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ORTHODOX SEMIGROUPS ON WHICH GREEN'S RELATIONS ARE COMPATIBLE, I. H-COMPATIBLE ORTHODOX SEMIGROUPS

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A semigroup S is said to be H [L,R,D]-compatible if the Green's H [L,R,D]-relation on S is a congruence. A strictly inversive semigroup introduced by the previous paper [5] of the author is of course H-compatible, and the structure of strictly inversive semigroups has been clarified by [5]. As a generalization of [5], the structure of H-compatible orthodox semigroups is studied in this paper. Further, we investigate L [R]-compatible orthodox semigroups.

For an orthodox semigroup S, it is easily verified that the following conditions are equivalent:

Let E_S be the band of idempotents of S.

(C1) If $e, f \in E_S$, $f \leqq e$, $x \in S$ and $xx^* = x^*x = e$ (where x^* denotes an inverse of x) then $xf = fx$. (Note. $f \leqq e$ means $ef = fe = f$.)

(C2) If $e, f \in E_S$ and $f \leqq e$, then $S(e) \cup S(f)$ (where $S(h)$ denotes the maximal subgroup of S containing h for each $h \in E_S$) is a subsemigroup of S.

(C3) The union G_S of all maximal subgroups of S is a strictly inversive semigroup (that is, a (orthodox) band of groups; see [5]).

(C4) The Green's H-relation (see [2]) on S is compatible (hence, it is a congruence; see [1]).

If S satisfies one of these conditions, then S is said to be <u>H-compatible</u>. If every maximal subgroup (that is, H-class containing an idempotent) of an orthodox semigroup A consists of a single element, then A is of course H-compatible. In this case, A is particularly called an <u>H-degenerated orthodox semi-</u>

group (abbrev., an H-d.o. semigroup). In this paper, as a generalization of the previous paper [5], we shall investigate the structure of H-compatible orthodox semigroups. The complete proofs are omitted and will be given in detail by [9].

§ 1. Spined product decomposition.

Let S be an H-compatible orthodox semigroup. Since G_S is a strictly inversive semigroup, there exist a semilattice Λ and a rectangular subgroup S_λ of G_S for each $\lambda \in \Lambda$ such that

$$G_S = \Sigma\{ S_\lambda : \lambda \in \Lambda \} \text{ (disjoint sum), and}$$

$$S_\alpha S_\beta \subset S_{\alpha\beta} \text{ for } \alpha, \beta \in \Lambda.$$

That is, G_S is decomposed into a semilattice Λ of rectangular groups S_λ. Further, such a decomposition of G_S is uniquely determined and is called the structure decomposition (see [5]); and this decomposition is denoted by $G_S \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$.

Now, define a relation σ_S on S as follows:

(C5) a σ_S b if and only if for any $e \in E_S^1$ (where E_S^1 denotes the adjunction of an identity 1 to E_S) there exist $\alpha, \beta \in \Lambda$ such that both aea^* and beb^* are contained in S_α and both a^*ea and b^*eb are contained in S_β.

Then,

Theorem 1. (1) σ_S is the finest H-degenerated inverse semigroup (abbrev., an H-d.i. semigroup) congruence on S,

(2) for the natural homomorphism $\phi : S \to S/\sigma_S$, Ker $\phi \equiv \{\bar{e}\phi^{-1} : \bar{e} \in E_{S/\sigma_S}\} = \{S_\lambda : \lambda \in \Lambda\}$, where $\bar{e}$ is the σ_S-class containing e, and $\bigcup$ Ker $\phi = G_S$,

(3) for the congruence ν_ϕ on S induced by ϕ, the restriction $\nu_\phi|G_S$

gives the structure decomposition of G_S.

Next, we extend the concept of spined products introduced by [5] (see also [6]) from strictly inversive semigroups to H-compatible orthodox semigroups. Let $\phi_1: A \rightarrow \Gamma$ and $\phi_2: B \rightarrow \Gamma$ be homomorphisms of H-compatible orthodox semigroups A and B onto an H-d.i. semigroup Γ such that the congruences ν_{ϕ_1}, ν_{ϕ_2} induced by ϕ_1, ϕ_2 give the finest H-d.i. semigroup decompositions of A,B respectively. Put $\gamma\phi_1^{-1} = A_\gamma$, $\gamma\phi_2^{-1} = B_\gamma$ for $\gamma \in \Gamma$. Define multiplication in $C = \bigcup \{A_\gamma \times B_\gamma : \gamma \in \Gamma\}$ as follows:

$$(a_\alpha, b_\alpha)(c_\beta, d_\beta) = (a_\alpha c_\beta, b_\alpha d_\beta)$$

for $(a_\alpha, b_\alpha) \in A_\alpha \times B_\alpha$ and $(c_\beta, d_\beta) \in A_\beta \times B_\beta$.

Then, C is also an H-compatible orthodox semigroup. This C is called the <u>spined product</u> of A and B with respect to (Γ, ϕ_1, ϕ_2), and denoted by $(A \bowtie B; \Gamma, \phi_1, \phi_2)$.

Under this definition, we have the following:

Lemma 2. Let S be an H-compatible orthodox semigroup, and σ_S the finest H-d.i. semigroup congruence on S. If H-compatible orthodox semigroup congruences σ_1 and σ_2 on S satisfy

(1) $\sigma_1 \subset \sigma_S$ and $\sigma_2 \subset \sigma_S$, and σ_1, σ_2 are permutable,

(2) $\sigma_1 \cap \sigma_2 = \iota_S$ (the identity congruence on S), and

(3) $\sigma_1 \vee \sigma_2 = \sigma_S$ (where $\sigma_1 \vee \sigma_2$ denotes the least congruence on S containing both σ_1 and σ_2),

then S is isomorphic to $(S/\sigma_1 \bowtie S/\sigma_2; S/\sigma_S, \phi_1, \phi_2)$, where ϕ_1, ϕ_2 are the homomorphisms of S/σ_1, S/σ_2 to S/σ_S respectively defined by $(x\sigma_1)\phi_1 = x\sigma_S$ and $(x\sigma_2)\phi_2 = x\sigma_S$.

Now, define two relations σ_1, σ_2 on an H-compatible orthodox semigroup S as follows:

(1) $a\ \sigma_1\ b$ if and only if $a\ \sigma_S\ b$ and $ab^*, b^*a \in E_S$ for some inverse b^* of b,

(2) $a\ \sigma_2\ b$ if and only if $a\ \sigma_S\ b$ and

$aea^*beb^* = beb^*$, $beb^*aea^* = aea^*$ for all $e \in E_S^1$ and for some inverses a^* and b^*;

$a^*eab^*eb = a^*ea$, $b^*eba^*ea = b^*eb$ for all $e \in E_S^1$ and for some inverses a^* and b^*.

Then, these σ_1, σ_2 satisfy the conditions (1),(2) and (3) of Lemma 2. Hence, S is isomorphic to $(S/\sigma_1 \bowtie S/\sigma_2;\ S/\sigma_S,\ \phi_1,\ \phi_2)$, where ϕ_1, ϕ_2 are the homomorphisms of S/σ_1, S/σ_2 to S/σ_S respectively defined by $(x\sigma_1)\phi_1 = x\sigma_S$ and $(x\sigma_2)\phi_2 = x\sigma_S$. In this case, it is easily seen that S/σ_1 and S/σ_2 are an H-compatible inverse semigroup and an H-d.o. semigroup respectively.

Thus, we have the following theorem.

Theorem 3. An orthodox semigroup S is H-compatible if and only if $S \cong (A \bowtie B;\ \Gamma,\ \phi_1,\ \phi_2)$, where A is an H-compatible inverse semigroup, B an H-d.o. semigroup, Γ an H-d.i. semigroup, and ϕ_1, ϕ_2 homomorphisms of A,B onto Γ respectively.

§ 2. Construction theorems.

The concept of a regular extension of an inversive semigroup [a band] by an inverse semigroup has been introduced by [8][by [7]] as follows:

If a regular semigroup R contains an inversive semigroup [a band] T as its subsemigroup and if there exists a homomorphism ϕ of R onto an inverse semigroup Γ such that

(1) $\bigcup \mathrm{Ker}\ \phi \equiv \bigcup\{\lambda\phi^{-1} : \lambda \in E_\Gamma\} = T$, and

(2) the structure decomposition of T is $T \sim \Sigma\{\lambda\phi^{-1}: \lambda \in E_\Gamma\}$, then R is called a <u>regular</u> <u>extension</u> of T by Γ.

Further, the previous paper [8] [[7]] has given a description of all possible regular extensions of T by Γ for a given inversive semigroup [band] T and for a given inverse semigroup Γ.

Since every H-d.o. semigroup B can be obtained as a regular extension of a band by an H-d.i. semigroup (see [7]), the construction of H-compatible orthodox semigroups is reduced to the following problems:

I. Construction of H-d.i. semigroups.

II. Construction of H-compatible inverse semigroups.

Now, we shall consider about these two problems.

I. Let E be a semilattice, and put $\{(e,f): eE \cong fE\} = \Delta$. Let Ω be a subset of Δ and $\phi_{(e,f)}$ an isomorphism of eE onto fE for each $(e,f) \in \Omega$. Assume that Ω and the family $F_\Omega(E) = \{\phi_{(e,f)}: (e,f) \in \Omega\}$ satisfy the following:

(C6) (1) $(e,e) \in \Omega$ for all $e \in E$,

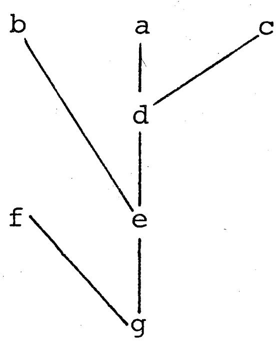
(2) $(e,f) \in \Omega$ implies $(f,e) \in \Omega$,

(3) $\phi_{(e,e)}$ is the identity mapping on eE for each $e \in E$,

(4) $(e,f),(h,t) \in \Omega$ implies $((fh)\phi_{(f,e)},(fh)\phi_{(h,t)}) \in \Omega$ and $\phi_{((fh)\phi_{(f,e)},(fh)\phi_{(h,t)})} = \phi_{(e,f)}\phi_{(h,t)}|(fh)\phi_{(f,e)}E$.

For example, consider the case where E satisfies the following condition:

(C7) For any $e \in E$, eE is a totally ordered set (with respect to the usual ordering); and when $e,f \in E$, $e \neq f$ and $eE \cong fE$, an isomorphism θ of eE onto fF is unique. (For example, see the diagram below.)



In this case, select Ω and $\phi_{(e,f)}$ as follows:

(1) $\Omega = \Delta$,

(2) $\phi_{(e,e)}$ is the identity mapping on eE for each $e \in E$; and $\phi_{(e,f)}$ is the unique isomorphism of eE onto fE for each $(e,f) \in \Delta$ such that $e \neq f$.

Then, the system $\{\Delta , \phi_{(e,f)}\}$ satisfies (1)-(4) of (C6).

Now, $F_\Omega(E)$ above is an H-d.i. subsemigroup of the symmetric inverse semigroup $\mathcal{J}_E$ on E. This $F_\Omega(E)$ is denoted by the symbol $HD(E; \Omega , \phi_{(e,f)})$.

Then, we have the following result:

Theorem 4. Any H-d.i. semigroup A is isomorphic to some $HD(E; \Omega, \phi_{(e,f)})$. In this case, the semilattice E_A can be selected as E. In particular, if E_A satisfies (C7) then A is isomorphic to a full regular subsemigroup of $HD(E_A; \Delta , \phi_{(e,f)})$ (where $\Delta = \{(e,f): e,f \in E_A, eE \cong fE\}$; $\phi_{(e,e)}$ = the identity mapping on eE for each $e \in E_A$; and $\phi_{(e,f)}$ = the unique isomorphism of eE onto fE for each $(e,f) \in \Delta$ such that $e \neq f$).

Remark. This result can be obtained by slightly modifying the results given by [3].

II. It is easily verified that an inverse semigroup is H-compatible if and only if it is a regular extension of a weakly C-inversive semigroup (that is, a semilattice of groups; see [6]) by an H-d.i. semigroup. Hence, hereafter we consider the problem of constructing all possible regular extensions of G by Γ for any given H-d.i. semigroup Γ having Λ as the semilattice of its idempotents and for any given weakly C-inversive semigroup G having $G \sim \Sigma\{G_\lambda : \lambda \in \Lambda\}$ as its structure decomposition (that is, G is a semilattice Λ of groups G_λ).

Let e_λ be the identity element of G_λ .
The regular extensions of G by Γ are obtained by slightly modifying the concept of a Schreier extension in the group theory.

Let $\Psi: \Gamma \to End(G)$ (the semigroup of endomorphisms on G) and $\phi: \Gamma \times \Gamma \to G$ be mappings such that each $\bar{\gamma} = \gamma\Psi$ maps G_λ into $G_{\gamma\lambda(\gamma\lambda)^*}$, especially maps $G_{\gamma^*\gamma}$ onto $G_{\gamma\gamma^*}$, and each $C(\gamma,\tau) = (\gamma,\tau)\phi$ is an element of $G_{\gamma\tau(\gamma\tau)^*}$. If $\{\bar{\gamma} : \gamma \in \Gamma\}$ and $\{C(\gamma,\tau) : (\gamma,\tau) \in \Gamma \times \Gamma\}$ satisfy

(C8) (1) $C(\lambda,\delta) = e_{\lambda\delta}$ for $\lambda,\delta \in E_\Gamma$, and $C(\gamma\gamma^*,\gamma) = C(\gamma,\gamma^*\gamma) = e_{\gamma\gamma^*}$ for each $\gamma \in \Gamma$,

(2) $C(\tau,\xi)^{\bar{\gamma}} C(\gamma,\tau\xi) = C(\gamma,\tau)C(\gamma\tau,\xi)$, where $x^{\bar{\gamma}} = x\bar{\gamma}$,

(3) $\bar{\gamma}\,\bar{\tau} = \overline{\tau\gamma}\;\overline{C(\tau,\gamma)}$ (where, for each $a \in G$, $\bar{a}$ denotes the endomorphism on G defined by $z^{\bar{a}} = aza^*$, $z \in G$),

(4) $e_\lambda b = b^{\bar{\lambda}}$ for $\lambda, \delta \in E_\Gamma$ and $b \in G_\delta$,

then the system $\{\bar{\gamma}, C(\gamma,\xi)\}$ is called a (<u>normalized</u>) <u>factor</u> <u>set</u> of G belonging to Γ.

Now, under this definition we have the following theorem:

Theorem 5. Let G, Γ and Λ be as above. Let $\{\bar{\gamma}, C(\tau,\xi)\}$ be a factor set of G belonging to Γ. If multilplication is defined in $G \divideontimes \Gamma = \{(a,\gamma) : a \in G_{\gamma\gamma^*}, \gamma \in \Gamma\}$ by

$$(a,\gamma)(b,\tau) = (ab^{\bar{\gamma}}C(\gamma,\tau), \gamma\tau),$$

then

(1) $\underline{G} = \{ (a,\lambda) : a \in G_{\lambda}, \lambda \in E_{\Gamma} \}$ is a weakly C-inversive semigroup which is isomorphic to G, and

(2) G ※ Γ is a regular extension of $\underline{G}$ by Γ.

Further, every regular extension of G by Γ is obtained in this fashion, up to isomorphism.

§ 3. L-compatible regular semigroups.

A semigroup S is said to be <u>L-compatible</u> [<u>R-compatible</u>] if the Green's L-relation [R-relation] on S is compatible (that is, a congruence). Firstly, we have the following theorem.

Theorem 6. A regular semigroup S is L-compatible if and only if it is a right regular band of left groups. Of course, in this case S is a union of groups.

A band B is called a <u>regular</u> [<u>left regular</u>, <u>right regular</u>] <u>band</u> if B satisfies the identity xyxzx = xyzx [xyx = xy, xyx = yx] (see [4]). If a semigroup S is an orthodox semigroup and is a right regular band of left groups, then S is called an <u>orthodox right regular band of left groups</u>. An orthodox band of groups, an orthodox regular band of groups, etc. are similarly defined.

Under these definitions and by using Theorem 6, we have the following result:

Corollary 7. A semigroup is an L-compatible orthodox semigroup if and only if it is an orthodox right regular band of left groups.

By using Theorem 6 and its dual result, the following is also obtained:

Theorem 8. A regular semigroup is both L-compatible and R-compatible if and only if it is a regular band of groups.

Corollary 9. A semigroup is a both L- and R-compatible orthodox semigroup if and only if it is both a semilattice of rectangular groups and a regular band of groups.

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