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ARCHIMEDEAN CLASSES IN AN ORDERED SEMIGROUP IV*)

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By an ordered semigroup we mean a semigroup S with a simple order $\leqq$ which satisfies

for $x, y, z \in S$, $x \leqq y$ imples $xz \leqq yz$ and $zx \leqq zy$.

The archimedean equivalence $\mathcal{A}$ on an ordered semigroup S is defined by:

for $x, y \in S$, $x \mathcal{A} y$ if and only if there exist natural numbers p, q, r and s such that $x^p \leqq y^q$ and $y^r \leqq x^s$.

The difficulty occurs because of the fact that the archimedean equivalence is not necessarily a congruence relation. In our previous papers [3], [4] and [5], we discussed the behavior of set products of two archimedean classes of an ordered semigroup. The purpose of the present paper is to give some supplementary properties to preceding papers and also some applications.

We use the terminology and notations in our previous papers [3], [4] and [5] freely.

1. In this section, we give some properties of archimedean classes which will be needed in the following discussion.

LEMMA 1. *Suppose* $A, B \in \mathcal{C}$, $B\delta \leqq A\delta$ *and* $B\delta$ *is periodic of L-type. Let* g *be the idempotent of* $A * B$ *and* f *the idempotent of* B. *Then*

(1) $ag = g$ *for every* $a \in A$;

(2) *if* $B\delta < A\delta$, *then* $af = g$ *for every* $a \in A$.

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*) In the seminar, we gave a talk which covers our papers "Archimedean classes in an ordered semigroup I-IV". But Part I-III was published recently and, accordingly, we publish here only Part IV.

PROOF. Since $(A * B)\delta = A\delta \wedge B\delta = B\delta$, $A * B$ is a periodic archimedean class and so really contains the unique idempotent g. In the proof, we only consider the case when $A \leqq B$. Then $A \leqq A * B \leqq B$. First suppose that $B\delta = A\delta$. Then

$$A * B = \min\{ X \in C \ ; \ A \leqq X \leqq B \text{ and } X \in A\delta \wedge B\delta = A\delta \} = A.$$

Since g is the zero element of $A * B = A$, we have $ag = g$ for every $a \in A$. Next suppose that $B\delta \prec A\delta$. Let $a \in A$ and put $h = ag$. Let D be the archimedean class containing the element h. Then, by [3] Lemma 5.2, h is an idempotent and

$$D = \min\{ X \in C \ ; \ A \leqq X \text{ and } X \in B\delta \}.$$

Since $A < A * B$ and $(A * B)\delta = A\delta \wedge B\delta = B\delta$, we have $D \leqq A * B$. On the other hand, $h = ag \leqq f^2 = f$ and so $A \leqq D \leqq B$ and also $D \in B\delta = A\delta \wedge B\delta$. Hence $A * B \leqq D$ and so $D = A * B$. Hence $h = g$ and so $ag = g$. Moreover, since $A < A * B$, we have $a < g$. Hence

$$g = ag \leqq af \leqq gf = g$$

and so $af = g$.

THEOREM 2. *Suppose that* $A, B \in C$, $A\delta \wedge B\delta \prec A\delta$ *and* $A\delta \wedge B\delta$ *is periodic of L-type. Then*

(1) *if* $A \leqq B$, *then* $AB \subseteq (A * B)_+$;

(2) *if* $B \leqq A$, *then* $AB \subseteq (A * B)_-$.

PROOF. Here we only show the assertion (1). Suppose $A \leqq B$. Since $A\delta \wedge B\delta \prec A\delta$, we have $A < A * B \leqq B$. Since $(A * B)\delta = A\delta \wedge B\delta$, $A * B$ is a periodic archimedean class. We denote by g the idempotent of $A * B$. Let $a \in A$ and $b \in B$. First suppose $g \leqq b$. Then, since $a < g$, we have $ab \leqq gb = g$. On the other hand, $(A * B)\delta = A\delta \wedge B\delta \prec A\delta$ and, by [3] Lemma 5.10, $A * (A * B) = A * B$. Hence, by Lemma 1, $g = ag \leqq ab$. Hence $ab = g \in (A * B)_+$. Next suppose that $b \leqq g$. Then we have $B \leqq A * B$ and so $B = A * B$. Hence B is a periodic archimedean class with idempotent g and so $b^n = g$ for some natural number n. Hence if $ab \leqq ba$, then, by Lemma 1,

$$g = a^{n+1}g = a^{n+1}b^{n+1} \leqq (ab)^{n+1} \leqq ab^n a^n b = aga^n b = ag = g,$$

and if $ba \leqq ab$, then

$$g = ag = aga^n b = ab^n a^n b \leqq (ab)^{n+1} \leqq a^{n+1}b^{n+1} = a^{n+1}g = g.$$

Thus we have $(ab)^{n+1} = g$ and so $ab \in A * B$. Also, since $A < A * B$, we have $ab \leqq gb = g$ and so $ab \in (A * B)_+$.

In the proof of Theorem 2, we incidentally proved

<u>COROLLARY 3</u>. <u>Suppose that</u> $A, B \in \mathcal{C}$, $A\delta \wedge B\delta \prec A\delta$, $A\delta \wedge B\delta$ <u>is periodic of</u> L-<u>type and</u> $A \leqq B$. <u>Let</u> g <u>be the idempotent of</u> $A * B$. <u>Then</u> $ab = g$ <u>for every</u> $a \in A$ <u>and</u> $b \in B$ <u>such that</u> $g \leqq b$. <u>In particular, if</u> $A * B \neq B$, <u>then</u> $ab = g$ <u>for every</u> $a \in A$ <u>and</u> $b \in B$.

2. Let S be an ordered semigroup. S is called a-<u>regular</u> if the archimedean equivalence on S is a congruence relation. S is called <u>nonnegatively ordered</u> if $a \leqq a^2$ for every $a \in A$. A criterion of a-regularity for a nonnegatively ordered semigroup was given in [2] Theorem 2.8. The purpose of this section is to give a criterion of a-regularity for a general ordered semigroup.

The next Lemma was given in our Lecture Note [6]. But, for the sake of convenience we give it with proof.

<u>LEMMA 4</u>. <u>Let</u> a <u>be an element of finite order</u> n <u>of an ordered semigroup</u> S. <u>If there exists an idempotent</u> g <u>of</u> S <u>such that</u> $a^n \mathcal{D}_E g$ <u>and</u> a <u>lies between</u> a^n <u>and</u> g, <u>then</u> $n \leqq 2$.

<u>PROOF</u>. Suppose $n < 1$. We consider only the case when a is positive, that is, $a < a^2$. Then we have $g < a < a^2 \leqq a^n$. By [3] Lemmas 1.6 and 1.7, we have $a^n \mathcal{L} g$ or $a^n \mathcal{R} g$. For the sake of definiteness, we assume $a^n \mathcal{R} g$ and so $a^n g = g$, $ga^n = a^n$. Then $g = g^2 \leqq ag \leqq a^n g = g$ and so $g = ag$. Hence $gaga = ga$ and $ga^2 ga^2 = ga^2$ and so ga and ga^2 are idempotents of S. We have $a < ga^2$, since $ga^2 \leqq a$ would imply $a^n = ga^n \leqq \ldots \leqq ga^2 \leqq a$,

which is a contradiction. If $a \leqq ga$, then

$$a^3 \leqq (ga)^3 = ga \leqq a^2 \leqq a^3,$$

and if $ga \leqq a$, then

$$a^3 \leqq (ga^2)^3 = ga^2 = (ga)a \leqq a^2 \leqq a^3.$$

Hence, in both cases, we have $a^2 = a^3$.

THEOREM 5. *The archimedean equivalence in an ordered semigroup* S *is not a congruence relation, if and only if either*

(1) *there exist torsion-free archimedean classes* A *and* B *in* S *such that* $A \neq B$ *and* $A \,\delta\, B$, *or*

(2) S *contains a subsemigroup* o-*isomorphic to either one of the ordered semigroups* K_1, K_2, K_3 *and* K_4:

K_1:

	e	f	a	g
e	e	e	e	e
f	f	f	f	f
a	f	g	g	g
g	g	g	g	g

$e < f < a < g$

K_2:

	e	f	a	g
e	e	f	f	g
f	e	f	g	g
a	e	f	g	g
g	e	f	g	g

$e < f < a < g$

K_3:

	g	a	f	e
g	g	g	g	g
a	g	g	g	f
f	f	f	f	f
e	e	e	e	e

$g < a < f < e$

K_4:

	g	a	f	e
g	g	g	f	e
a	g	g	f	e
f	g	g	f	e
e	g	f	f	e

$g < a < f < e$.

PROOF. "Only if" part. Suppose that the archimedean equivalence on S is not a congruence relation. Then there exist archimedean classes A and B such that AB is not contained in a single archimedean class. First suppose that $A\delta \wedge B\delta$ is torsion-free.

Then, by [3] Corollary 6.2, we have $A \neq B$ and $A \,\delta\, B$. Then $A\delta = A\delta \wedge B\delta = B\delta$ and so A and B are torsion-free archimedean classes. Hence we have the condition (1). Next suppose that $A\delta \wedge B\delta$ is periodic. First we consider the case when $A\delta \wedge B\delta$ is of L-type and $A \leqq B$. Then, by Theorem 2, we have $A\delta = A\delta \wedge B\delta$. We denote by g and e the idempotents of the periodic archimedean classes A and $B * A$, respectively. Then $A\delta = A\delta \wedge B\delta = (B * A)\delta$ and, by [3] Theorem 3.3, we have $g\, \mathcal{D}_E\, e$. Also, by [3] Lemma 6.7, there exists an idempotent f of S such that $g < f < e$ and $g\, \mathcal{D}_E\, f$, and also there exists $a \in A \setminus \{g\}$ such that $ae = f$. Since A is a periodic archimedean class with idempotent g, we have $a^n = g$ for some natural number $n > 1$. Also $g = a^n < a < f$ and, by Lemma 4, we have $a^2 = g$. Now we can verify that $\{g, a, f, e\}$ forms a subsemigroup o-isomorphic to K_3. In a similar way, if $A\delta \wedge B\delta$ is of L-type and $B \leqq A$ or $A\delta \wedge B\delta$ is of R-type and $A \leqq B$ or $A\delta \wedge B\delta$ is of R-type and $B \leqq A$, we can prove that S contains a subsemigroup o-isomorphic to K_1 or K_2 or K_4.

"If" part. Suppose that there exist torsion-free archimedean classes A and B in S such that $A \neq B$ and $A \,\delta\, B$. Then, by [3] Theorem 2.4, $AB \cap A \neq \square$ and $AB \cap B \neq \square$. Hence AB is not contained in a single archimedean class and so the archimedean equivalence is not a congruence relation. If S contains a subsemigroup o-isomorphic to K_1 or K_2 or K_3 or K_4, then clearly the archimedean equivalence is not a congruence relation.

3. As an application, in this section we give a result that a finite product of elements of an ordered semigroup is archimedean equivalent under certain conditions to a product of at most two of these factors.

LEMMA 6. *In an ordered semigroup* S, *let* $a = x_1 \ldots x_n$ *and let* $X_1, \ldots, X_n$ *and* A *be archimedean classes containing* $x_1, \ldots, x_n$ *and* a, *respectively. Then*

$$X_1\delta \wedge \ldots \wedge X_n\delta \leqq A\delta.$$

PROOF. If $n = 1$, the assertion is trivial. Suppose $n = 2$. If $x_1 \leqq x_2$, then

$$x_1^2 \leqq x_1x_2 = a \leqq x_2^2 \quad \text{with} \quad x_1^2 \in X_1, \quad a \in A \quad \text{and} \quad x_2^2 \in X_2$$

and if $x_2 \leqq x_1$, then

$$x_2^2 \leqq x_1x_2 = a \leqq x_1^2 \quad \text{with} \quad x_2^2 \in X_2, \quad a \in A \quad \text{and} \quad x_1^2 \in X_1.$$

Hence, by [3] Lemma 5.6, we have $X_1\delta \wedge X_2\delta \leqq A\delta$. Suppose $n > 2$. Put $y = x_2 \ldots x_n$ and let Y be the archimedean class containing y. Then, by induction hypothesis,

$$X_2\delta \wedge \ldots \wedge X_n\delta \leqq Y\delta$$

and, since $a = x_1x_2 \ldots x_n = x_1y$, we have

$$X_1\delta \wedge X_2\delta \wedge \ldots \wedge X_n\delta \leqq X_1\delta \wedge Y\delta \leqq A\delta.$$

THEOREM 7. *In an ordered semigroup* S, *let* $a = x_1 \ldots x_n$ *and let* $X_1, \ldots, X_n$ *and* A *be archimedean classes containing* $x_1, \ldots, x_n$ *and* a, *respectively. If* $X_1\delta \wedge \ldots \wedge X_n\delta = A\delta$ *and* a *is an element of infinite order, then* $a\, \mathcal{A}\, x_i$ *for some* $1 \leqq i \leqq n$.

PROOF. If $n = 1$, then the assertion is trivial. Suppose $n = 2$. If $X_1 = X_2$, then $a = x_1x_2 \in X_1X_2 = X_1^2 \subseteq X_1$ and so $a\, \mathcal{A}\, x_1$. If $X_1 \neq X_2$ and $X_1\delta = X_2\delta$, then $A\delta = X_1\delta \wedge X_2\delta = X_1\delta = X_2\delta$ and, by [3] Theorem 3.5, $A = X_1$ or $A = X_2$. Hence we have $a\, \mathcal{A}\, x_1$ or $a\, \mathcal{A}\, x_2$. If $X_1\delta \neq X_2\delta$, then, since $X_1\delta \wedge X_2\delta = A\delta$ is a torsion-free δ-class, it follows from [3] Theorem 6.1 that either $a = x_1x_2 \in X_1X_2 \subseteq X_1$ or $a = x_1x_2 \in X_1X_2 \subseteq X_2$ and so $a\, \mathcal{A}\, x_1$ or $a\, \mathcal{A}\, x_2$. Finally suppose $n > 2$. Let Y be the archimedean class containing the element $y = x_2 \ldots x_n$. Then, since $a = x_1x_2 \ldots x_n = x_1y$, we have $X_1\delta \wedge Y\delta \leqq A\delta$ and $X_2\delta \wedge \ldots \wedge X_n\delta \leqq Y\delta$ by Lemma 6. Hence

$$A\delta = X_1\delta \wedge X_2\delta \wedge \ldots \wedge X_n\delta \leqq X_1\delta \wedge Y\delta \leqq A\delta$$

and so $X_1\delta \wedge Y\delta = A\delta$. Also we have $a = x_1y$ and so $a \mathcal{A} x_1$ or $a \mathcal{A} y$. But, if $a \mathcal{A} y$, then we have $A = Y$ and so

$$Y\delta = A\delta = X_1\delta \wedge X_2\delta \wedge \ldots \wedge X_n\delta \leqq X_2\delta \wedge \ldots \wedge X_n\delta \leqq Y\delta.$$

Hence $Y\delta = X_2\delta \wedge \ldots \wedge X_n\delta$ and, by induction hypothesis, we have $a \mathcal{A} y \mathcal{A} x_i$ for some $2 \leqq i \leqq n$. This completes the proof.

LEMMA 8. *Let A, B and C be archimedean classes in an ordered semigroup S such that $AB \subseteq C$. Then we have $C = A * B$ and $C\delta = A\delta \wedge B\delta$.*

PROOF. If $A = B$, then $AB = A^2 \subseteq A$ and so $C = A$. Hence, by [3] Lemma 5.8, $C = A = A * A = A * B$ and also $C\delta = A\delta = A\delta \wedge B\delta$. Next suppose that $A \neq B$ and $A\delta \wedge B\delta$ is torsion-free. Then, by [3] Corollary 6.2, $A\ \delta\ B$ does not hold. Hence, by [3] Theorem 6.1, we have either $A \gamma B$ or $B \gamma A$. Also, if $A \gamma B$, then, since $AB \subseteq A = A * B$, we have $C = A = A * B$ and $C\delta = A\delta = A\delta \wedge B\delta$, and if $B \gamma A$, then, since $AB \subseteq B = A * B$, we have $C = B = A * B$ and $C\delta = B\delta = A\delta \wedge B\delta$. Finally suppose that $A \neq B$ and $A\delta \wedge B\delta$ is periodic. For the sake of definiteness we assume $A \leqq B$ and $A\delta \wedge B\delta$ is of L-type. Let $a \in A$ and $b \in B$. Then, since $A < B$, we have $a < b$ and so $a^2 \leqq ab \leqq b^2$ with $a^2 \in A$, $ab \in C$ and $b^2 \in B$. Hence $A \leqq C \leqq B$ and, by [3] Lemma 5.6, we have $A\delta \wedge B\delta \leqq C\delta$. On the other hane, we have $a^2b = a(ab) \in C \cap AC$ and $ab^2 = (ab)b \in C \cap CB$. Hence $C \gamma A$ and $C \gamma B$ and so $C\delta \leqq A\delta \wedge B\delta$. Hence we have $C\delta = A\delta \wedge B\delta$. Also, since $A \leqq C \leqq B$ and $C \in A\delta \wedge B\delta$, we have

$$A * B = \min\{ D \in \mathcal{C} ;\ A \leqq D \leqq B \text{ and } D \in A\delta \wedge B\delta \} \leqq C.$$

On the other hand, since $A * B \in A\delta \wedge B\delta$, $A * B$ is a periodic archimedean class and so contains an idempotent, say g. Then, since $A \leqq A * B$, we have $a_1 \leqq g$ for some $a_1 \in A$. Since $(A * B)\delta = A\delta \wedge B\delta \leqq B\delta$, we have $(A * B) \gamma B$ and so, by [3] Theorem 2.7, $a_1b \leqq gb = g$. Hence we have $C \leqq A * B$ and thus $C = A * B$.

LEMMA 9. In a nonnegatively ordered semigroup S, suppose that $a = x_1 \ldots x_n$. Let $X_1, \ldots, X_n$ and A be archimedean classes containing $x_1, \ldots, x_n$ and a, respectively. Then $X_1\delta \wedge \ldots \wedge X_n\delta = A\delta$.

PROOF. If $n = 1$, the assertion is trivial. Suppose $n = 2$. If X_1X_2 is contained in a single archimedean class, then, by Lemma 8, we have $X_1\delta \wedge X_2\delta = A\delta$. Next consider the case when X_1X_2 is not contained in a single archimedean class. If $X_1 \leqq X_2$, then, since S is nonnegatively ordered, it follows from [3] Lemma 1.8 that X_2 is a periodic archimedean class of R-type with idempotent, say e, and there exists an idempotent f such that $f \mathcal{R} e$ and $X_1X_2 \subseteq \{f\} \cup X_2$. We denote by Y the archimedean class containing the element f. Then, by [3] Theorem 3.3, we have $Y\delta = X_2\delta$. Since $a = x_1x_2 \in \{f\} \cup X_2$, we have $A\delta = X_2\delta$. On the other hand,

$$x_1e = x_1(x_2e) = x_1x_2e \in (\{f\} \cup X_2)\{e\} = \{fe\} \cup X_2\{e\} = \{e\}$$

and so $e = x_1e \in X_1X_2 \cap X_2$. Hence $X_2 \gamma X_1$ and so $X_2\delta \leqq X_1\delta$. Hence we have $A\delta = X_2\delta = X_1\delta \wedge X_2\delta$. If $X_2 \leqq X_1$, we can similarly prove that $A\delta = X_1\delta \wedge X_2\delta$.

Now suppose $n > 2$. We put $y = x_2 \ldots x_n$ and denote by Z the archimedean class containing y. Then, by induction hypothesis, $Z\delta = X_2\delta \wedge \ldots \wedge X_n\delta$. Also, since $a = x_1x_2 \ldots x_n = x_1y$, we have $A\delta = X_1\delta \wedge Z\delta$. Hence

$$A\delta = X_1\delta \wedge X_2\delta \wedge \ldots \wedge X_n\delta.$$

COROLLARY 10. In a nonnegatively ordered semigroup S, suppose that $a = x_1 \ldots x_n$ and a is an element of infinite order. Then $a \mathcal{A} x_i$ for some $1 \leqq i \leqq n$.

<u>LEMMA 11</u>. <u>In an ordered semigroup</u> S, <u>let</u> $a = x_1x_2x_3$ <u>and</u> <u>let</u> X_1, X_2, X_3 <u>and</u> A <u>be archimedean classes containing</u> x_1, x_2 x_3 <u>and</u> a, <u>respectively</u>. <u>If</u> $A\delta$ <u>is periodic of</u> L-<u>type and</u> $X_1\delta \wedge X_2\delta \wedge X_3\delta = A\delta < X_1\delta \wedge X_2\delta$, <u>then</u> $a \mathcal{A} x_1x_3$ <u>or</u> $a \mathcal{A} x_2x_3$.

<u>PROOF</u>. Put $y = x_1x_2$ and let Y be the archimedean class containing y. Then, by Lemma 6, $X_1\delta \wedge X_2\delta \leqq Y\delta$ and, since $a = x_1x_2x_3 = yx_3$, we also have $Y\delta \wedge X_3\delta \leqq A\delta = X_1\delta \wedge X_2\delta \wedge X_3\delta \leqq Y\delta \wedge X_3\delta$. Hence

$$Y\delta \wedge X_3\delta = A\delta < X_1\delta \wedge X_2\delta \leqq Y\delta.$$

Hence, by Theorem 2, $a = yx_3 \in Y * X_3$. Now, by way of contradiction, we assume that x_3 lies between x_1 and x_2, that is, either $x_1 \leqq x_3 \leqq x_2$ or $x_2 \leqq x_3 \leqq x_1$. Then we have either $X_1 \leqq X_3 \leqq X_2$ or $X_2 \leqq X_3 \leqq X_1$. Hence, by [3] Lemma 5.6, we have $X_1\delta \wedge X_2\delta \leqq X_3\delta$ and so $X_1\delta \wedge X_2\delta \wedge X_3\delta = X_1\delta \wedge X_2\delta$, which is a contradiction. Hence either x_1 lies between x_2 and x_3 or x_2 lies between x_1 and x_3. For the sake of definiteness, we assume $x_2 \leqq x_1 \leqq x_3$. Then $X_2 \leqq X_1 \leqq X_3$ and, by [3] Lemma 5.6, we have $X_2\delta \wedge X_3\delta \leqq X_1\delta$ and so $X_2\delta \wedge X_3\delta = X_1\delta \wedge X_2\delta \wedge X_3\delta = A\delta < X_1\delta \wedge X_2\delta \leqq X_2\delta$. Hence, by Theorem 2, $x_2x_3 \in X_2 * X_3$. Now, since $x_2 \leqq x_1$, we have $x_2^2 \leqq x_1x_2 \leqq x_1^2$ and so $X_2 \leqq Y \leqq X_1 \leqq X_3$. Also, for every $Z \in \mathcal{C}$ such that $X_2 \leqq Z \leqq Y$, we have $X_2 \leqq Z \leqq Y \leqq X_1$ and, again by [3] Lemma 5.6, $X_2\delta \wedge X_3\delta = A\delta < X_1\delta \wedge X_2\delta \leqq Z\delta$. Hence

$$\begin{aligned} Y * X_3 &= \min\{\, U \in \mathcal{C} \;;\; Y \leqq U \leqq X_3 \text{ and } U \in A\delta \,\} \\ &= \min\{\, U \in \mathcal{C} \;;\; X_2 \leqq U \leqq X_3 \text{ and } U \in A\delta \,\} = X_2 * X_3. \end{aligned}$$

Thus we obtain $a, x_2x_3 \in Y * X_3 = X_2 * X_3$ and so $a \mathcal{A} x_2x_3$. In the remaining cases, we can similarly prove that either $a \mathcal{A} x_1x_3$ or $a \mathcal{A} x_2x_3$.

LEMMS 12. In a nonnegatively ordered semigroup S, suppose that $A, B \in \mathcal{C}$ such that $A\delta \leqq B\delta$ and $A\delta$ is periodic of L-type and $a \in A$, $b \in B$ and e and g are idempotents of the periodic archimedean classes A and $B * A$, respectively. Then $ab \in A$ if and only if $ag = e$.

PROOF. First suppose that $AB \subseteq A$. Then we have $ab \in A$ for every $a \in A$ and $b \in B$. On the other hand, if $A = B * A$, then we have $ag = ae = e$. Also, if $A \neq B * A$, then, since $B * A$ lies between A and B, the element g in $B * A$ lies between a and b' for some $b' \in B$. Hence ag lies between a^2 and ab' with $a^2 \in A$ and $ab' \in AB \subseteq A$. Hence $ag \in A$ and, since $(B * A)\delta = A\delta \wedge B\delta = A\delta$, we have $ga = g$ by [3] Theorem 2.7 and so ag is an idempotent of A. Hence $ag = e$. Next suppose that AB is not contained in A. Since $eb = e \in AB \cap A$, AB is not contained in a single archimedean class. Hence, by [3] Lemma 1.8, $B < A$ and $AB \subseteq \{f\} \cup A$, where f is an idempotent of S such that $f < e$ and $f \mathcal{L} e$. Again by [3] Lemma 1.8, BA is contained in a single archimedean class and, by Lemma 8, $BA \subseteq B * A$. Since be is an idempotent and also $be \in BA \subseteq B * A$, we have $be = g$. Now we suppose $ab \in A$. Then $ag = a(be) = (ab)e = e$. Next we suppose $ab \notin A$. Then, since $ab \in AB \subseteq \{f\} \cup A$, we have $ab = f$. Hence $ag = a(be) = (ab)e = fe = f \neq e$.

LEMMA 13. In a nonnegatively ordered semigroup S, let $a = x_1x_2x_3$ and let X_1, X_2, X_3 and A be archimedean classes containing x_1, x_2, x_3 and a, respectively. If $A\delta$ is periodic, then $a \mathcal{A} x_1x_2$ or $a \mathcal{A} x_2x_3$ or $a = x_1x_3$.

PROOF. For the sake of definiteness we assume $A\delta$ is of L-type. By Lemma 9, we have $A\delta = X_1\delta \wedge X_2\delta \wedge X_3\delta$. If $A\delta < X_1\delta \wedge X_2\delta$, then the assertion follows from Lemma 11. In what follows, we assume

$A\delta = X_1\delta \wedge X_2\delta$. We denote by Y the archimedean class containing x_1x_2. By Lemma 9, $Y\delta = X_1\delta \wedge X_2\delta = A\delta$ and so Y is a periodic archimedean class with idempotent, say e. If $x_1x_2x_3 \in Y$, then we clearly have $a = x_1x_2x_3 \, \mathcal{A} \, x_1x_2$. Suppose $x_1x_2x_3 \notin Y$. Then, since $e = ex_3 \in Y \cap YX_3$, YX_3 does not contained in a single archimedean class. Hence, by [3] Lemma 1.8, we have $X_3 < Y$, every element of Y is of order at most two, there exists an idempotent f of S such that $f < e$, $f \, \mathcal{L} \, e$ and e and f are consecutive in $e\mathcal{L}$, there exists a periodic archimedean class U with idempotent g which satisfies $g \, \mathcal{L} \, e$, $g < e$, $X_3 \leqq U$, X_3U, $UX_3 \subseteq U$, $Yg = \{f, e\}$ and $YX_3 \subseteq \{f\} \cup Y$. Since $x_1x_2x_3 \in YX_3$ and $x_1x_2x_3 \notin Y$, we have $x_1x_2x_3 = f$.

(a) The case: $X_1 \leqq X_2$.

Since $X_1\delta \wedge X_2\delta = A\delta$ is of L-type, X_1X_2 is contained in a single archimedean class by [3] Lemma 1.8. Hence, by Lemma 8, $X_1 \leqq Y = X_1 * X_2 \leqq X_2$. Since

$$X_1 * X_2 = \min\{ X \in \mathcal{C} \ ; \ X_1 \leqq X \leqq X_2 \text{ and } X \in X_1\delta \wedge X_2\delta = A\delta \},$$

there is no archimedean class $X \in A\delta$ such that $X_1 \leqq X < Y$. Since $e \in Y$ and $Y\delta = X_1\delta \wedge X_2\delta \leqq X_1\delta$, we have $ex_1 = e$ and so x_1e is an idempotent and $x_1e \, \mathcal{L} \, e$. Since S is nonnegatively ordered, e is the greatest element of Y and so $x_1 \leqq e$. Hence $x_1^2 \leqq x_1e \leqq e$ and so the archimedean class containing x_1e belongs to $A\delta$ and lies between X_1 and Y. Hence it coincides with Y and so $x_1e = e$. Now, since $g \, \mathcal{L} \, e$, we have $U\delta = Y\delta = A\delta \leqq X_3\delta$ and so $gx_3 = g$ by [3] Lemma 2.7. Hence x_3g is an idempotent and also $x_3g \in X_3U \subseteq U$. Hence $x_3g = g$ and so $x_1x_2g = x_1x_2x_3g = fg = f < e = x_1e$. Hence $x_2g < e = eg$ and so $x_2 < e$. Hence $X_2 \leqq Y \leqq X_2$ and so we obtain $X_2 = Y$. Since $X_3U \subseteq U$, it follows from Lemma 8 that $U = X_3 * U$. Also, since $X_3 \leqq U < Y = X_2$, $U\delta = A\delta$ and $(X_3 * X_2)\delta = X_3\delta \wedge X_2\delta = A\delta$,

$U = X_3 * U = \min\{ X \in \mathcal{C} ;\ X_3 \leqq X \leqq U$ and $X \in A\delta \}$

$= \min\{ X \in \mathcal{C} ;\ X_3 \leqq X \leqq X_2$ and $X \in A\delta \} = X_3 * X_2$

and so g is the idempotent of $X_3 * X_2$. Now e is the idempotent of X_2 and $x_2 g \neq e$, it follows from Lemma 12 that $x_2 x_3 \notin X_2$. Hence $X_2 X_3$ is not contained in a single archimedean class and, by [3] Lemma 1.8, we have $x_2 x_3 = f$ and so $x_2 x_3 = f = x_1 x_2 x_3 = a$.

(b) The case: $X_2 \leqq X_1$ and $X_1 X_2$ is contained in a single archimedean class.

We have $X_2 \leqq X_1 * X_2 \leqq X_1$ and, by Lemma 8, $Y = X_1 * X_2$. Also $x_1 x_2 x_3 = f < e = e x_2 x_3$ and so $x_1 < e$. Hence we have $X_1 \leqq X_1 * X_2$ and so $X_1 = X_1 * X_2$. If $x_2 \leqq x_3$, then

$$f = f x_2^2 \leqq x_1 x_2^2 \leqq x_1 x_2 x_3 = f$$

and so $x_1 x_2^2 = f \notin X_1$. Since $X_1\delta = (X_1 * X_2)\delta = X_1\delta \wedge X_2\delta \leqq X_2\delta$, it follows from Lemma 12 that $x_1 h \neq e$, where h is the idempotent of $X_2 * X_1$. Hence, again by Lemma 12, we have $x_1 x_2 \notin X_1$. But, by [3] Lemma 1.8, $X_1 X_2 \subseteq \{f\} \cup X_1$ and so $x_1 x_2 = f = x_1 x_2 x_3 = a$. If $x_3 \leqq x_2$, we can prove in a similar way that $a = x_1 x_3$.

(c) The case: $X_2 \leqq X_1$ and $X_1 X_2$ is not contained in a single archimedean class.

By [3] Lemma 1.8, X_1 is a periodic archimedean class with idempotent, say e_1, there exists an idempotent f_1 of S such that $f_1 < e_1$, $f_1 \mathcal{L} e_1$ and f_1 and e_1 are consecutive in $e_1\mathcal{L}$, there exists a periodic archimedean class T with idempotent k such that $k \mathcal{L} e_1$, $X_2 \leqq T$, TX_2, $X_2 T \subseteq T$ and $X_1 X_2 \subseteq \{f_1\} \cup X_1$. Hence $x_1 x_2 \in X_1 X_2 \subseteq \{f_1\} \cup X_1$. If $x_1 x_2 = f_1$, then f_1 is an idempotent of Y and so $x_1 x_2 = f_1 = e$. Hence $a = x_1 x_2 x_3 = e \neq f$, which is a contradiction. Hence $x_1 x_2 \in X_1$. Then we have $X_1 = Y$ and so $e_1 = e$, $f_1 = f$. Since $X_2 T \subseteq T$, we have $T = X_2 * T$ by Lemma 8 and so, since $T\delta = X_1\delta = Y\delta = A\delta = X_1\delta \wedge X_2\delta$ and $X_2 \leqq T < X_1$,

$$T = X_2 * T = \min\{ X \in \mathcal{C} \ ; \ X_2 \leqq X \leqq T \text{ and } X \in A\delta \}$$
$$= \min\{ X \in \mathcal{C} \ ; \ X_2 \leqq X \leqq X_1 \text{ and } X \in A\delta \} = X_2 * X_1.$$

Hence k is the idempotent of $X_2 * X_1$ and, since $x_1x_2 \in X_1$, we have $x_1k = e$ by Lemma 12. Hence, again by Lemma 12, $x_1x_2^2 \in X_1 = Y$ and so $x_1x_2x_3 = f < x_1x_2^2$. Hence we have $x_3 < x_2$ and so $x_1x_3^2 \leqq x_1x_2x_3 = f < e$. Since $X_1\delta = A\delta \preceq X_3\delta$, it follows from [3] Theorem 2.7 that $ex_3 = e \in X_1X_3 \cap X_1$. Hence X_1X_3 is not contained in a single archimedean class and so, by [3] Lemma 1.8, $X_1X_3 \subseteq \{f\} \cup X_1$. Since $x_1x_3^2 \leqq f$, we have $x_1x_3^2 \notin X_1$ and so $x_1h \neq e$ by Lemma 12, where h is the idempotent of $X_3 * X_1$. Hence, again by Lemma 12, $x_1x_3 \notin X_1$. Since $x_1x_3 \in X_1X_3 \subseteq \{f\} \cup X_1$, we have $x_1x_3 = f = x_1x_2x_3 = a$.

THEOREM 14. *In a nonnegatively ordered semigroup* S, *let* $a = x_1 \ldots x_n$ *with* $n \geqq 2$ *and let* $X_1, \ldots, X_n$ *and* A *be archimedean classes containing* $x_1, \ldots, x_n$ *and* a, *respectively. If* a *is an element of finite order, then* $a \mathcal{A} x_ix_j$ *for some* i, j *such that* $1 \leqq i < j \leqq n$.

PROOF. If $n = 2$, the assertion is trivial. If $n = 3$, then the assertion is given by Lemma 13. If $n > 3$, then put $y = x_3 \ldots x_n$. Then $a = x_1x_2y$ and, by Lemma 13, $a \mathcal{A} x_1x_2$ or $a \mathcal{A} x_2y = x_2x_3 \ldots x_n$ or $a = x_1y = x_1x_3 \ldots x_n$. Now we obtain the assertion by induction hypothesis.

COROLLARY 15 ([1] Théorème 1). *In an ordered idempotent semigroup* S, *the product of a finite number of elements of* S *is equal to a product of at most two of these factors.*

PROOF. If S is an ordered idempotent semigroup, each element is of finite order and each archimedean class is constituted by a single element. Hence the corollary follows from Theorem 14.

COROLLARY 16. *In a nonnegatively ordered semigroup* S, *the product of a finite number of elements of* S *is archimedean equivalent to a product of at most two of these factors.*

PROOF. The corollary follows from Corollary 10 and Theorem 14.

Example 17. Let S be an ordered semigroup consisting of seven elements $e < a < u < g < v < b < f$ with the multiplication table:

	e	a	u	g	v	b	f
e	e	e	e	e	e	e	e
a	e	e	e	e	a	u	g
u	e	a	u	g	g	g	g
v	g	g	g	g	v	b	f
b	g	v	b	f	f	f	f
f	f	f	f	f	f	f	f

In S, $A = \{e, a\}$, $U = \{u\}$, $G = \{g\}$, $V = \{v\}$ and $B = \{b, f\}$ are archimedean classes and $A\delta = G\delta = B\delta$, $A\delta \prec U\delta$, $A\delta \prec V\delta$. Since $ab = u$, Lemma 9 does not hold in general without the assumption that S is nonnegatively ordered. Also we have $ab^2 = g$ but neither $ab = u \mathcal{A} g$ nor $b^2 = f \mathcal{A} g$. Hence Lemma 13 does not hold in general without the assumption that S is nonnegatively ordered.

4. In [7] Žožikašvili and Loginov proved that in an ordered semigroup S which satisfies the conditions (1) for $x, y, z \in S$, $x < y$ implies $xz < yz$ and $zx < zy$, and (2) $xz > x$ and $zx > x$ for every $x, z \in S$ such that z is not the identity of S, the subsemigroup generated by a well-ordered subset of S is also a well-ordered subset of S.

As an application of the preceding section, in this section we extend the result of Žožikašvili and Loginov for nonnegatively ordered semigroups. The proof is carried out in a similar way to that in [7].

In this section, we denote by S a nonnegatively ordered semigroup.

LEMMA 18. If M_1 and M_2 are well-ordered subsets of S, then the set product M_1M_2 is also a well-ordered subset of S.

PROOF. By way of contradiction, we assume M_1M_2 is not well ordered. Then there exists an infinite sequence

$$x_1 > x_2 > x_3 > \cdots$$

of elements of M_1M_2. Since $x_i \in M_1M_2$, we have $x_i = x_{i1}x_{i2}$ for some $x_{i1} \in M_1$ and $x_{i2} \in M_2$. Since M_1 is well-ordered, $\{ x_{i1} ;\ i = 1, 2, 3, \ldots \}$ has the least element y_1. We put $I_1 = \{ i ;\ x_{i1} = y_1 \}$. By way of contradiction, we assume I_1 is infinite and consists of $i_{11} < i_{12} < i_{13} < \ldots$. Then

$$x_{i_{11}} = x_{i_{11}1}x_{i_{11}2} = y_1x_{i_{11}2} > x_{i_{12}} = x_{i_{12}1}x_{i_{12}2} = y_1x_{i_{12}2}$$
$$> x_{i_{13}} = x_{i_{13}1}x_{i_{13}2} = y_1x_{i_{13}2} > \ldots,$$

whence we have an infinite sequence $x_{i_{11}2} > x_{i_{12}2} > x_{i_{13}2} > \cdots$ of elements of M_2, which contradicts the fact that M_2 is well-ordered. Hence I_1 is finite and so we can take $n_1 = \max I_1$. Then if $i > n_1$, then $i \notin I_1$ and so $x_{i1} > y_1$. Also we have $x_{n_1} > x_i$. Hence

$$y_1x_{n_12} = x_{n_11}x_{n_12} = x_{n_1} > x_i = x_{i1}x_{i2} \geqq y_1x_{i2}$$

and so $x_{n_12} > x_{i2}$. Thus we have shown that $x_{n_12} > x_{i2}$ for every $i > n_1$.

Now suppose that m is a natural number such that $x_{m2} > x_{i2}$ for every $i > m$. Since M_1 is well-ordered, $\{ x_{i1} ; i \geqq m + 1 \}$ has the least element y_m. We put

$$I_m = \{ i ; i \geqq m + 1 \text{ and } x_{i1} = y_m \}.$$

By way of contradiction, we assume I_m is infinite and consists of elements $i_{m1} < i_{m2} < i_{m3} < \ldots.$ Then

$$x_{i_{m1}} = x_{i_{m1}1}x_{i_{m1}2} = y_m x_{i_{m1}2} > x_{i_{m2}} = x_{i_{m2}1}x_{i_{m2}2} = y_m x_{i_{m2}2}$$
$$> x_{i_{m3}} = x_{i_{m3}1}x_{i_{m3}2} = y_m x_{i_{m3}2} > \ldots,$$

whence we have an infinite sequence $x_{i_{m1}2} > x_{i_{m2}2} > x_{i_{m3}2} > \ldots$ of elements of M_2, which contradicts the fact that M_2 is well-ordered. Hence I_m is finite and so we can take $n_m = \max I_m$. For $i > n_m$, we have $i \notin I_m$ and also $i > n_m \geqq m + 1$ and so and so $x_{i1} > y_m$. Also we have $x_{n_m} > x_i$. Hence

$$y_m x_{n_m 2} = x_{n_m 1}x_{n_m 2} = x_{n_m} > x_i = x_{i1}x_{i2} \geqq y_m x_{i2}$$

and so $x_{n_m 2} > x_{i2}$. Also, since $n_m \geqq m + 1 > m$, we have $x_{m2} > x_{n_m 2}$. Thus we have shown that there exists a natural number n_m such that $n_m > m$, $x_{m2} > x_{n_m 2}$ and $x_{n_m 2} > x_{i2}$ for every $i > n_m$. Hence we obtain an infinite sequence $x_{n_1 2} > x_{n_2 2} > x_{n_3 2} > \ldots$ of elements of M_2, which contradicts that M_2 is well-ordered. This proves that $M_1 M_2$ is well-ordered.

From Lemma 18, we have, by induction

COROLLARY 19. *If* $M_1, M_2, \ldots, M_k$ *are well-ordered subsets of* S, *then* $M_1 M_2 \ldots M_k$ *is also a well-ordered subset of* S.

LEMMA 20. *Let* M *be a well-ordered subset of* S, *let* L *be the subsemigroup generated by* M *and let* $\mathcal{B}$ *be the set of all archimedean classes of the semigroup* L. *Then* $\mathcal{B}$ *is a well-ordered set.*

PROOF. Let $\mathcal{B}'$ be a nonempty subset of $\mathcal{B}$. Let $B \in \mathcal{B}'$ and let $x \in B$. Then, since $x \in B \subseteq L$, there exists a finite number of elements $x_1, x_2, \ldots, x_n$ of M such that $x = x_1x_2\ldots x_n$. By Corollary 16, there exists $y \in M \cup M^2$ such that $x \mathcal{A} y$. Thus we have shown that each archimedean class B in $\mathcal{B}'$ contains a representative $x(B) \in M \cup M^2$. Then $X = \{ x(B) ; B \in \mathcal{B}' \}$ is a nonempty subset of $M \cup M^2$. But $M \cup M^2$ is a well-ordered subset of S by Lemma 18 and so there exists $B_0 \in \mathcal{B}'$ such that $x(B_0) = \min X$. Then clearly B_0 is the least element of $\mathcal{B}'$. This proves that $\mathcal{B}$ is well-ordered.

<u>LEMMA 21</u>. <u>Let</u> M <u>be a well-ordered subset of</u> S <u>and let</u> L <u>be the subsemigroup generated by</u> M. <u>Then every archimedean class of the semigroup</u> L <u>is a well-ordered subset of</u> S.

PROOF. By way of contradiction, we assume that there exists an archimedean class of L which is not well-ordered. Then, by Lemma 20, there exists the least archimedean class X which is not well-ordered. As above we denote by $\mathcal{B}$ the set of all archimedean classes of the semigroup L. Thus, if $Y \in \mathcal{B}$ and $Y < X$, then Y is well-ordered.

We put $U = \cup \{ Y \in \mathcal{B} ; Y < X \}$.

(a) If $U \neq \square$, then U is a subsemigroup of S.

In fact, let $y, z \in U$. Then $y \in Y$ and $z \in Z$ for some $Y, Z \in \mathcal{B}$ such that $Y < X$ and $Z < X$. If $y \leqq z$, then $yz \leqq z^2 \in Z$ and so $yz \in W$ for some $W \in \mathcal{B}$ such that $W \leqq Z < X$, whence $yz \in U$. If $z \leqq y$, then we can obtain $yz \in U$ in a similar way.

Similarly we can prove

(b) $X \cup U$ is a subsemigroup of S.

(c) If $U \neq \square$, then U is a well-ordered subset of S.

In fact, let V be a nonempty subset of U and let

$\mathcal{B}' = \{ B \in \mathcal{B} ;\ B \cap V \neq \Box \}$. Then $\mathcal{B}'$ is a nonempty subset of $\mathcal{B}$. Hence, by Lemma 20, we can take $B_0 = \min \mathcal{B}'$. Then $B_0 \in \mathcal{B}'$ and, by the definition of U, $B_0 < X$ and so, by assumption, B_0 is well-ordered. Hence we can take $b_0 = \min(B_0 \cap V)$. Now it is clear that b_0 is the least element of V.

Put $T = M \cup UM \cup MU \cup UMU \cup M^2 \cup UM^2 \cup M^2U \cup UM^2U$.

(d) Every element $x \in X$ can be written in the form $x = t_1t_2\ldots t_m$ with $t_1, t_2, \ldots , t_m \in T \cap X$.

In fact, let $x \in X$. Then, since $x \in X \subseteq L$, $x = x_1x_2\ldots x_k$ for some $x_1, x_2, \ldots , x_k \in M$. By Corollary 16, there exists $y \in M \cup M^2$ such that $x \mathcal{A} y$. Then $y \in X$. Also, since S is nonnegatively ordered, there exists a natural number s such that $x \leqq y^s$. We denote by X_i the archimedean class containing the element x_i. Suppose $X < X_i$ for some i. Then we have $x \leqq y^s < x_i$. Putting $p = x_1\ldots x_{i-1}$ and $q = x_{i+1}\ldots x_k$ (p or q may be the empty symbol), we have $px_iq = x \leqq y^s < x_i$. Since S is nonnegatively ordered, we have $p \leqq p^2$ and $q \leqq q^2$ and so

$$px_iq \leqq p^2x_iq^2 = pxq \leqq py^sq \leqq px_iq.$$

Hence $x = px_iq = py^sq$. Applying this procedure several times, we obtain an expression $x = y_1y_2\ldots y_h$ with $y_1, y_2, \ldots , y_h \in (M \cup M^2) \cap (X \cup U)$. Finally, by (a) and (b), we obtain an expression $x = t_1t_2\ldots t_m$ with $t_1, t_2, \ldots , t_m \in T \cap X$.

Now we return to the proof of the lemma. Since the archimedean class X of L is not well-ordered, there exists an infinite sequence $z_1 < z_2 < z_3 < \ldots$ of elements of X. Since X contains at most one idempotent and, if X contains an idempotent, then the idempotent is the greatest element of X, we can assume that each one of $z_1,\ z_2,\ z_3, \ldots$ is not an idempotent. Since U and M are well-ordered subsets of S, it follows from Corollary 19 that T is well-ordered. Hence $T \cap X$ contains the least element t_0. Then, since z_1 is not an idempotent and z_1 and t_0 are archimedean equivalent, there exists a natural number k such that $z_1 < t_0^k$. By (d), for each z_i, there is a representation $z_i = t_1 t_2 \ldots t_m$ with $t_1, t_2, \ldots, t_m \in T \cap X$. If $k < m$ were true, then $t_0^{k+1} \leqq t_0^m \leqq t_1 t_2 \ldots t_m = z_i \leqq z_1 < t_0^k$, which contradicts the fact that S is nonnegatively ordered. Hence $m \leqq k$ and so $z_i \in \bigcup_{j=1}^k T^j$. But, by Corollary 19, T is well-ordered and so also T^j is well-ordered for every j such that $1 \leqq j \leqq k$. Hence $\bigcup_{j=1}^k T^j$ is a well-ordered subset of S and also contains an infinite sequence $z_1 < z_2 < z_3 < \ldots$, which is a contradiction. This proves Lemma 21.

THEOREM 22. *In a nonnegatively ordered semigroup* S, *let* M *be a well-ordered subset and let* L *be the subsemigroup of* S *generated by* M. *Then* L *is also a well-ordered subset of* S.

PROOF. Suppose $\Box \neq N \subseteq L$. We denote by $\mathcal{B}$ the set of all archimedean classes of the semigroup L. We put

$$\mathcal{B}' = \{ X \in \mathcal{B} \ ; \ X \cap N \neq \Box \}.$$

Then, by Lemma 20, there exists $X_0 = \min \mathcal{B}'$ and $X_0 \cap N \neq \Box$. Also, by Lemma 21, there exists $x_0 = \min(X_0 \cap N)$. Then clearly x_0 is the least element of N. Hence L is well-ordered.

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