数理解析研究所講究録
107巻 1971年 78-108

An Implicit One-Step Method of High-Order Accuracy for the Numerical Integration of Ordinary Differential Equations

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(To appear in Numerische Mathematik)

Abstract. For a differential equation $dx/dt = f(t,x)$ with $f_t(t,x)$, $f_x(t,x)$ computable, the author presents a new one-step method of high-order accuracy. A rule of controlling the mesh-size is given and the method is compared with the Runge-Kutta method in two numerical examples.

Key words. Numerical integration. Ordinary differential equations. One-step method. Control of the mesh-size. Convergence of a numerical integration method.

1. Introduction. Let

$$\frac{dx}{dt} = f(t,x) \tag{1.1}$$

be a differential equation such that the function

$$(1.2) \qquad g(t,x) = f_t(t,x) + f_x(t,x)f(t,x)$$

can be evaluated on a computer. In the present paper we shall present a new one-step method of high-order accuracy for solving differential equations having the above property.

Let

$$(1.3) \qquad t_i = t_0 + ih \qquad (i = 0, 1, 2),$$

$$(1.4) \qquad f_i = f(t_i, x_i), \quad g_i = g(t_i, x_i) \qquad (i = 0, 1),$$

then our integration formula reads as follows:

$$(1.5) \qquad x_1 = L_1(x_0, f_0, f_1, \hat{f}_2, g_0, g_1, \hat{g}_2)$$
$$\triangleq x_0 + \frac{h}{240}(101f_0+128f_1+11\hat{f}_2)$$
$$+ \frac{h^2}{240}(13g_0-40g_1-3\hat{g}_2) \ ,$$

where

$$(1.6) \qquad \hat{f}_2 = f(t_2, \hat{x}_2), \qquad \hat{g}_2 = g(t_2, \hat{x}_2)$$

and

$$(1.7)\qquad \hat{x}_2 = L_2(x_0,\ x_1,\ f_0,\ f_1,\ g_0,\ g_1)$$

$$\triangleq -31x_0 + 32x_1$$

$$-h(14f_0+16f_1) + h^2(-2g_0+4g_1) \quad .$$

Suppose that $(t_0,\ x_0)$ is known. Then by means of a well-known iterative method, one can solve equation (1.5) numerically with respect to x_1, if necessary, using the formula

$$(1.8)\qquad x_1 = x_0 + hf_0 + \frac{h^2}{2}g_0$$

for starting the iterative process. The value of x_1 found gives a desired approximation to $x(t_1)$, where $x(t)$ is a solution of equation (1.1) such that $x(t_0) \approx x_0$. The value of $\hat{x}_2$ obtained in the course of solution of equation (1.5) gives an approximation to $x(t_2)$, but it is not adopted in our method as a final approximation to $x(t_2)$ since it is inferior in the accuracy. Once the value of x_1 has been found, one repeats the above process replacing t_1, x_1, f_1 and g_1 by t_0, x_0, f_0 and g_0 respectively, and so on. When one proceeds from any step to the next one, the value of $\hat{x}_2$ found in the first step gives an approximation to x_1 in the second step if the mesh size h is not changed. In such a case, clearly the formula (1.8) is unnecessary for starting the iterative process in the second step.

In the numerical solution of equation (1.5), solution x_1 should be computed as accurately as possible. As will be shown in 4, from this requirement follows a rule of controlling the mesh size h.

In the present paper, the derivation of the integration formula (1.5) will be given in 2 and the convergence of our one-step method will be proved in 3. A rule of controlling the mesh size h will be given in 4 and two numerical examples will be presented in 5.

Computations in the present paper have been carried out by the use of TOSBAC 3400 at Research Institute for Mathematical Sciences, Kyoto University. The author wishes to acknowledge the assistance of Mrs. S. Asako who has written the program.

2. The derivation of the integration formula. Let $x(t)$ be a solution of (1.1) and put

$$t_i = t_0 + ih \qquad (i=0,1,2\ ), \tag{2.1}$$

$$x_i = x(t_i) \qquad (i=0,1,2\ ). \tag{2.2}$$

If $x(t)$ is analytic in t, then we have the equality of the following form :

$$\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 \tag{2.3}$$

$$= h(\beta_0\dot{x}_0 + \beta_1\dot{x}_1 + \beta_2\dot{x}_2) + h^2(\gamma_0\ddot{x}_0 + \gamma_1\ddot{x}_1 + \gamma_2\ddot{x}_2) + T \qquad (\cdot = d/dt) ,$$

where T is of the form

$$T = Ch^p\left(\frac{d}{dt}\right)^p x(t_0) + \cdots . \tag{2.4}$$

In order to determine the coefficients $\alpha_i, \beta_i, \gamma_i$ $(i=0,1,2)$, C and the positive integer p, we consider the following polynomials:

$$\begin{cases} \rho(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 , \\ \sigma(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 , \\ \tau(\lambda) = \gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 . \end{cases} \tag{2.5}$$

Let E and D be the operators such that

$$Ex(t) = x(t+h), \qquad Dx(t) = \dot{x}(t) . \tag{2.6}$$

Then we can write equality (2.3) in the following form:

$$[\rho(E) - h\sigma(E)D - h^2\tau(E)D^2]x(t_0) = Ch^pD^px(t_0) + \cdots .$$

Since

$$Ex(t) = x(t + h) = e^{hD}\,x(t)\ ,$$

we then have formally

$$\rho(e^{hD}) - \sigma(e^{hD})hD - \tau(e^{hD})h^2D^2 = Ch^pD^p + \cdots\ , \tag{2.7}$$

which is equivalent formally to

$$\rho(\zeta) - \sigma(\zeta)\log\zeta - \tau(\zeta)\log^2\zeta = C\log^p\zeta + \cdots\ . \tag{2.8}$$

In (2.8), clearly $\zeta \to 1$ and $\log\zeta \to 0$ as $h \to 0$. Hence in the limit we have

$$\rho(1) = 0. \tag{2.9}$$

Put

$$\zeta = 1 + \xi\ , \tag{2.10}$$

then clearly $\xi \to 0$ as $h \to 0$, and from (2.8) we have

$$\frac{\rho(1+\xi)}{\log(1+\xi)} - \sigma(1+\xi) - \tau(1+\xi)\cdot\log(1+\xi) \tag{2.11}$$

$$= C\log^{p-1}(1+\xi) + \cdots\ .$$

By (2.9), $\rho(1+\xi)$ is of the form

$$(2.12)\qquad \rho(1+\xi) = \xi(\rho_0 + \rho_1\xi)\ .$$

Clearly $\sigma(1+\xi)$ and $\tau(1+\xi)$ are of the forms

$$(2.13)\qquad \begin{cases} \sigma(1+\xi) = \sigma_0 + \sigma_1\xi + \sigma_2\xi^2, \\ \tau(1+\xi) = \tau_0 + \tau_1\xi + \tau_2\xi^2. \end{cases}$$

Now it is clear that

$$(2.14)\qquad \frac{\xi}{\log(1+\xi)} = \int_0^1 (1+\xi)^u du = \sum_{r=o}^{\infty} c_r\xi^r$$

where

$$(2.15)\qquad c_r = \int_0^1 \frac{u(u-1)\cdots(u-r+1)}{r!}\,du \qquad (r=0,1,2,\cdots).$$

By elementary calculations, it is easily seen that

$$(2.16)\qquad \begin{cases} c_0 = 1, \quad c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{12}, \quad c_3 = \frac{1}{24}, \\ c_4 = -\frac{19}{720}, \quad c_5 = \frac{3}{160}, \quad c_6 = -\frac{863}{60480}\ . \end{cases}$$

Now

$$\log(1+\xi) = \xi - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 - \cdots ,$$

hence we can write (2.11) in the following form :

$$(2.17) \qquad \begin{aligned} &(\rho_0 + \rho_1\xi)(1+c_1\xi+c_2\xi^2+\cdots) \\ &\quad - (\sigma_0+\sigma_1\xi+\sigma_2\xi^2) \\ &\quad\quad - (\tau_0+\tau_1\xi+\tau_2\xi^2)(\xi-\tfrac{1}{2}\xi^2+\tfrac{1}{3}\xi^3-\cdots) \\ &\quad\quad\quad = C\xi^{p-1} + \cdots . \end{aligned}$$

Let us consider the two cases.

Case I : $\rho(\lambda) = -1 + \lambda$.   In this case

$$\rho(1 + \xi) = -1 + (1 + \xi) = \xi ,$$

therefore by (2.12) we see that

$$\rho_0 = 1, \qquad \rho_1 = 0.$$

Then from (2.17) it follows that

$$(2.18) \qquad p \geqq 7$$

if and only if the following equations are satisfied by σ_i and τ_i $(i=0,1,2)$:

$$
(2.19)\quad \begin{cases} 1 - \sigma_0 = 0 \,, \\ c_1 - \sigma_1 - \tau_0 = 0 \,, \\ c_2 - \sigma_2 - (-\frac{1}{2}\tau_0 + \tau_1) = 0 \,, \\ c_3 - (\frac{1}{3}\tau_0 - \frac{1}{2}\tau_1 + \tau_2) = 0 \,, \\ c_4 - (-\frac{1}{4}\tau_0 + \frac{1}{3}\tau_1 - \frac{1}{2}\tau_2) = 0 \,, \\ c_5 - (\frac{1}{5}\tau_0 - \frac{1}{4}\tau_1 + \frac{1}{3}\tau_2) = 0 \,. \end{cases}
$$

Solving (2.19) with respect to σ_i and τ_i, we find that σ_i and τ_i satisfying (2.19) are

$$
(2.20)\quad \begin{cases} \sigma_0 = 1, \quad \sigma_1 = \frac{5}{8}, \quad \sigma_2 = \frac{11}{240}, \\ \tau_0 = -\frac{1}{8}, \quad \tau_1 = -\frac{23}{120}, \quad \tau_2 = -\frac{1}{80} \,. \end{cases}
$$

For these values of σ_i and τ_i, from (2.17) we have

$$
(2.21)\qquad p = 7
$$

and

$$
(2.22)\qquad C = c_6 - (-\frac{1}{6}\tau_0 + \frac{1}{5}\tau_1 - \frac{1}{4}\tau_2) = \frac{1}{9450} \,.
$$

Since

$$
\sigma(\lambda) = \sigma_0 + \sigma_1(\lambda - 1) + \sigma_2(\lambda - 1)^2 \,,
$$

$$
\tau(\lambda) = \tau_0 + \tau_1(\lambda - 1) + \tau_2(\lambda - 1)^2
$$

by (2.13), for the values given by (2.20) we have

(2.23)
$$\begin{cases} \sigma(\lambda) = \frac{1}{240}(101 + 128\lambda + 11\lambda^2) , \\ \tau(\lambda) = \frac{1}{240}(13 - 40\lambda - 3\lambda^2) , \end{cases}$$

which by (2.5) means

$$(\beta_0, \beta_1, \beta_2) = \frac{1}{240}(101, 128, 11) ,$$

$$(\gamma_0, \gamma_1, \gamma_2) = \frac{1}{240}(13, -40, -3) .$$

Now for solution $x(t)$ of (1.1), it is evident that

$$\dot{x}(t) = f[t, x(t)],$$
$$\ddot{x}(t) = g[t, x(t)].$$

Hence from (2.3) we have

(2.24)
$$x_1 = x_0 + \frac{h}{240}[101f(t_0, x_0) + 128f(t_1, x_1) + 11f(t_2, x_2)]$$
$$+ \frac{h^2}{240}[13g(t_0, x_0) - 40g(t_1, x_1) - 3g(t_2, x_2)]$$
$$+ \frac{1}{9450}h^7D^7x(t_0) + \cdots ,$$

from which one obtains the integration formula (1.5) by neglecting the residual terms.

Case II : $\rho(\lambda) = \alpha_0 + \alpha_1\lambda + \lambda^2$, $\beta_2 = \gamma_2 = 0$. By (2.9),

$$\alpha_1 = -(1 + \alpha_0) ,$$

therefore we have

$$\rho(1 + \xi) = \xi[(1-\alpha_0) + \xi],$$

which implies

(2.25) $$\rho_0 = 1 - \alpha_0, \qquad \rho_1 = 1$$

in (2.12). Since $\beta_2 = \gamma_2 = 0$, we have

(2.26) $$\sigma_2 = \tau_2 = 0$$

in (2.13). Then corresponding to (2.17), we have:

(2.27) $$\begin{aligned}(\rho_0 + \xi)(1 + c_1\xi + c_2\xi^2 + \cdots) \\ -(\sigma_0 + \sigma_1\xi) \\ -(\tau_0 + \tau_1\xi)(\xi - \tfrac{1}{2}\xi^2 + \tfrac{1}{3}\xi^3 - \cdots) \\ = C\xi^{p-1} + \cdots ;\end{aligned}$$

Similarly to Case I, we then have :

$$(2.28) \qquad p = 6,$$

$$(2.29) \qquad \begin{cases} \rho_0 = -30, & \\ \sigma_0 = -30, & \sigma_1 = -16, \\ \tau_0 = 2, & \tau_1 = 4, \end{cases}$$

$$(2.30) \qquad C = \frac{1}{90} \; .$$

From (2.29) readily follows

$$(2.31) \qquad \begin{cases} \rho(\lambda) = 31 - 32\lambda + \lambda^2, \\ \sigma(\lambda) = -14 - 16\lambda, \\ \tau(\lambda) = -2 + 4\lambda \; . \end{cases}$$

Thus by (2.28) and (2.30), from (2.3) we have

$$(2.32) \qquad \begin{aligned} x_2 = &-31x_0 + 32x_1 \\ &- h[14f(t_0, x_0) + 16f(t_1, x_1)] \\ &+ h^2[-2g(t_0, x_0) + 4g(t_1, x_1)] \\ &+ \frac{1}{90} h^6 D^6 x(t_0) + \cdots , \end{aligned}$$

from which one obtains the formula (1.7) by neglecting the residual terms.

3. Convergence of the method. Let $x(t)$ be a solution of (1.1) and suppose that

$$(3.1)\qquad \begin{cases} t_{n+1} = t_n + h_n , \\ \tilde{x}_n = x(t_n) \end{cases} \qquad (n=0,1,2,\cdots) .$$

Then by (2.24) and (2.32), we have the equalities of the following forms :

$$(3.2)\qquad \tilde{x}_{n+1} = \tilde{x}_n + h_n[\beta_0 f(t_n, \tilde{x}_n) + \beta_1 f(t_{n+1}, \tilde{x}_{n+1}) + \beta_2 f(t_{n+2}, \tilde{x}_{n+2})]$$
$$+ h_n^2[\gamma_0 g(t_n, \tilde{x}_n) + \gamma_1 g(t_{n+1}, \tilde{x}_{n+1})$$
$$+ \gamma_2 g(t_{n+2}, \tilde{x}_{n+2})] + T_n ,$$

$$(3.3)\qquad \tilde{x}_{n+2} = -31\tilde{x}_n + 32\tilde{x}_{n+1}$$
$$+ h_n[\beta_0' f(t_n, \tilde{x}_n) + \beta_1' f(t_{n+1}, \tilde{x}_{n+1})]$$
$$+ h_n^2[\gamma_0' g(t_n, \tilde{x}_n) + \gamma_1' g(t_{n+1}, \tilde{x}_{n+1})]$$
$$+ T_n' ,$$

where

(3.4) $$T_n = \frac{1}{9450} h_n{}^7 D^7 x(t_n) + \cdots ,$$

(3.5) $$T_n' = \frac{1}{90} h_n{}^6 D^6 x(t_n) + \cdots .$$

Let x_n be approximations to $\tilde{x}_n = x(t_n)$ obtained by our numerical method. Then by (1.5) and (1.7), corresponding to (3.2) and (3.3), we have

(3.6) $$\begin{aligned} x_{n+1} = x_n &+ h_n[\beta_0 f(t_n,x_n) + \beta_1 f(t_{n+1},x_{n+1}) + \beta_2 f(t_{n+2},\hat{x}_{n+2}) \\ &+ h_n{}^2[\gamma_0 g(t_n,x_n) + \gamma_1 g(t_{n+1},x_{n+1}) + \gamma_2 g(t_{n+2},\hat{x}_{n+2} \\ &+ R_n , \end{aligned}$$

(3.7) $$\begin{aligned} \hat{x}_{n+2} = -31x_n &+ 32x_{n+1} \\ &+ h_n[\beta_0' f(t_n,x_n) + \beta_1' f(t_{n+1},x_{n+1})] \\ &+ h_n{}^2[\gamma_0' g(t_n,x_n) + \gamma_1' g(t_{n+1},x_{n+1})] , \end{aligned}$$

where R_n is a round-off error arising in computation of x_{n+1}.

We assume that functions $f(t,x)$ and $g(t,x)$ satisfy a Lipschitz condition with respect to x, that is, there are positive constants K_1 and K_2 such that

(3.8) $$\begin{cases} |f(t,x') - f(t,x'')| \leqq K_1|x'-x''| , \\ |g(t,x') - g(t,x'')| \leqq K_2|x'-x''| \end{cases}$$

for any x' and x''..

Put

$$(3.9) \qquad x_n - \tilde{x}_n = e_n \qquad (n=0,1,2,\cdots),$$

then subtracting (3.2) from (3.6) and making use of (3.8), we have

$$(3.10) \quad |e_{n+1}| \leqq |e_n| + |h_n| K_1(|\beta_0|\cdot|e_n| + |\beta_1|\cdot|e_{n+1}| + |\beta_2|\cdot|\hat{x}_{n+2} - \tilde{x}_{n+2}|) + |h_n|^2 K_2(|\gamma_0|\cdot|e_n| + |\gamma_1|\cdot|e_{n+1}| + |\gamma_2|\cdot|\hat{x}_{n+2} - \tilde{x}_{n+2}|) + |R_n| + |T_n| .$$

However subtracting (3.3) from (3.7) and making use of (3.8), we have

$$(3.11) \quad |\hat{x}_{n+2} - \tilde{x}_{n+2}| \leqq 31|e_n| + 32|e_{n+1}| + |h_n| K_1(|\beta_0'|\cdot|e_n| + |\beta_1'|\cdot|e_{n+1}|) + |h_n|^2 K_2(|\gamma_0'|\cdot|e_n| + |\gamma_1'|\cdot|e_{n+1}|) + |T_n'| .$$

Hence substituting (3.11) into (3.10), we have

$$|e_{n+1}| \leqq |e_n| + |h_n| K_1(|\beta_0|\cdot|e_n| + |\beta_1|\cdot|e_{n+1}|)$$

$$+ |h_n|^2 K_2(|\gamma_0|\cdot|e_n| + |\gamma_1|\cdot|e_{n+1}|)$$

$$+ (|h_n|K_1|\beta_2| + |h_n|^2K_2|\gamma_2|)[31|e_n| + 32|e_{n+1}|$$

$$+|h_n|K_1(|\beta_0'|\cdot|e_n| + |\beta_1'|\cdot|e_{n+1}|)$$

$$+ |h_n|^2K_2(|\gamma_0'|\cdot|e_n| + |\gamma_1'|\cdot|e_{n+1}|) + |T_n'|]$$

$$+ |R_n| + |T_n| .$$

Then we have

$$\begin{aligned}(3.12)\qquad & [1 - |h_n|K_1(|\beta_1| + |\beta_2|M_n) \\ & \quad - |h_n|^2K_2(|\gamma_1| + |\gamma_2|M_n)]|e_{n+1}| \\ \leqq\ & [1 + |h_n|K_1(|\beta_0| + |\beta_2|N_n) \\ & \quad + |h_n|^2K_2(|\gamma_0| + |\gamma_2|N_n)]|e_n| + \rho_n ,\end{aligned}$$

where

$$(3.13)\qquad \begin{cases} M_n = 32 + |h_n|K_1|\beta_1'| + |h_n|^2K_2|\gamma_1'|, \\ N_n = 31 + |h_n|K_1|\beta_0'| + |h_n|^2K_2|\gamma_0'|, \\ \rho_n = |R_n| + |T_n| + |h_n|(K_1|\beta_2| + |h_n|K_2|\gamma_2|)|T_n'|. \end{cases}$$

Suppose that

$$(3.14)\qquad |h_n| \leqq H \qquad (n=0,1,2,\cdots) .$$

Then for small $H>0$, by (3.4) and (3.5), we may suppose that

(3.15) $$|T_n| \leqq C_0|h_n|^7, \quad |T_n'| \leqq C_1|h_n|^6 .$$

For R_n, let us suppose that

(3.16) $$|R_n| \leqq \varepsilon(|h_n|)\cdot|h_n| \qquad (n=0,1,2,\cdots)$$

where $\varepsilon(h)$ is a monotonously increasing function such that $\varepsilon(h)\downarrow 0$ as $h\downarrow 0$. By (3.15) and (3.16), we then have

(3.17) $$\rho_n \leqq \rho(|h_n|)\cdot|h_n| ,$$

where $\rho(h)$ is a monotously increasing function such that $\rho(h)\downarrow 0$ as $h\downarrow 0$. Now from (3.13), we have

$$\begin{cases} M_n \leqq 32 + H(K_1|\beta_1'| + HK_2|\gamma_1'|), \\ N_n \leqq 31 + H(K_1|\beta_0'| + HK_2|\gamma_0'|) \end{cases} \qquad (n=0,1,2,\cdots) ,$$

therefore by (3.14) and (3.17), from (3.12) we have the inequality of the following form :

$$(1-|h_n|K_3)|e_{n+1}| \leqq (1+|h_n|K_4)|e_n| + \rho(H)|h_n| ,$$

that is,

$$(3.18) \qquad |e_{n+1}| \leqq \frac{1+|h_n|K_4}{1-|h_n|K_3}|e_n| + \hat{\rho}|h_n| \qquad (n=0,1,2,\cdots),$$

where K_3 and K_4 are positive constants and

$$(3.19) \qquad \hat{\rho} = \frac{\rho(H)}{1-HK_3} .$$

By elementary manipulations, it is easily seen that

$$\frac{1+|h_n|K_4}{1-|h_n|K_3} \leqq 1+|h_n|K_5 \qquad (n=0,1,2,\cdots)$$

where

$$K_5 = \frac{K_3 + K_4}{1 - HK_3} .$$

Hence from (3.18), we have

$$(3.20) \qquad |e_{n+1}| \leqq (1+|h_n|K_5)|e_n| + \hat{\rho}|h_n| \qquad (n=0,1,2,\cdots),$$

from which by induction readily follows

$$(3.21) \qquad |e_n| \leqq \prod_{i=0}^{n-1} (1 + |h_i| K_5) \cdot |e_0| + \hat{\rho} \sum_{j=0}^{n-1} [|h_j| \prod_{i=j+1}^{n-1} (1 + |h_i| K_5)]$$

$$(n=0,1,2,\cdots).$$

However

$$\sum_{j=0}^{n-1} [|h_j| \prod_{i=j+1}^{n-1} (1 + |h_i| K_5)]$$

$$= \frac{1}{K_5} \sum_{j=0}^{n-1} [\{(1 + |h_j| K_5) - 1\} \prod_{i=j+1}^{n-1} (1 + |h_i| K_5)]$$

$$= \frac{1}{K_5} \sum_{j=0}^{n-1} [\prod_{i=j}^{n-1} (1 + |h_i| K_5) - \prod_{i=j+1}^{n-1} (1 + |h_i| K_5)]$$

$$= \frac{1}{K_5} [\prod_{i=0}^{n-1} (1 + |h_i| K_5) - 1] \quad .$$

Hence from (3.21) we have

$$(3.22) \qquad |e_n| \leqq \prod_{i=0}^{n-1} (1 + |h_i| K_5) \cdot |e_0| + \frac{\hat{\rho}}{K_5} [\prod_{i=0}^{n-1} (1 + |h_i| K_5) - 1]$$

$$(n=0,1,2,\cdots) \quad .$$

Let ℓ be the length of the interval I on which the numerical integration of (1.1) is carried out. Then

$$\sum_{i=0}^{n-1} |h_i| \leqq \ell ,$$

therefore we have

$$\prod_{i=0}^{n-1} (1 + |h_i| K_5) \leqq \prod_{i=0}^{n-1} \exp[|h_i| K_5]$$

$$= \exp[\sum_{i=0}^{n-1} |h_i| \cdot K_5]$$

$$\leqq \exp(\ell K_5) .$$

Then from (3.22) follows

$$(3.23) \qquad |e_n| \leqq |e_0| \cdot \exp(\ell K_5) + \frac{\hat{\rho}}{K_5} [\exp(\ell K_5) - 1]$$

$$(n=0,1,2,\cdots\cdot) ,$$

which by (3.19) implies

$$|e_n| \rightarrow 0 \quad (n=0,1,2,\cdots) \quad \text{as} \quad |e_0|, H \rightarrow 0 .$$

This proves the convergence of our integration method.

Remark 1. For the proof of the convergence of an integration method, it is clear that no generality is lost by the assumption.

Remark 2. By (3.4) and (3.5), from (3.13) we have

$$\rho_n = |R_n| + \frac{1}{9450}|h_n|^7 \, |D^7x(t_n)| + \frac{11}{240}\times\frac{1}{90}\,|h_n|^7 \cdot K_1 |D^6x(t_n)|$$

$$+ o(|h_n|^7)$$

$$= |R_n| + \frac{1}{9450}|h_n|^7 [\,|D^7x(t_n)| + \frac{77}{16} K_1 |D^6x(t_n)|\,]$$

$$+ o(|h_n|^7) \quad .$$

Hence by (3.16) and (3.19), we have

$$\hat{\rho} \leqq \varepsilon(H)[1+O(H)] \tag{3.24}$$

$$+ \frac{1}{9450} H^6 [\max|D^7x(t)| + \frac{77}{16} K_1 \cdot \max|D^6x(t)|\,]$$

$$+ o(H^6),$$

by which from (3.23) we see that our integration method possesses the high accuracy.

4. A rule of controlling the mesh size. In the function in the right-hand side of (1.5), fix x_o and substitute the right-hand side of (1.7) for $\hat{x}_2$. Let $\varphi(x_1)$ be a function obtained. Then equation (1.5) can be written briefly as

$$x_1 = \varphi(x_1) \tag{4.1}$$

and the iterative process for solving equation (1.5) can be written as

$$x_1^{(m+1)} = \varphi[x_1^{(m)}] \qquad (m = 0, 1, 2, \dots). \tag{4.2}$$

Now from (1.5) we have

$$\varphi'(x_1) = \frac{8}{15}hf_x(t_1, x_1) + \frac{11}{240}hf_x(t_2, \hat{x}_2)\frac{d\hat{x}_2}{dx_1}$$
$$- \frac{1}{6}h^2g_x(t_1, x_1) - \frac{1}{80}h^2g_x(t_2, \hat{x}_2)\frac{d\hat{x}_2}{dx_1},$$

where $' = d/dx_1$. From (1.7) we have

$$\frac{d\hat{x}_2}{dx_1} = 32 - 16hf_x(t_1, x_1) + 4h^2g_x(t_1, x_1).$$

Hence we have

$$(4.3) \quad \varphi'(x_1) = \frac{8}{15} hf_x(t_1, x_1) - \frac{1}{6} h^2 g_x(t_1, x_1)$$

$$+ h[\frac{11}{240} f_x(t_2, \hat{x}_2) - \frac{1}{80} h\, g_x(t_2, \hat{x}_2)] \times$$

$$\times [32 - 16\, h\, f_x(t_1, x_1) + 4\, h^2\, g_x(t_1, x_1)]$$

$$= h(\frac{8}{15} + \frac{11}{240} \cdot 32) f_x(t_1, x_1) + O(h^2)$$

$$= 2hf_x(t_0, x_0) + O(h^2).$$

Taking a small positive number H, let us restrict the mesh size h so that

$$(4.4) \qquad |h| \leqq H.$$

Then from (4.3) it follows that if

$$(4.5) \qquad 2\,|h| \cdot |f_x(t_0, x_0)| < k,$$

then

$$(4.6) \qquad |\varphi(x_1') - \varphi(x_1'')| \leqq k\,|x_1' - x_1''|$$

for any x_1', x_1'' near x_0.

Suppose that h is chosen so that (4.5) may be valid for

for $k < 1$. Then an approximate value to the solution x_1 of equation (4.1) can be indeed obtained by the iterative process (4.2) (for details, see [1]). Let ε be a bound of a round-off error in the evaluation of $\varphi(x)$ and take a positive number α such that

$$\alpha \geqq \frac{2\varepsilon}{1-k} . \tag{4.7}$$

Then as is proved in [2], we can stop the iterative process (4.2) by the criterion

$$\left| x_1^{(m+1)} - x_1^{(m)} \right| \leqq \alpha \tag{4.8}$$

and moreover, if k satisfies

$$k\alpha \leqq \varepsilon , \tag{4.9}$$

a final value $x_1^{(m+1)}$ obtained by the iterative process possesses the similar accuracy as the value in the state of oscillating numerical convergence, that is, the best approximate solution that one can get by the iterative process (4.2).

From the above results, we get a following method of controlling the mesh-size h.

Namely, taking into account inequalities (3.24), (4.7) and (4.9), we specify the numbers H, α and k before excution of of computation. In the course of computation, we then determine the mesh size h by halving or doubling so that h may be a maximal mesh size satisfying both of inequalities (4.4) and (4.5).

5. Numerical examples.

Example 1. $\frac{dx}{dt} = \frac{1}{5} x^2$, $x(0) = 1$.

In this example, it is evident that the exact solution is

$$x(t) = \frac{5}{5 - t} .$$

Specifying H, k and α so that

$$H = 0.125, \quad k = 0.1, \quad \alpha = 10^{-8} \quad \text{and} \quad 10^{-9},$$

we applied our method to the given equation. The values of the solutions obtained are shown in Table 1. The errors of the solutions multiplied by 10^9 are also shown in Table 1. In order to compare our method with the Runge-Kutta method, we computed the solution by both methods for a constant mesh size $h = 0.0625$. In our method we specified α so that $\alpha = 10^{-9}$. Errors multiplied by 10^9 for the solutions obtained by both methods are shown in Table 2.

Example 2. $\frac{dx}{dt} = 5t(\frac{1}{2} - x)^{4/5}$, $x(-1) = \frac{15}{32} = 0.46875$.

In this example, it is easily seen that the exact solution is

$$x(t) = \frac{1}{2} - (1 - \frac{1}{2} t^2)^5.$$

Specifying H, k and α so that

$$H = 0.0625, \quad k = 0.1, \quad \alpha = 10^{-8} \text{ and } 10^{-9},$$

we applied our method to the given equation. The values of the solutions obtained are shown in Table 3. The errors of the solutions multiplied by 10^{10} are also shown in Table 3. In order to compare our method with the Runge-Kutta method, we computed the solution by both methods for a constant mesh size h = 0.03125. In our method we specified α so that $\alpha = 10^{-9}$. Errors multiplied by 10^{10} for the solutions obtained by both methods are shown in Table 4.

Tables 1 ~ 4 show the practical usefulness of our method.

Table 1

h	t	Solutions (Errors × 10^9) $\alpha = 10^{-8}$	Solutions (Errors × 10^9) $\alpha = 10^{-9}$
	0	1.00000 0000	1.00000 0000
0.125	0.125	1.02564 1025(-1)	1.02564 1026(0)
0.0625	0.1875	1.03896 1038(-1)	1.03896 1039(0)
⋮	⋮	⋮	⋮
0.0625	2.5625	2.05128 2025(-26)	2.05128 2047(-4)
0.03125	2.59375	2.07792 2050 (-28)	2.07792 2074(-4)
⋮	⋮	⋮	⋮
0.03125	3.78125	4.10256 3972(-131)	4.10256 4081(-22)
0.015625	3.796875	4.15584 4022(-134)	4.15584 4134(-22)
⋮	⋮	⋮	⋮
0.015625	4.390625	8.20512 7654(-551)	8.20512 8110(-95)
0.0078125	4.3984375	8.31168 7746(-566)	8.31168 8214(-98)
⋮	⋮	⋮	⋮
0.0078125	4.6953125	16.41025 417(-2240)	16.41025 602(-390)
0.00390625	4.69921875	16.62337 433(-2290)	16.62337 622(-400)
⋮	⋮	⋮	⋮
0.00390625	4.75	19.99999 667(-3330)	19.99999 941(-590)

Table 2

t	Errors × 10^9	
	Our one-step method	Runge-Kutta method
0	0	0
0.5	0	0
1.0	-1	-1
1.5	-1	-1
2.0	-2	-4
2.5	-3	-10
3.0	-7	-32
3.5	-17	-138
4.0	-132	-1051
4.5	-19499	-33167
4.5625	-55860	-64280
4.625	-194950	-137680
4.6875	-910920	-337710
4.75	-6882850	-1005330

Table 3

h	t	Solutions (Errors × 10^{10})	
		$\alpha = 10^{-8}$	$\alpha = 10^{-9}$
	-1.0	0.46875 00000	0.46875 00000
0.00390625	-0.99609 375	0.46751 25336(3)	0.46751 25333(0)
⋮	⋮	⋮	⋮
0.00390625	-0.91796 875	0.43511 52891(49)	0.43511 52840(-2)
0.0078125	-0.91015 625	0.43101 16566(55)	0.43101 16509(-2)
⋮	⋮	⋮	⋮
0.0078125	-0.63671 875	0.17782 33175(398)	0.17782 32782(5)
0.015625	-0.62109 375	0.15747 36963(413)	0.15747 36555(5)
⋮	⋮	⋮	⋮
0.015625	-0.37109 375	-0.19998 03843(674)	-0.19998 04515(2)
0.03125	-0.33984 375	-0.24274 17954(703)	-0.24274 18654(3)
⋮	⋮	⋮	⋮
0.03125	-0.18359 375	-0.41852 61361(822)	-0.41852 62182(1)
0.0625	-0.12109 375	-0.46387 43081(843)	-0.46387 43934(-10)
⋮	⋮	⋮	⋮
0.0625	0.25390 625	-0.34888 98249(830)	-0.34888 99026(53)
0.03125	0.28515 625	-0.31258 61465(799)	-0.31258 62213(51)
⋮	⋮	⋮	⋮
0.03125	0.37890 625	-0.18903 76135(708)	-0.18903 76799(44)
0.015625	0.39453 125	-0.16690 02795(690)	-0.16690 03442(43)
⋮	⋮	⋮	⋮
0.015625	0.64453 125	0.18780 92069(373)	0.18780 91719(23)
0.0078125	0.65234 375	0.19766 31599(362)	0.19766 31259(22)
⋮	⋮	⋮	⋮
0.0078125	0.92578 125	0.43905 38383(99)	0.43905 38291(7)
0.00390625	0.92968 75	0.44096 19448(96)	0.44096 19359(7)
⋮	⋮	⋮	⋮
0.00390625	1.0	0.46875 00053(53)	0.46875 00004(4)

Table 4

t	Errors × 10^{10}	
	Our one-step method	Runge-Kutta method
-1	0	0
-0.75	58	39602
-0.5	144	92887
-0.25	216	1 40186
0.0	243	1 59239
0.25	214	1 40193
0.5	141	92918
0.75	58	39237
1.0	3	3849

References

[1] URABE, M.: Convergence of numerical iteration in solution of equations. J. Sci. Hiroshima Univ. Ser. A, 19, 479-489(1956).

[2] URABE, M.: Error estimation in numerical solution of equations by iteration process. J. Sci. Hiroshima Univ. Ser. A-I, 26, 77-91(1962).