint unioun of groups.

(6) For each element of $SeS \setminus P$, there exists a two-sided identity.

Proof. From the fact that $P \cap SeS$ is a maximal ideal of the semigroup SeS, equivalency of the conditions (2) $\sim$ (6) is an immediate consequence of Corollary 2 to Theorem 1 in [1].

It is evident that (1) implies (2), (3) and (4). We shall show that (2) implies (1). Let a and b be elements of S such that $a, b \notin P$. Since P is a prime ideal of S and $e \notin P$, there exist $x, y \in S$ such that $exa \notin P$ and $bye \notin P$. As the elements exa and bye are contained in $SeS \setminus P$, (2) implies that $(exa)(bye) \notin P$. This shows that $ab \notin P$, so that P is a completely prime ideal.

Theorem 2.5. Let Q be a q-ideal of S and let e be a Q-primitive idempotent. If Q is completely semiprime, then the open prime ideal $J_o(S \setminus e)$ is completely prime.

Proof. We denote $J_o(S \setminus e)$ by P, and let a be an arbitrary element of $SeS \setminus P$. We shall show that $a^2 \in SeS \setminus P$. As Q is completely semiprime and $a \notin Q$, we have $a^2 \notin Q$, that is, $a^2 \in SeS \setminus Q$. From Lemma 2.3 we can conclude that $SeS \setminus Q = SeS \setminus P$, and so $a^2 \in SeS \setminus P$. Therefore, P is completely prime by Theorem 2.4.

§3. Let P be a prime ideal and I an ideal of a semigroup. P is said to be a minimal prime ideal belonging to I, if $I \subset P$, and if there is no other prime ideal containing I and properly contained in P.

It was shown in [4] that an open semiprime ideal Q of a compact semigroup S is an intersection of minimal prime ideals belonging to Q, each of which is necessarily open (see [4; Theorems 3.5 and 3.12]).

Remark. Let I be an open ideal of a compact semigroup S. If P' is a prime ideal of S containing I, then there exists a minimal prime ideal P belonging to I such that $P \subset P'$. Moreover P is necessarily open.

Now we ask the question : Is an open semiprime ideal of S expressible as an intersection of a finite number of minimal (open) prime ideals belonging to it ? The answer is "no" , as the following example shows.

Example 3.1. Let S be the set consists of rational numbers $0, \pm 1/n$; $n = 1, 2, \cdots\cdots$: $S = \{ 0, \pm 1/n : n = 1, 2, \cdots\cdots \}$. S is a compact Hausdorff space as a subspace of the real line with the usual topology.

We write e_n in place of $1/n$ $(n = 1, 2, \cdots\cdot)$, f_n in place of $-1/n$ $(n = 1, 2, \cdots\cdot)$, and e_∞ in place of 0. Define a multiplication on S in the following way :

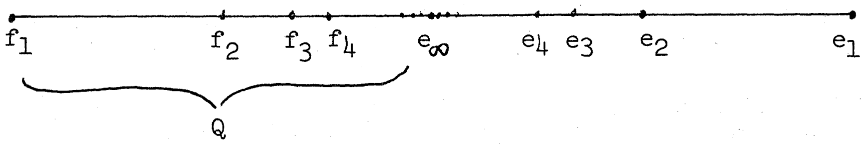
$e_i{}^2 = e_i$ for $i = \infty, 1, 2, \cdots\cdots$,

$e_i e_j = f_k$ for $i \neq j$ and $i, j = \infty, 1, 2, \cdots\cdots$,

$f_i f_j = f_k$ for $i, j = 1, 2, \cdots\cdots$, and

$$e_i f_j = f_k = f_j e_i \quad \text{for} \quad i = \infty, 1, 2, \cdots ; \ j = 1, 2, \cdots ,$$

where $k = \min(i, j)$.



It is not difficult to see that the multiplication defined above satisfies the associative law, and so with this multiplication S is an (abstract) semigroup.

We can easily see that the multiplication defined on S is continuous with respect to the topology. Thus S becomes a compact semigroup.

We denote by Q the set $\{ f_n : n = 1, 2, \cdots\cdots \}$. It can be easily shown that Q is an open semiprime ideal of the semigroup S. All the sets of the form $S \setminus e_i$, $i = \infty, 1, 2, \cdots\cdots$, are minimal prime ideals belonging to Q, and there is no minimal prime ideal belonging to Q except them. It is obvious that the intersection of any finite number of ideals of the form $S \setminus e_i$ does not coincide with Q. From this we see that Q can not be expressed as an intersection of a finite number of minimal prime ideals belonging to Q.

We shall now consider the condition under which an open semiprime ideal Q of S is expressible as an intersection of a finite number of minimal prime ideals belonging to Q.

Lemma 3.2. Let Q be an open semiprime ideal of S. If f is an idempotent of S not contained in Q, then there exists a Q-primitive idempotent e such that $ef = e = fe$.

Proof. By Proposition 1.6, there exists a Q-primitive idempotent g in SfS. From $g \in SfS$, g can be written in the form $g = ab$, where $a \in gSf$ and $b \in fSg$. Let us set $e = ba$. We shall show that e is one of the required idempotents. Firstly,

$$e^2 = b(ab)a = bga = ba = e,$$

that is, e is an idempotent. Secondly,

$$fe = f(ba) = ba = e \quad \text{and} \quad ef = (ba)f = ba = e.$$

And lastly, from $SeS = Se^2S = Sb(ab)aS \subset SabS = SgS = Sg^2S = Sa(ba)bS \subset SbaS = SeS$, we obtain $SeS = SgS$. Using Proposition 1.5, we can conclude that e is a Q-primitive idempotent.

Theorem 3.3. Let Q be an open semiprime ideal of S. If there exists an infinite number of minimal (open) prime ideals belonging to Q, then $E_Q \cap (\overline{E_Q^2 \cap Q}) \neq \emptyset$, where E_Q is the set of all Q-primitive idempotents and $\overline{E_Q^2 \cap Q}$ denotes the topological closure of the set $E_Q^2 \cap Q$.

Proof. Let $\{ P_\alpha : \alpha \in \Lambda \}$ be the collection of all minimal (open) prime ideals belonging to Q. It is obvious that each P_α is properly contained in S. Of course we assume that $P_\alpha \neq P_\beta$ if $\alpha \neq \beta$. By the assumption the index set Λ is infinite.

To each $\alpha \in \Lambda$, choose an idempotent e_α such that $P_\alpha = J_0(S \setminus e_\alpha)$. Then e_α is a Q-primitive idempotent, because P_α is a minimal prime ideal belonging to Q, so that the correspondence $\alpha \longrightarrow e_\alpha$ is a one-to-one mapping from Λ into E_Q. Therefore, the set $\{ e_\alpha : \alpha \in \Lambda \}$ is an infinite subset of E_Q.

Let E be the set of all idempotents of S. It is well-known that E is a closed subset of S. As Q is open, $E\setminus Q$ is also a closed subset of S, and so it is compact. The set $\{ e_\alpha : \alpha \in \Lambda \}$ has a complete limit point (point d'accumlation maximée) f in $E\setminus Q$, since this is an infinite subset of the compact set $E\setminus Q$. By Lemma 3.2 there exists a Q-primitive idempotent e such that ef = e = fe.

We shall show that the idempotent e is containdd in $\overline{E_Q^2 \cap Q}$. Let U be an arbitrary neighborhood of e in S. From ef = e there exists a neighborhood V of f such that $eV \subset U$. As f is a complete limit point of the set $\{ e_\alpha : \alpha \in \Lambda \}$, the set $V \cap \{ e_\alpha : \alpha \in \Lambda \}$ is an infinite set. Therefore, there exists an e_ν, $\nu \in \Lambda$, in V such that $P_\nu = J_o(S\setminus e_\nu)$ is different from the open prime ideal $J_o(S\setminus e)$. From $J_o(S\setminus e_\nu) \neq J_o(S\setminus e)$ it follows that $Se_\nu S \neq SeS$. Since both $Se_\nu S$ and SeS are minimal ideals not contained in Q, the product $(SeS)(Se_\nu S)$ is contained in Q. Hence $ee_\nu \in Q$, and so $ee_\nu \in E_Q^2 \cap Q$. On the other hand, we have $ee_\nu \in eV \subset U$. Therefore we obtain $U \cap (E_Q^2 \cap Q) \neq \emptyset$. As U is taken arbitrary, we have $e \in \overline{E_Q^2 \cap Q}$. This completes the proof.

As immediate consequences of Theorem 3.3 we obtain the following corollaries. We denote by E the set of all idempotents of S.

Corollary 3.4. Let Q be an open semiprime ideal of S. If $E^2 = E$ and if $E \cap Q$ is a closed subset of S, then there exists only a finite number of minimal (open) prime ideals belonging to Q.

Corollary 3.5 ([8 :Theorem 17]). Let Q be an open semiprime ideal of S. If Q is closed, then there exists only a finite number of minimal (open) prime ideals belonging to Q.

Corollary 3.6. If S is a compact N-semigroup in which $E^2 = E$ holds, then the radical is the intersection of a finite number of open prime ideals.

(A semigroup S with 0 is said to be an N-semigroup if the set N of all the nilpotent elements is an open subset of S.)

Proof. It is well-known that the radical $J_o(N)$, the largest ideal contained in N, of S is an open semiprime ideal. Therefore the corollary is a consequence of Corollary 3.4, because $E \cap J_o(N) = \{ 0 \}$.

Problem. Is Corollary 3.6 valid without the assumption $E^2 = E$?

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