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CERTAIN RIGHT REGULAR BANDS

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0. Abstract. A band is called a <u>right regular band</u> if it satisfies xyx = yx. Theorem 1 in Kimura [2] proved that a right regular band is a semilattice of right zero semigroups. Hence the fine structure of a right regular band is obtained by constructing all $\mathcal{S}$-compositions of a system of right zero semigroups. Theorem 4.3 in Yamada [6] discussed its problem using the concept of the free product. We shall discuss the same problem by more effective method. Furthermore we shall describe certain right regular bands. We use the natations of Clifford and Preston [1] without comment.

1. Preliminaries. Let S be a semigroup. $\Lambda=\{\lambda\}$ $[P=\{\rho\}$, $\Lambda^{(0)}$, $P^{(0)}]$ denote the full left [right, inner left, inner right] translation semigroup of S. A structure semilattice means a lower semilattice. $\underline{1}$ denotes the identity mapping. A <u>chain</u> C means a semilattice in which $\beta\gamma = \beta$ or γ in C.

To each σ in a semilattice [chain] Γ assign a pairwise disjoint semigroup $S_\sigma=\{x_\sigma, y_\sigma, \cdots\}$. Let $S=\bigcup\{S_\sigma\colon \sigma\in\Gamma\}$. If S($\circ$) becomes a semigroup by defining a composite ($\circ$) by

$$x_\sigma \circ y_\tau \begin{cases} =x_\sigma y_\sigma & \text{if } \sigma=\tau, \\ \in S_{\sigma\tau} & \text{otherwise,} \end{cases}$$

then S($\circ$) is said to be a <u>semilattice</u> [<u>chain</u>] or an $\mathcal{S}$- [$\mathcal{C}$-] <u>composition of a system</u> $\{S_\sigma\colon \sigma\in\Gamma\}$ <u>of semigroups</u>.

An I- [V-, Y-] <u>semilattice</u> means a semilattice having

	δ	γ	β	α
δ	δ	δ	δ	δ
γ	δ	γ	γ	γ
β	δ	γ	β	γ
α	δ	γ	γ	α

Table 1

the structure isomorphic with $\{\delta,\gamma,\alpha\}$ [$\{\gamma,\beta,\alpha\}$, $\{\delta,\gamma,\beta,\alpha\}$] defined by Table 1. An $\mathcal{S}_I$- [$\mathcal{S}_V$-, $\mathcal{S}_Y$-] composition means an $\mathcal{S}$-composition of which the structure semilattice is I- [V-, Y-] semilattice. We write $\rho \mathcal{L}k \lambda$ if λ and ρ are linked.

2. $\mathcal{S}_I$-compositions. Let S_β and S_γ be any semigroups. If to each element t of S_β we let correspond a left translation $\lambda_\gamma(t)$ and a right translation $\rho_\gamma(t)$ of S_γ, respectively, such that the following conditions (C), then $\{(\lambda_\gamma(t), \rho_\gamma(t)): t \in S_\beta\}$ is called a <u>system of strictly linked translations</u>:

$$\text{(C)} \quad \begin{cases} \lambda_\gamma(t_2)\ \lambda_\gamma(t_1)=\lambda_\gamma(t_1t_2), & \rho_\gamma(t) \mathcal{L}k \lambda_\gamma(t), \\ \rho_\gamma(t_1)\rho_\gamma(t_2)=\rho_\gamma(t_1t_2), & \lambda_\gamma(t_1)\rho_\gamma(t_2)=\rho_\gamma(t_2)\lambda_\gamma(t_1). \end{cases}$$

Theorem 2.1. [7, Theorem 2.1] Let C be a chain, and to every element α of C assign any semigroup $S_\alpha=\{a_\alpha, b_\alpha, \cdots\}$. Let $S = \bigcup\{S_\alpha:\ \alpha \in C\}$. Assume that the following conditions hold:

(2.1.1) To each pair $\gamma,\beta \in C$, $\gamma<\beta$, there is a system of strictly linked translations.

(2.1.2) If $\delta<\gamma<\alpha$, their translations satisfy the next equations:

$$\begin{cases} \lambda_\delta(g_\gamma\cdot\rho_\gamma(b_\alpha))=\lambda_\delta(b_\alpha)\lambda_\delta(g_\gamma), & \lambda_\delta(g_\gamma\cdot\lambda_\gamma(a_\alpha))=\lambda_\delta(g_\gamma)\lambda_\delta(a_\alpha), \\ \rho_\delta(g_\gamma\cdot\lambda_\gamma(b_\alpha))=\rho_\delta(b_\alpha)\rho_\delta(g_\gamma), & \rho_\delta(g_\gamma\cdot\rho_\gamma(a_\alpha))=\rho_\delta(g_\gamma)\rho_\delta(a_\alpha), \\ \lambda_\delta(g_\gamma)\rho_\delta(a_\alpha)=\rho_\delta(a_\alpha)\lambda_\delta(g_\gamma), & \lambda_\delta(a_\alpha)\rho_\delta(g_\gamma)=\rho_\delta(g_\gamma)\lambda_\delta(a_\alpha). \end{cases}$$

Then $S(\circ)$ becomes a $\mathcal{C}$-composition of $\{S_\alpha:\ \alpha\in C\}$ by defining a composite $(\circ)$ in S as follows:

$$\text{(2.1.3)} \qquad y_\tau\circ u_\sigma = \begin{cases} u_\sigma\cdot\lambda_\sigma(y_\tau) & \text{if } \sigma<\tau, \\ y_\tau\cdot\rho_\tau(u_\sigma) & \text{if } \tau<\sigma, \\ y_\sigma u_\sigma\ . & \text{if } \sigma=\tau. \end{cases}$$

Conversely, every $\mathcal{C}$-composition of $\{S_\sigma:\ \sigma\in C\}$ can be found in this way.

Theorem 2.1 is rewrited in the following for a system of left reductive semigroups. $P^{(k)}$ denotes the set of all right translations having a linked left translation. Δ denotes the correspondence which prescribes for each ρ in $P^{(k)}$ left translations linked with ρ.

(2.2) If a semigroup S is left reductive, then the correspondence Δ is one-valued.

Corollary 2.3. [8, Corollary 2'] To every α of a chain C, assign a left reductive semigroup S_α. Let Δ_α be the mapping of $P_\alpha^{(k)}$ into Λ_α described in (2.2). Then every $\mathcal{C}$-composition of $\{S_\alpha : \alpha \in C\}$ is completely determined by homomorphisms $\psi_{\beta,\gamma}$, $\gamma < \beta$, of S_β into $P_\gamma^{(k)}$ satisfying

$$\left.\begin{array}{ll} (2.3.1) & [g_\gamma \cdot (a_\alpha \cdot \psi_{\alpha,\gamma})] \cdot \psi_{\gamma,\delta} = (g_\gamma \cdot \psi_{\gamma,\delta})(a_\alpha \cdot \psi_{\alpha,\delta}) \\ (2.3.2) & [g_\gamma \cdot \{(a_\alpha \cdot \psi_{\alpha,\gamma}) \cdot \Delta_\gamma\}] \cdot \psi_{\gamma,\delta} = (a_\alpha \cdot \psi_{\alpha,\delta})(g_\gamma \cdot \psi_{\gamma,\delta}) \end{array}\right\} \text{ for } \delta < \gamma < \alpha \text{ in } C.$$

Now we shall treat a system of right zero semigroups.

(2.4) Let S be a right zero semigroup. Then

(i) $\Lambda = \{\underline{1}\}$ and $P = \mathcal{T}_S$;

(ii) $\alpha \not{k} \underline{1}$ for all $\alpha \in \mathcal{T}_S$.

(2.5) Let C be a chain. For every $\alpha \in C$ let S_α be a right zero semigroup. Let $S = \bigcup \{S_\alpha : \alpha \in C\}$. To each pair $\delta < \gamma$ of C, we let correspond a homomorphism $\chi_{\gamma,\delta}$ of S_γ into $\mathcal{T}_{S_\delta}$ satisfying

$$\left.\begin{array}{ll} (2.5.1) & [g_\gamma \cdot (a_\alpha \cdot \chi_{\alpha,\gamma})] \cdot \chi_{\gamma,\delta} = (g_\gamma \cdot \chi_{\gamma,\delta})(a_\alpha \cdot \chi_{\alpha,\delta}) \\ (2.5.2) & g_\gamma \cdot \chi_{\gamma,\delta} = (a_\alpha \cdot \chi_{\alpha,\delta})(g_\gamma \cdot \chi_{\gamma,\delta}) \end{array}\right\} \text{ for } \delta < \gamma < \alpha \text{ in } C.$$

Then $S(\circ)$ becomes a $\mathcal{C}$-composition of $\{S_\alpha : \alpha \in C\}$ by defining a product $(\circ)$ in S as follows:

$$(2.5.3) \qquad y_\tau \circ u_\sigma = \begin{cases} y_\tau \cdot (u_\sigma \cdot \chi_{\sigma,\tau}) & \text{if } \tau < \sigma, \\ u_\sigma & \text{if } \tau \geqq \sigma. \end{cases}$$

Conversely every $\mathcal{C}$-composition of $\{S_\alpha : \alpha \in C\}$ can be obtained in this way.

Hereafter we shall consider a partition π of a semigroup S as the equivalence relation on S defined by $s\pi t$ $(s,t\in S)$ if s and t belong to the same member of π.

(2.6) Let S_γ and S_δ be two right zero semigroups. Then we take a partition $\pi_{\gamma,\delta}$ of S_δ where $\pi_{\gamma,\delta}: S_\delta = \bigcup\{S_\mu : \mu\in M\}$. To every $g\in S_\gamma$, fix an element $\bar{u}_\mu(g)$ from each equivalence class S_μ of S_δ mod $\pi_{\gamma,\delta}$. We define $\chi_{\gamma,\delta}$ by $u_\mu\cdot(g\cdot\chi_{\gamma,\delta})=\bar{u}_\mu(g)$ for all $u_\mu\in S_\mu$. Then $\chi_{\gamma,\delta}$ is a homomorphism of S_γ into $\mathcal{J}_{S_\delta}$.

Conversely every homomorphism $\chi_{\gamma,\delta}$ of S_γ into $\mathcal{J}_{S_\delta}$ can be obtained in this fashion.

Theorem 2.7. Let C be a chain. For each $\alpha\in C$ assign a pairwise disjoint right zero semigroup S_α.

(i) For every pair γ,α in C, $\gamma<\alpha$, we take a partition $\pi_{\alpha,\gamma}$ of S_γ such that

(2.7.1) if $\delta<\gamma<\alpha$ then $\pi_{\gamma,\delta} \geq \pi_{\alpha,\delta}$;

that is, let $\pi_{\alpha,\gamma}$: $\{S_\xi : \xi\in\Xi\}$, $\pi_{\gamma,\delta}$: $\{S_\mu : \mu\in M\}$ and $\pi_{\alpha,\delta}$: $\{S_{\nu(\mu)} : \nu(\mu)\in N(\mu),\ \mu\in M\}$.

(ii) For each $a\in S_\alpha$ [$g\in S_\gamma$, $a\in S_\alpha$], let $\{\bar{z}_\xi(a) : \xi\in\Xi\}$ [$\{\bar{u}_\mu(g) : \mu\in M\}$, $\{\bar{\bar{u}}_{\nu(\mu)}(a) : \nu(\mu)\in N(\mu),\ \mu\in M\}$] be fixed elements from equivalence classes S_ξ [S_μ, $S_{\nu(\mu)}$] for indices in (i). Suppose that they have the next connection:

(2.7.2) If $g\ \pi_{\alpha,\gamma}\ h$, then $\bar{u}_\mu(g)\ \pi_{\alpha,\delta}\ \bar{u}_\mu(h)$ for all $\mu\in M$.

(2.7.3) Let $\mu\in M$ and $a\in S_\alpha$. Then

$$\bar{u}_\mu(\bar{z}_\xi(a))\ \pi_{\alpha,\delta}\ \bar{u}_\mu(\bar{z}_{\xi'}(a)) \text{ implies } \bar{u}_\mu(\bar{z}_\xi(a)) = \bar{u}_\mu(\bar{z}_{\xi'}(a)).$$

(2.7.4) Let $a\in S_\alpha$. If $u_\mu(z_\xi(a))\in S_{\nu(\mu)}$, then $\bar{\bar{u}}_{\nu(\mu)}(a)=u_\mu(z_\xi(a))$.

Then we can construct a $\mathscr{C}$-composition of $\{S_\alpha : \alpha \in C\}$.

Conversely every $\mathscr{C}$-composition of $\{S_\alpha: \alpha \in C\}$ can be so constructed.

Proof. Sufficiency. Define $\chi_{\alpha,\gamma}$, $\chi_{\gamma,\delta}$, $\chi_{\alpha,\delta}$ by

$$(2.7.5)\quad \begin{cases} z_\xi \cdot (a \cdot \chi_{\alpha,\gamma}) = \bar{z}_\xi(a),\ z_\xi \in S_\xi; \quad u_\mu \cdot (g \cdot \chi_{\gamma,\delta}) = \bar{u}_\mu(g),\ u_\mu \in S_\mu; \\ \qquad u_{\nu(\mu)} \cdot (a \cdot \chi_{\alpha,\delta}) = \bar{\bar{u}}_{\nu(\mu)}(a),\ u_{\nu(\mu)} \in S_{\nu(\mu)}. \end{cases}$$ ∎

3. $\mathscr{S}_V$-compositions. Let $\Gamma = \{\gamma, \beta, \alpha\}$ be a V-semilattice. To each element of Γ, assign mutually disjoint any semigroup $S_\gamma = \{g, h, \ldots\}$, $S_\beta = \{c, d, \ldots\}$, $S_\alpha = \{a, b, \ldots\}$.

Theorem 3.1. [7, Theorem 2.2] Let $S = S_\gamma \cup S_\beta \cup S_\alpha$. Assume that the following conditions hold.

(3.1.1) There is a system of strictly linked translations between S_α and S_γ [S_β and S_γ].

(3.1.2) There is a mapping θ [ψ] of $S_\beta \times S_\alpha$ [$S_\alpha \times S_\beta$] into S_γ.

(3.1.3) The connection between their translations and their mappings is given by

$$\begin{cases} \lambda(c)\lambda(a) = \lambda_{(a,c)\cdot\psi}, & \lambda(a)\lambda(c) = \lambda_{(c,a)\cdot\theta}, \\ \rho(a)\rho(c) = \rho_{(a,c)\cdot\psi}, & \rho(c)\rho(a) = \rho_{(c,a)\cdot\theta}, \end{cases}$$

$$\lambda(c)\rho(a) = \rho(a)\lambda(c), \qquad \lambda(a)\rho(c) = \rho(c)\lambda(a),$$

$$\begin{cases} (ab,c)\cdot\psi = [(b,c)\cdot\psi]\cdot\lambda(a),\ (c,ab)\cdot\theta = [(c,a)\cdot\theta]\cdot\rho(b), \\ \qquad [(a,c)\cdot\psi]\cdot\rho(b) = [(c,b)\cdot\theta]\cdot\lambda(a), \end{cases}$$

$$\begin{cases} (cd,a)\cdot\theta = [(d,a)\cdot\theta]\cdot\lambda(c),\ (a,cd)\cdot\psi = [(a,c)\cdot\psi]\cdot\rho(d), \\ \qquad [(a,d)\cdot\psi]\cdot\lambda(c) = [(c,a)\cdot\theta]\cdot\rho(d). \end{cases}$$

Then we can construct an $\mathscr{S}_V$-composition of $\{S_\gamma, S_\beta, S_\alpha\}$.

Conversely every $\mathscr{S}_V$-composition of $\{S_\gamma, S_\beta, S_\alpha\}$ can be so constructed.

Proof. Sufficiency. Define a composite ($\circ$) by

$$(3.1.4)\quad \begin{cases} c \circ a = (c,a)\cdot\theta, & a \circ c = (a,c)\cdot\psi, \\ a \circ g = g\cdot\lambda(a),\quad c \circ g = g\cdot\lambda(c), & g \circ a = g\cdot\rho(a),\quad g \circ c = g\cdot\rho(c) \end{cases}$$ ∎

We have the next theorem for a system of right zero semigroups.

Theorem 3.2. Assign mutually disjoint right zero semigroups S_γ, S_β, S_α. We take two partitions of S_γ:

(3.2.1) $\pi_{\alpha,\gamma}$: $S_\gamma = \bigcup\{S_\xi : \xi \in \Xi\}$, (3.2.2) $\pi_{\beta,\gamma}$: $S_\gamma = \bigcup\{S_\eta : \eta \in H\}$.

For any element a in S_α, we fix $\bar{z}_\xi(a)$ from each S_ξ such that

(3.2.3) there is $\eta(a) \in H$ satisfying $\bar{z}_\xi(a) \in S_{\eta(a)}$ for all $\xi \in \Xi$.

For any c in S_β, we fix $\bar{z}_\eta(c)$ from each S_η such that

(3.2.4) there is $\xi(c) \in \Xi$ satisfying $\bar{z}_\xi(c) \in S_{\xi(c)}$ for all $\eta \in H$.

Then we can construct an $\mathscr{S}_V$-composition $S(\circ) = S_\gamma \cup S_\beta \cup S_\alpha$.

Conversely every $\mathscr{S}_V$-composition of $\{S_\gamma, S_\beta, S_\alpha\}$ can be so constructed.

4. $\mathscr{S}$-compositions. Let $\Gamma = \{\alpha, \beta, \gamma, \delta, \cdots, \sigma, \cdots\}$ be any semilattice. $\mathscr{H}(S)$ denotes the translational hull of a semigroup S and let $\mathscr{D}(S) = \{(\lambda_s, \rho_s) : s \in S\}$ be the translational diagonal of S. The following theorem is due to M. Petrich.

Theorem 4.1. [5, Theorem 7.8.13] For every σ in a semilattice Γ, let $S_\sigma = \{a_\sigma, b_\sigma, c_\sigma, \cdots\}$ be a weakly reductive semigroup where S_σ are pairwise disjoint. For every pair $\delta, \gamma \in \Gamma$, $\delta \leqq \gamma$, let $\bar{\chi}_{\gamma,\delta}$ be a mapping of S_γ into $\mathscr{H}(S_\delta)$ satisfying

(4.1.1) $s_\sigma \cdot \bar{\chi}_{\sigma,\sigma} = (\lambda_{s_\sigma}, \rho_{s_\sigma})$ for all $s_\sigma \in S_\sigma$,

(4.1.2) if $\alpha \not\leqq \beta$, $\beta \not\leqq \alpha$, then $(S_\alpha \cdot \bar{\chi}_{\alpha,\alpha\beta})(S_\beta \cdot \bar{\chi}_{\beta,\alpha\beta}) \subseteq \mathscr{D}(S_{\alpha\beta})$,

(4.1.3) if $\delta < \alpha\beta$, then

$$[(a_\alpha \cdot \bar{\chi}_{\alpha,\alpha\beta})(c_\beta \cdot \bar{\chi}_{\beta,\alpha\beta})] \cdot \bar{\chi}_{\alpha\beta,\alpha\beta}^{-1} \bar{\chi}_{\alpha\beta,\delta} = (a_\alpha \cdot \bar{\chi}_{\alpha,\delta})(c_\beta \cdot \bar{\chi}_{\beta,\delta}).$$

Define a composite $(\circ)$ in $S = \bigcup\{S_\sigma : \sigma \in \Gamma\}$ by

(4.1.4) $$a_\alpha \circ c_\beta = [(a_\alpha \cdot \bar{\chi}_{\alpha,\alpha\beta})(c_\beta \cdot \bar{\chi}_{\beta,\alpha\beta})] \cdot \bar{\chi}_{\alpha\beta,\alpha\beta}^{-1}.$$

Then $S(\circ)$ becomes an $\mathscr{S}$-composition. Conversely every $\mathscr{S}$-composition of a system of weakly reductive semigroups can be so constructed.

(4.2) If S is a right zero semigroup, then $\mathcal{H}(S)$ is isomorphic onto $\mathcal{J}_S$ by $(\underline{1},\alpha)\mapsto\alpha$.

Theorem 4.3. Let Γ be a semilattice. To each $\sigma\in\Gamma$ assign a right zero semigroup S_σ, where S_σ are pairwise disjoint. To every pair $\tau,\sigma\in\Gamma$, $\tau<\sigma$, we take a partition $\pi_{\sigma,\tau}$ and fixed elements satisfying the following condition.

Let $\{\delta,\gamma,\beta,\alpha\}$ be a Y-subsemilattice of Γ. Let $S_\delta=\{u,\bar{u},\bar{\bar{u}},\cdots\}$, $S_\gamma=\{g,h,\cdots z,\bar{z},\cdots\}$, $S_\beta=\{c,\cdots\}$ and $S_\alpha=\{a,\cdots\}$. Assume that (4.3.1)-(4.3.3) hold.

(4.3.1) As in Theorem 3.2, we take two partitions $\pi_{\alpha,\gamma}$ and $\pi_{\beta,\gamma}$ of S_γ and fixed elements $\bar{z}_\xi(a)$ $[\bar{z}_\eta(c)]$ from each S_ξ $[S_\eta]$ satisfying (3.2.3) [(3.2.4)].

(4.3.2) We take a partition $\pi_{\gamma,\delta}$ of S_δ: $S_\delta=\bigcup\{S_\mu:\ \mu\in M\}$, and we fix $\bar{u}_\mu(g)\in S_\mu$ for $g\in S_\gamma$ and $\mu\in M$.

(4.3.3) As in Theorem 2.7, we take a partition $\pi_{\alpha,\delta}$ of S_δ satisfying (2.7.1)-(2.7.3). Similarly for $\pi_{\beta,\delta}$. We fix $\bar{\bar{u}}_{\nu(\mu)}(a)$ $\in S_{\nu(\mu)}$ as follows:

$$(4.3.3.1)\quad \bar{\bar{u}}_{\nu(\mu)}(a)\begin{cases} = \bar{u}_\mu(\bar{z}_\xi(a)) & \text{if } \bar{u}_\mu(\bar{z}_\xi(a))\in S_{\nu(\mu)}, \\ \in S_{\nu(\mu)}\cap[\bar{u}_\mu(\bar{z}_{\eta(a)}(c))\pi_{\beta,\delta}] & \text{otherwise}. \end{cases}$$

Similarly for $\bar{\bar{u}}_{\kappa(\mu)}(c)$. Then we can construct an $\mathcal{S}$-composition $S(\circ)=\bigcup\{S_\sigma:\ \sigma\in\Gamma\}$.

Conversely every $\mathcal{S}$-composition of a system of right zero semigroups can be found in this fashion.

5. Certain $\mathcal{C}$-compositions. We shall discuss certain $\mathcal{C}$-compositions of right zero semigroups.

Let C be a chain, $3\leq|C|$. To each $\alpha\in C$ assign a pairwise disjoint right zero semigroup S_α. Let $S(\circ)=\bigcup\{S_\alpha:\ \alpha\in C\}$ be a $\mathcal{C}$-composition. Let $\gamma_0\in C$. A (ω,ι,ω)- $[(\omega,\iota,\iota)$-, (ι,ι,ω)-, (ι,ω,ω)-$]$

right regular band means a $\mathscr{C}$-composition $S(\circ)$ such that $\pi_{\alpha,\delta}$, $\delta<\alpha$ [$\pi_{\alpha,\gamma}$, $\gamma<\alpha$, $\pi_{\gamma,\delta}$, $\delta<\gamma$, $\pi_{\alpha,\gamma}$, $\alpha<\gamma$] described in Theorem 2.7 satisfies $\pi_{\alpha,\delta}=\begin{cases}\iota_\delta & \text{if } \delta<\gamma_0<\alpha,\\ \omega_\delta & \text{otherwise}\end{cases}$ $\left[\pi_{\alpha,\gamma}=\begin{cases}\omega_\gamma & \text{if } \gamma_0\leqq\gamma<\alpha,\\ \iota_\gamma & \text{otherwise},\end{cases}\right.$

$\pi_{\gamma,\delta}=\begin{cases}\omega_\delta & \text{if } \delta<\gamma<\gamma_0,\\ \iota_\delta & \text{otherwise},\end{cases}$ $\left.\pi_{\alpha,\gamma}=\begin{cases}\iota_\gamma & \text{if } \gamma_0<\gamma<\alpha,\\ \omega_\gamma & \text{otherwise}.\end{cases}\right]$

Let $\delta,\gamma\in C$, $\delta<\gamma$. Fix $u_0\in S_\delta$. Define a constant mapping $\varphi_{\gamma,\delta}(u_0)$ by $g\cdot\varphi_{\gamma,\delta}(u_0)=u_0$ for all $g\in S_\gamma$.

Theorem 5.1. Let C be a chain. To every $\alpha\in C$ associate a right zero semigroup S_α, and suppose that S_α are pairwise disjoint. Let $S=\bigcup\{S_\alpha:\ \alpha\in C\}$. Fix $\gamma_0\in C$. Let $\{\varphi_{\alpha,\gamma}:\ \gamma_0\leqq\gamma<\alpha\}$ be an inductive system of mappings. Let $\{\varphi_{\delta,\gamma}:\ \delta<\gamma\leqq\gamma_0\}$ be an inductive system of mappings such that $\varphi_{\gamma_0,\delta}=\varphi_{\gamma_0,\delta}(u_0)$, $u_0\in S_\delta$. Let $u\in S_\delta$, $a\in S_\alpha$. Define a composite $(\circ)$ by

$$u\circ a=\begin{cases}u & \text{if } \delta<\gamma_0<\alpha,\\ a\cdot\varphi_{\alpha,\delta} & \text{otherwise}.\end{cases}$$

Then we can construct an (ω,ι,ω)-right regular band.

Conversely every (ω,ι,ω)-right regular band can be so constructed.

		u	v	⋯	k	l	⋯	g	h	⋯	c	d	⋯	a	b	⋯
S_δ	u v ⋮				$k\cdot\varphi_{\gamma,\delta}$	$l\cdot\varphi_{\gamma,\delta}$	⋯	u_0	u_0	⋯	u v ⋮	u v ⋮	⋯	u v ⋮	u v ⋮	⋯
S_γ	k l ⋮	u	v	⋯				k_0	k_0	⋯	k l ⋮	k l ⋮	⋯	k l ⋮	k l ⋮	⋯
S_{γ_0}	g h ⋮	u	v	⋯	k	l	⋯				$c\cdot\varphi_{\beta,\gamma_0}$	$d\cdot\varphi_{\beta,\gamma_0}$	⋯	$a\cdot\varphi_{\alpha,\gamma_0}$	$b\cdot\varphi_{\alpha,\gamma_0}$	⋯
S_β	c d ⋮	u	v	⋯	k	l	⋯	g	h	⋯				$a\cdot\varphi_{\alpha,\beta}$	$b\cdot\varphi_{\alpha,\beta}$	⋯
S_α	a b ⋮	u	v	⋯	k	l	⋯	g	h	⋯	c	d	⋯			

Table 2

Theorem 5.2. With same situation as in Theorem 5.1, let $\{\mathcal{G}_{\alpha,\gamma} : \gamma_0 \leqq \gamma < \alpha\}$ be an inductive system of mappings. Let $k \in S_\gamma$, $a \in S_\alpha$, $\gamma < \alpha$. We define a composite

$$(\circ) \text{ by } k \circ a = \begin{cases} a \cdot \mathcal{G}_{\alpha,\gamma} & \text{if } \gamma_0 \leqq \gamma < \alpha, \\ k & \text{otherwise.} \end{cases}$$

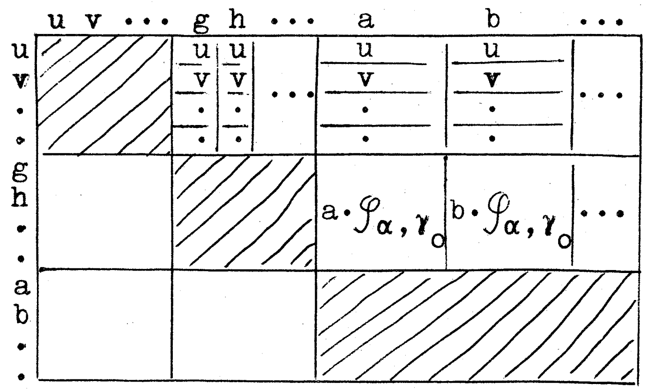

Table 3

Then we can construct an (ω, ι, ι)-right regular band.

Conversely every (ω, ι, ι)-right regular band can be so constructed.

Theorem 5.3. With same situation in Theorem 5.1, let $\{\mathcal{G}_{\gamma,\delta} : \delta < \gamma \leqq \gamma_0\}$ be an inductive system of mappings. Let $u \in S_\delta$, $k \in S_\gamma$, $\delta < \gamma$. We define a composite

$$(\circ) \text{ by } u \circ k = \begin{cases} k \cdot \mathcal{G}_{\gamma,\delta} & \text{if } \delta < \gamma \leqq \gamma_0, \\ u & \text{otherwise.} \end{cases}$$

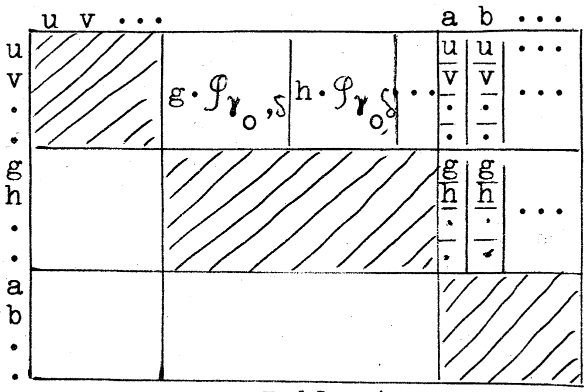

Table 4

Then we can construct an (ι, ι, ω)-right regular band.

Conversely every (ι, ι, ω)-right regular band can be so constructed.

Theorem 5.4. With same situation in Theorem 5.3, we take $\gamma_0 \in C$. Fix $u_0 \in S_\delta$ for all δ where $\delta < \gamma_0$.

Let $\delta < \gamma < \alpha < \gamma_0$. Let k_0, u_0 be the fixed element in S_γ, S_δ, respectively. Select an inductive system $\{\mathcal{G}_{\gamma,\delta} : \delta < \gamma < \gamma_0\}$ of mappings satisfying $k_0 \cdot \mathcal{G}_{\gamma,\delta} = u_0$. Let $u \in S_\delta$, $k \in S_\gamma$, $a \in S_\alpha$. Define a composite

$$(\circ) \text{ by } \begin{cases} u \circ a = u_0 & \text{if } \delta < \gamma_0 \leqq \alpha, \\ u \circ k = k \cdot \mathcal{G}_{\gamma,\delta} & \text{if } \delta < \gamma < \gamma_0, \\ k \circ a = k & \text{if } \gamma_0 \leqq \gamma < \alpha. \end{cases}$$

Then we can construct a (ι,ω,ω)-right regular band.

Conversely every (ι,ω,ω)-right regular band can be so constructed.

	u v ...	k l ...	g h ...	c d ...	a b ...
S_δ {u, v, ., .}		$k\cdot\mathcal{G}_{\gamma,\delta}$ | $l\cdot\mathcal{G}_{\gamma,\delta}$ | ...	u_0 | u_0 | ...	u_0 | u_0 | ...	u_0 | u_0 | ...
S_γ {k, l, ., .}			k_0 | k_0 | ...	k_0 | k_0 | ...	k_0 | k_0 | ...
S_{γ_0} {g, h, ., .}				g, h, ., . | g, h, ., . | ...	g, h, ., . | g, h, ., . | ...
S_β {c, d, ., .}					c, d, ., . | c, d, ., . | ...
S_α {a, b, ., .}					

Table 5

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Addenda. After I wrote this report, I read B.D.Arendt "Semisimple bands", Transaction Amer. Math. Soc. 143 (1969), 133-143. And I knew that my Theorem 2.7 and Theorem 3.2 have similar contents as Arendt's Theorem 27 and Theorem 24, respectively.