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AN APPLICATION OF THE EPSILON TECHNIQUE TO THE SOLUTION OF PURSUIT AND EVASION PROBLEMS

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ABSTRACT

A necessary and sufficient condition for attaining capture in pursuit and evasion problems described by nonlinear differential equations are presented. It is shown that the "epsilon technique" developed by A. V. Balakrishnan for computing optimal control can be applied for solving the pursuit and evasion problems. The existence of the solution for the epsilon problem is proved, and the relation between the original pursuit problem and the epsilon problem is shown.

1. Introduction

In this paper, we shall discuss pursuit and evasion problems related to a max-min problem first considered by Kelendzheridze [1]. A necessary and sufficient condition for attaining capture will be presented, on the basis of an inclusion relation between two attainable sets of a pursuer and an evader. The "epsilon technique" which was developed by A. V. Balakrishnan [2], [3] for computing optimal control will be applied for solving the pursuit and evasion problems. The advantage of the epsilon technique lies in the fact that the optimization problem containing differential equation constrains can be reduced to a nondynamic optimization problem. It will be shown that the solution of the pursuit and evasion problems described by differential equations can be obtained by solving a nondynamic sup-inf problem. The existence of the solution of the epsilon problem is proved, and the relation between the original pursuit and evasion problem and the epsilon problem is shown.

2. Statement of the Problem

Let there be two players, the one called pursuer and the other called evader. The states of the pursuer and the evader at any time t, $0 \leq t < \infty$, are represented by m-dimensional vectors $x(t)$ and $y(t)$, respectively. The dynamics are given by the following differential equations,

$$dx(t)/dt = f\Big(x(t),\ u(t),\ t\Big),\quad x(0) = x_0, \tag{1}$$

$$dy(t)/dt = g\Big(y(t),\ v(t),\ t\Big),\quad y(0) = y_0, \tag{2}$$

where x_0 and y_0 are initial states of the pursuer and the evader, respectively.

Let U be a nonempty compact subset of an r-dimensional Euclidean space R^r, and let V be a nonempty compact subset of an s-dimensional Euclidean space R^s. The control $u(\cdot)$ of the pursuer is said to be admissible if $u(\cdot)$ is measurable on $[0, T]$ and for each $t \in [0, T]$

$$u(t) \in U. \tag{3}$$

Let Ω_u denote the set of admissible controls of the pursuer defined on $[0, T]$. Analogously, the control $v(\cdot)$ of the evader is said to be admissible if $v(\cdot)$ is measurable on $[0, T]$ and for each $t \in [0, T]$

$$v(t) \in V. \tag{4}$$

Let Ω_v denote the set of admissible controls of the evader defined on $[0, T]$.

For the functions f and g, the following assumptions will be made:

Assumption 1. The function $f(x, u, t)$ is continuous on $E^m \times U \times [0, T]$ and continuously differentiable in x. Similarly, the function $g(y, v, t)$ is continuous on $E^m \times V \times [0, T]$ and continuously differentiable in y.

Assumption 2. There exists a positive constant c such that

$$[x,\ f(x,u,t)] \leqq c(1+\|x\|^2) \tag{5}$$

for all $x \in E^m$, $u \in U$, and $t \in [0, T]$, and that

$$[y,\ g(y,v,t)] \leqq c(1+\|y\|^2)$$

for all $y \in E^m$, $v \in V$, and $t \in [0, T]$, where $[\ \cdot\ ,\ \cdot\ ]$ denotes the inner product.

Assumption 3'. The set $F(x, t)$ defined by

$$F(x,t) = \{f(x,u,t) : u \in U\} \tag{6}$$

is convex for every x and $t \in [0, T]$.

With Assumptions 1 and 2, for each $u \in \Omega_u$, (1) has a unique solution uniformly bounded on $[0, T]$ which will be denoted by $x(\cdot, u)$, and for each $v \in \Omega_v$, (2) has a unique solution uniformly bounded on $[0, T]$ which will be denoted by $y(\cdot, v)$ [4], [5].

Let $A_x(T, x_0)$, or in short $A_x(T)$, denote the attainable set of the pursuer defined by

$$A_x(T, x_0) = \left\{ x_0 + \int_0^T f\Big(x(t,u),\ u(t),\ t\Big)dt : u(\cdot) \in \Omega_u \right\}. \tag{7}$$

In the same manner, the attainable set of the evader, denoted by $A_y(T, y_0)$ or $A_y(T)$, is defined by

$$A_y(T, y_0) = \left\{ y_0 + \int_0^T g\Big(y(t,v),\ v(t),\ t\Big)dt : v(\cdot) \in \Omega_v \right\}. \tag{8}$$

Under Assumption 3', the attainable set $A_x(T)$ of the pursuer turns out to be compact [4], [5].

Now, let π be an $n \times m$ $(n \leqq m)$ matrix corresponding to the orthogonal projection from R^m onto an n-dimensional linear subspace. We say that the capture is attained from the initial states x_0 and y_0 if, no matter what admissible control may be chosen by the evader, the pursuer can choose an admissible control corresponding to the evader's control such that

$$\| \pi x(T) - \pi y(T) \| \leqq \delta \tag{9}$$

for some finite time T, where $\delta \geqq 0$ is a given constant. Let $B_x(T)$ and $B_y(T)$ denote the projections of the attainable sets $A_x(T)$ and $A_y(T)$ into the n-dimensional linear subspace, i.e.,

$$\left.\begin{aligned} B_x(T) &= \pi A_x(T) = \{\pi x : x \in A_x(T)\} \subset R^n, \\ B_y(T) &= \pi A_y(T) = \{\pi x : x \in A_y(T)\} \subset R^n . \end{aligned}\right\} \tag{10}$$

Let $\bar{S}_\delta$ denote a closed sphere in R^n of radius δ about the origin, i.e.,

$$\bar{S}_\delta = \{x \in R^n : \|x\| \leqq \delta\}, \tag{11}$$

Then, it is clear that the pursuit-evasion game can be completed if and only if

$$B_x(T) + \bar{S}_\delta \supset B_y(T). \tag{12}$$

3. Completion of the Game

Let η be an arbitrary point of E^n. The distance between a point η and a set $B_x(T)$ is defined by

$$\rho\left(\eta, B_x(T)\right) = \inf \left\{ \| \eta - \xi \| : \xi \in B_x(T) \right\}. \tag{13}$$

Further, let us define an asymmetrical distance between two sets $B_y(T)$ and $B_x(T)$ as follows:

$$\begin{aligned} \rho^*\left(B_y(T), B_x(T)\right) &= \sup \left\{ \rho\left(\eta, B_x(T)\right) : \eta \in B_y(T) \right\} \\ &= \sup_{\eta \in B_y(T)} \inf_{\xi \in B_x(T)} \| \eta - \xi \|. \end{aligned} \tag{14}$$

By using the asymmetrical distance between the attainable sets of pursuer and evader in R^n, the necessary and sufficient condition for completion of the game is obtained.

Theorem 1. Under Assumptions 1, 2, and 3', the necessary and sufficient condition for completion of the game is that the relation

$$\rho^*\left(B_y(T),\ B_x(T)\right) = \sup_{\eta \in B_y(T)} \inf_{\xi \in B_x(T)} \|\eta - \xi\|$$

$$= \sup_{y \in A_y(T)} \inf_{x \in A_x(T)} \|\pi y - \pi x\| \leqq \delta \tag{15}$$

holds for some finite time T.

Proof. To prove the necessity, let us assume

$$B_x(T) + \bar{S}_\delta \supset B_y(T) . \tag{12}$$

The above relation implies that for all $\eta \in B_y(T)$ there exists a $\xi \in B_x(T)$ such that

$$\|\eta - \xi\| \leqq \delta .$$

Therefore, it follows that

$$\inf_{\xi \in B_x(T)} \|\eta - \xi\| \leqq \delta \quad \text{for all} \quad \eta \in B_y(T) . \tag{16}$$

Hence we obtain

$$\sup_{\eta \in B_y(T)} \inf_{\xi \in B_x(T)} \|\eta - \xi\| \leqq \delta .$$

For proving the sufficiency, let us assume that there is a point $\hat{\eta} \in B_y(T)$ such that $\hat{\eta} \notin B_x(T) + \bar{S}_\delta$. Since the set $B_x(T)$ is compact, it follows that

$$\rho\left(\hat{\eta},\ B_x(T)\right) = \inf\left\{\|\hat{\eta} - \xi\| \ :\ \xi \in B_x(T)\right\} > \delta . \tag{17}$$

Therefore

$$\rho^*\left(B_y(T),\ B_x(T)\right) \geqq \rho\left(\hat{\eta},\ B_x(T)\right) > \delta \ .$$

This is a contradiction. (Q. E. D.)

4. Application of the Epsilon Technique

Now, the problem has been reduced to computing

$$\sup_{\eta \in B_y(T)} \ \inf_{\xi \in B_x(T)} \|\eta - \xi\| = \sup_{y \in A_y(T)} \ \inf_{x \in A_x(T)} \|\pi x - \pi y\| \ . \tag{18}$$

Since the sets $A_x(T)$ and $A_y(T)$ are the attainable sets, most known methods for computing (18) will involve the solution of the dynamic equations (1) and (2) as an essential step.

If the epsilon technique [2], [3] is applied, however, (18) can be computed without solving the dynamic equations. Thus we formulate a non-dynamic problem for fixed $\epsilon' > 0$ and $\epsilon'' > 0$. We seek a sup-inf of the following functional, the time T being fixed,

$$\begin{aligned} h_T\left(\epsilon',\ \epsilon'' ;\ x,\ u;\ y,\ v\right) &= \|\pi x(T) - \pi y(T)\| \\ &+ \frac{1}{2\epsilon'} \int_0^T \left\|\dot{x}(t) - f\left(x(t),\ u(t),\ t\right)\right\|^2 dt \\ &- \frac{1}{2\epsilon''} \int_0^T \left\|\dot{y}(t) - g\left(y(t),\ v(t),\ t\right)\right\|^2 dt \ , \end{aligned} \tag{19}$$

over the class of absolutely continuous state functions $x(\cdot)$ and $y(\cdot)$ satisfying the given initial conditions, and over the class of control functions $u(\cdot)$ and $v(\cdot)$ subject to $u(t) \epsilon U$ and $v(t) \epsilon V$ for each time $t \epsilon [0, T]$. It will be shown in Theorem 3 that the solutions of this problem approximate as closely as desired the original sup-inf problem (18) for sufficiently small ϵ' and ϵ''. The epsilon problem can be solved computationally in many ways, e. g., by use of the gradient method, or Rayleigh-Ritz procedure, or Newton-Raphson method, or the combination thereof [3].

Now let us formulate the epsilon problem more exactly. Let X_1 be the class of absolutely continuous functions $x(\cdot)$ over $[0, T]$ subject to $x(0) = x_0$, with the derivative square integrable over $[0, T]$. Likewise, let Y_1 be the class of absolutely continuous functions $y(\cdot)$ over $[0, T]$ subject to $y(0) = y_0$, with the derivative square integrable over $[0, T]$. Let us define product spaces X and Y by

$$X = X_1 \times \Omega_u \ , \quad Y = Y_1 \times \Omega_v . \tag{20}$$

We introduce new notations defined by

$$\left.\begin{aligned} \phi(\cdot) = \big(x(\cdot),\ u(\cdot)\big) \ , \ \psi(\cdot) = \big(y(\cdot),\ v(\cdot)\big), \\ \epsilon = (\epsilon',\ \epsilon'') . \end{aligned}\right\} \tag{21}$$

Further let us define subsets of X and Y, respectively, by

$$\left.\begin{aligned} P = \{\phi(\cdot) = \big(x(\cdot, u),\ u(\cdot)\big) : u \in \Omega_u\} \subset X, \\ E = \{\psi(\cdot) = \big(y(\cdot, v),\ v(\cdot)\big) : v \in \Omega_v\} \subset Y, \end{aligned}\right\} \tag{22}$$

where $x(\cdot, u)$ and $y(\cdot, v)$ are solutions of the differential equations (1) and (2) corresponding to controls $u \in \Omega_u$ and $v \in \Omega_v$.
By using the notation (21), (19) can be abbreviated as

$$h_T(\varepsilon', \varepsilon''; x,u; y,v) = h_T(\varepsilon ; \phi ; \psi).$$

For proving the existence of the solution of the epsilon problem, we shall make another assumption:

<u>Assumption 3.</u> The sets $f(x,U,t)$ and $g(x,V,t)$ defined by

$$f(x,U,t) = \Big\{ f(x,u,t) \ : \ u \in U \Big\} \ , \ g(x,V,t) = \Big\{ g(x,v,t) \ : \ v \in V \Big\} \tag{23}$$

are convex, respectively, for every x and t.

Concerning the existence of the solution of the epsilon problem, we obtain the following theorem. The way of proving the theorem follows Balakrishnan [3] and Choudhury [6].

Theorem 2. Let us denote the sup-inf of (19) by $h_T(\epsilon', \epsilon'')$ or $h_T(\epsilon)$, i.e.,

$$h_T(\epsilon) = h_T(\epsilon', \epsilon'') = \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon', \epsilon''; \phi; \psi), \tag{24}$$

where the terminal time T is fixed. Let the sets U and V be compact. Then, under Assumptions 1, 2, and 3, the sup-inf is attained for each $\epsilon' > 0$ and $\epsilon'' > 0$; i.e., there exist $\phi^o \in X$ and $\psi^o \in Y$ such that

$$\begin{aligned} h_T(\epsilon', \epsilon'') &= \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon', \epsilon''; \phi; \psi) \\ &= h_T(\epsilon', \epsilon''; \phi^o; \psi^o). \end{aligned} \tag{25}$$

Proof. Let $\{x_n(\cdot), u_n(\cdot)\}$ be a minimizing sequence for (19), ψ being fixed. It can be shown that the sequence $x_n(\cdot)$ is equi-bounded and equi-continuous [2]. Hence, using an appropriate subsequence, we may take $x_n(\cdot)$ to converge uniformly to an absolutely continuous function $x^o(\cdot) \in X_1$. Also it can be shown that $\dot{x}_n(\cdot)$ converges weakly over $L_2(0, T)$ to $\dot{x}^o(\cdot)$ as in [2]. Now, as in Choudhury [6], let us define $\bar{x}_n(\cdot)$ by

$$\bar{x}_n(\cdot) = \frac{x_1(\cdot) + \cdots + x_n(\cdot)}{n} . \tag{26}$$

Then, since $\dot{x}_n(\cdot)$ converges weakly to $\dot{x}^o(\cdot)$, by the Mazur Theorem [7], it follows that $\dot{\bar{x}}_n(\cdot)$ converges strongly to $\dot{x}^o(\cdot)$. Also $\bar{x}_n(\cdot)$ converges uniformly to $x^o(\cdot)$.

By the convexity assumption (Assumption 3), there exists a $\bar{u}_n(t) \in U$ such that

$$\frac{1}{n} \sum_{i=1}^{n} f\left(\bar{x}_n(t), u_i(t), t\right) = f\left(\bar{x}_n(t), \bar{u}_n(t), t\right). \tag{27}$$

It can be shown that $\{\bar{x}_n, \bar{u}_n\}$ is a minimizing sequence [6]. In fact, let

$$z_n(t) = \dot{x}_n(t) - f\left(x_n(t),\ u_n(t), t\right),$$

$$\theta_n(t) = \frac{1}{n} \sum_{k=1}^{n} \left[f\left(x_k(t),\ u_k(t), t\right) - f\left(\bar{x}_n(t),\ u_k(t), t\right)\right]. \tag{28}$$

Then it is clear that

$$\dot{\bar{x}}_n(t) - f\left(\bar{x}_n(t),\ \bar{u}_n(t), t\right) = \theta_n(t) + \bar{z}_n(t), \tag{29}$$

where

$$\bar{z}_n(t) = \frac{z_1(t) + \cdots + z_n(t)}{n}$$

Now let us show that L_2- norm of $\theta_n(\cdot)$ defined by

$$\| \theta_n(\cdot) \|_2 = \left\{ \int_0^T \| \theta_n(t) \|^2 \, dt \right\}^{1/2}$$

converges to zero. Since $f(x,u,t)$ is continuously differentiable in x, the admissible controls are uniformly bounded, and $x_n(\cdot)$ and $\bar{x}_n(\cdot)$ converge uniformly to $x^o(\cdot)$, it follows that for arbitrary number $\varepsilon > 0$ there is an integer N such that if $k > N$, then

$$\left.\begin{aligned} &\left\| f\left(x_k(\cdot),\ u_k(\cdot), \cdot\right) - f\left(x^o(\cdot),\ u_k(\cdot), \cdot\right) \right\|_2 < \epsilon, \\ &\left\| f\left(\bar{x}_k(\cdot),\ u_k(\cdot), \cdot\right) - f\left(x^o(\cdot),\ u_k(\cdot), \cdot\right) \right\|_2 < \epsilon. \end{aligned}\right\} \tag{30}$$

Now it follows that

$$\begin{aligned} \| \theta_{N+n}(\cdot) \|_2 \leqq \frac{1}{N+n} \Bigg[&\sum_{k=1}^{N} \left\| f\left(x_k(\cdot),\ u_k(\cdot), \cdot\right) - f\left(\bar{x}_{N+n}(\cdot),\ u_k(\cdot), \cdot\right) \right\|_2 \\ &+ \sum_{k=N+1}^{N+n} \left\| f\left(x_k(\cdot),\ u_k(\cdot), \cdot\right) - f\left(x^o(\cdot),\ u_k(\cdot), \cdot\right) \right\|_2 \\ &+ \sum_{k=N+1}^{N+n} \left\| f\left(\bar{x}_{N+n}(\cdot),\ u_k(\cdot), \cdot\right) - f\left(x^o(\cdot),\ u_k(\cdot), \cdot\right) \right\|_2 \Bigg] \end{aligned}$$

$$< \frac{1}{N+n}\left[\sum_{k=1}^{N}\left\| f\left(x_k(\cdot),\ u_k(\cdot),\cdot\right) - f\left(\bar{x}_{N+n}(\cdot),\ u_k(\cdot),\cdot\right)\right\|_2 + 2n\epsilon\right]. \tag{31}$$

Letting $n \to \infty$, we can conclude that

$$\lim_{n\to\infty} \| \theta_n(\cdot) \|_2 = 0 . \tag{32}$$

From (29) and (32), we obtain

$$\lim_{n\to\infty} \left\| \dot{\bar{x}}_n(\cdot) - f\left(\bar{x}_n(\cdot),\ \bar{u}_n(\cdot),\cdot\right)\right\|_2 = \lim_{n\to\infty} \| \bar{z}_n(\cdot) \|_2 . \tag{33}$$

Since $z_n(\cdot)$ converges weakly to $z^o(\cdot)$ say, $\bar{z}_n(\cdot)$ converges strongly to $z^o(\cdot)$. Hence

$$\lim_{n\to\infty} \|\bar{z}_n(\cdot)\|_2 = \|z^o(\cdot)\|_2 \leqq \varliminf_{n\to\infty} \|z_n(\cdot)\|_2 . \tag{34}$$

Inequalities (33) and (34) show that $\bar{\phi}_n = \{\bar{x}_n,\ \bar{u}_n\}$ is a minimizing sequence for $h_T(\epsilon ; \phi ; \psi)$, ψ being fixed. Thus, we obtain

$$\inf_{\phi\in X} h_T(\epsilon; \phi ; \psi) = \lim_{n\to\infty} h_T(\epsilon ; \bar{x}_n,\ \bar{u}_n ; \psi)$$

$$= \lim_{n\to\infty} h_T(\epsilon ; x^o,\ \bar{u}_n ; \psi) . \tag{35}$$

The existence of an ordinary control that attains the infimum may be proved using the Blackwell theorem [8] on the range of a vector measure and Filippov lemma [4] as in Balakrishnan [3] and in Neustadt [9]. Now, since the function

$$F(t, u) = \frac{1}{2\varepsilon'} ||\dot{x}^o(t) - f(x^o(t), u, t)||^2$$

is continuous in u and the set U is compact, the set defined by

$$F(t, U) = \left\{ F(t, u) : u \varepsilon U \right\}$$

is compact. Hence, by the closure property of the range of vector integrals as proved by Blackwell [8], the set

$$\left\{ \int_0^T a(t)dt \;:\; a(t) \varepsilon F(t, U) \right\}$$

is closed. Therefore, by the same argument as in Blackwell [8] and in Neustadt [9], there is a measurable function $a(\cdot)$ such that

$$\int_0^T F(t, \bar{u}_n(t))dt \xrightarrow[n\to\infty]{} \int_0^T a(t)dt,$$

$$a(t) \varepsilon F(t, U) \quad \text{for every} \quad t \varepsilon [0, T].$$

By the Filippov lemma [4], [12], there is an admissible control $u^o(\cdot)$ such that

$$a(t) = F(t, u^o(t)) \quad \text{a.e. in} \quad [0, T].$$

Thus, we obtain that there is an admissible control $u^o \varepsilon \Omega_u$ such that

$$\lim_{n\to\infty} h_T(\varepsilon; x^o, \bar{u}_n; \psi) = h_T(\varepsilon; x^o, u^o; \psi)$$

$$= \inf_{\phi \varepsilon X} h_T(\varepsilon; \phi; \psi). \tag{36}$$

Since x^o and u^o are dependent on the value of $y(T)$, we write them as $x^o(\cdot, y(T))$ and $u^o(\cdot, y(T))$. Further let

$$\phi^o\big(y(T)\big) = \big(x^o(\cdot, y(T)), u^o(\cdot, y(T))\big).$$

Then

$$\inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) = h_T\big(\epsilon ; \phi^o(y(T)); \psi\big)$$

$$= \|\pi\, x^o\big(T, y(T)\big) - \pi\, y(T)\|$$

$$+ \frac{1}{2\epsilon'} \int_o^T \|\dot{x}^o\big(t, y(T)\big) - f\big(x^o(t, y(T)), u^o(t, y(T)), t\big)\|^2 dt$$

$$- \frac{1}{2\epsilon''} \int_o^T \| \dot{y}(t) - g\big(y(t), v(t), t\big)\|^2 dt . \tag{37}$$

If we define a function Φ_T by

$$\Phi_T\big(\epsilon' ; x, u ; y(T)\big) = \|\pi\, x(T) - \pi\, y(T)\|$$

$$+ \frac{1}{2\epsilon'} \int_o^T \| \dot{x}(t) - f\big(x(t), u(t), t\big)\|^2 dt , \tag{38}$$

then (37) can be rewritten as

$$\inf_{\phi \in x} h_T(\epsilon', \epsilon'' ; \phi; \psi) = h_T\big(\epsilon', \epsilon'' ; \phi^o(y(T)) ; y, v\big)$$

$$= \Phi_T\big(\epsilon' ; x^o(\cdot , y(T)), u^o(\cdot, y(T)); y(T)\big)$$

$$- \frac{1}{2\epsilon''} \int_o^T \| \dot{y}(t) - g\big(y(t), v(t), t\big)\|^2 dt . \tag{39}$$

Let $\{y_n(\cdot) , v_n(\cdot)\}$ denote a maximizing sequence for (39). In the same way as before, we may take the sequence $y_n(\cdot)$ to be converging uniformly to an absolutely continuous function $y^o(\cdot) \in Y_1$. Let us define another maximizing sequence $\{\bar{y}_n , \bar{v}_n\}$ such that

$$\left.\begin{aligned} \bar{y}_n(t) &= \frac{y_1(t) + \cdots + y_n(t)}{n}, \\ \frac{1}{n}\sum_{i=1}^{n} g\left(\bar{y}_n(t),\ v_i(t),\ t\right) &= g\left(\bar{y}_n(t),\ \bar{v}_n(t), t\right),\quad \bar{v}_n(t) \in V. \end{aligned}\right\} \tag{40}$$

For fixed $\bar{y}_n(T)$, since $\phi^o(\cdot, \bar{y}_n(T)) = (x^o(\cdot, \bar{y}_n(T)), u^o(\cdot, \bar{y}_n(T)))$ attains the infimum of $\Phi_T(\varepsilon'; x, u; \bar{y}_n(T))$, it follows that

$$\begin{aligned} &\Phi_T\left(\varepsilon';\ x^o(\cdot, \bar{y}_n(T)),\ u^o(\cdot, \bar{y}_n(T));\ \bar{y}_n(T)\right) \\ &\leqq \Phi_T\left(\varepsilon';\ x^o(\cdot, y^o(T)),\ u^o(\cdot, y^o(T));\ \bar{y}_n(T)\right). \end{aligned} \tag{41}$$

Since it holds from (41) that

$$\begin{aligned} &h_T(\varepsilon;\ \phi^o(\bar{y}_n(T));\ \bar{y}_n,\ \bar{v}_n) \\ &\leqq\ h_T(\varepsilon;\ \phi^o(y^o(T));\ \bar{y}_n,\ \bar{v}_n) \qquad \text{for all } n, \end{aligned} \tag{42}$$

we obtain

$$\begin{aligned} &\lim_{n\to\infty}\ h_T(\varepsilon;\ \phi^o(\bar{y}_n(T));\ \bar{y}_n,\ \bar{v}_n) \\ &\leqq \lim_{n\to\infty}\ h_T(\varepsilon;\ \phi^o(y^o(T));\ \bar{y}_n,\ \bar{v}_n). \end{aligned} \tag{43}$$

Since $\dot{\bar{y}}_n(\cdot)$ converges strongly to $\dot{y}^o(\cdot)$, and $y_n(\cdot)$ converges uniformly to $y^o(\cdot)$, it follows that

$$\begin{aligned} &\lim_{n\to\infty}\ h_T(\varepsilon;\ \phi^o(y^o(T));\ \bar{y}_n,\ \bar{v}_n) \\ &= \lim_{n\to\infty}\ h_T(\varepsilon;\ \phi^o(y^o(T));\ y^o,\ \bar{v}_n). \end{aligned}$$

Thus, from (43) we obtain

$$\lim_{n \to \infty} h_T(\varepsilon;\ \phi^o(\bar{y}_n(T));\ \bar{y}_n,\ \bar{v}_n)$$

$$\leqq \lim_{n \to \infty} h_T(\varepsilon;\ \phi^o(y^o(T));\ y^o,\ \bar{v}_n). \tag{44}$$

Since $\left\{ \bar{y}_n(\cdot),\ \bar{v}_n(\cdot) \right\}$ is a maximizing sequence for (39), (44) implies that

$$\lim_{n \to \infty} h_T(\varepsilon;\ \phi^o(\bar{y}_n(T));\ \bar{y}_n,\ \bar{v}_n)$$

$$= \lim_{n \to \infty} h_T(\varepsilon;\ \phi^o(y^o(T));\ y^o,\ \bar{v}_n)$$

$$= \sup_{\psi \varepsilon Y} \inf_{\phi \varepsilon X} h_T(\varepsilon;\ \phi;\ \psi). \tag{45}$$

The function $G(t, v) = \| \dot{y}^o(t) - g(y^o(t), v, t) \|^2/2\varepsilon''$ is continuous in v. Therefore, applying again the Blackwell theorem [8] and Filippov lemma [4], [12] as before, we obtain that there exists an admissible control $v^o(\cdot) \varepsilon \Omega_V$ such that

$$\lim_{n \to \infty} h_T\left(\epsilon\,;\phi^o(y^o(T))\,;y^o,\ \bar{v}_n\right)$$

$$= h_T\left(\epsilon;\phi^o(y^o(T));\ y^o, v^o\right)$$

$$= \sup_{\psi \epsilon Y} \inf_{\phi \epsilon X} h_T(\epsilon\,;\phi\,;\psi)\,. \qquad \text{(Q. E. D.)} \tag{46}$$

Now the relation between the solution of the epsilon problem (24) and the original problem (18) is given by the following theorem.

Theorem 3. Suppose there exist $\phi^o \epsilon X$ and $\psi^o \epsilon Y$, which are dependent on ε, such that (25) holds. Then

$$\lim_{\epsilon \to 0} h_T(\epsilon) = \lim_{\epsilon', \epsilon'' \to 0} h_T(\epsilon', \epsilon'') = \lim_{\epsilon \to 0} \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon; \phi; \psi)$$

$$= \sup_{y \in A_y(T)} \inf_{x \in A_x(T)} \|\pi x - \pi y\| . \tag{47}$$

<u>Proof.</u> $h_T(\epsilon ; \phi ; \psi)$ is defined by

$$h_T(\epsilon ; \phi ; \psi) = \| \pi x(T) - \pi y(T) \|$$

$$+ \frac{1}{2\epsilon'} \int_0^T \| \dot{x}(t) - f\big(x(t), u(t), t\big)\|^2 dt$$

$$- \frac{1}{2\epsilon''} \int_0^T \| \dot{y}(t) - g\big(y(t), v(t), t\big) \|^2 dt . \tag{48}$$

Now it is clear that the relation

$$\sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \geqq \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \tag{49}$$

holds for all $\psi \in Y$, and $\epsilon = (\epsilon', \epsilon'') > 0$. In particular, (49) holds for a $\psi = (y, v)$ which satisfies the differential Equation (2). Hence,

$$\sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \geqq \inf_{\phi \in X} \Big[\|\pi x(T) - \pi y(T) \|$$

$$+ \frac{1}{2\epsilon'} \int_0^T \| \dot{x}(t) - f\big(x(t), u(t), t\big)\|^2 dt \Big] \tag{50}$$

holds for any $\epsilon = (\epsilon', \epsilon'') > 0$ and $y(T) \in A_y(T)$. By [2, Theorem 3. 1] it follows that

$$\lim_{\epsilon' \to 0} \inf_{\phi \in X} \Big[\| \pi x(T) - \pi y(T)\| + \frac{1}{2\epsilon'} \int_0^T \| \dot{x}(t) - f\big(x(t), u(t), t\big)\|^2 dt \Big]$$

$$= \inf_{x \in A_x(T)} \| \pi x - \pi y(T) \| . \tag{51}$$

Letting $\epsilon \to 0$ in (50), we obtain

$$\lim_{\epsilon \to 0} \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \geqq \inf_{x \in A_x(T)} \| \pi x - \pi y(T) \| . \tag{52}$$

Since (52) holds for all $y(T) \in A_y(T)$, it follows that

$$\lim_{\epsilon \to 0} \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi)$$

$$\geqq \sup_{y \in A_y(T)} \inf_{x \in A_x(T)} \| \pi x - \pi y \| . \tag{53}$$

On the other hand, since $P \subset X$, where P is defined by (22), we obtain:

$$\inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \leqq \inf_{\phi \in P} h_T(\epsilon ; \phi ; \psi). \tag{54}$$

Further it is obvious that

$$\inf_{\phi \in P} \| \pi\, x(T) - \pi\, y(T) \| = \inf_{x \in A_x(T)} \| \pi x - \pi y(T) \| . \tag{55}$$

Therefore, the inequality

$$\inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \leqq \inf_{x \in A_x(T)} \| \pi x - \pi\, y(T) \|$$

$$- \frac{1}{2\epsilon''} \int_0^T \| \dot{y}(t) - g\Big(y(t), \ v(t), t\Big) \|^2 dt \tag{56}$$

holds for all $\epsilon = (\epsilon', \epsilon'') > 0$ and $\psi \in Y$. From (56) it follows that

$$\sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi) \leqq \sup_{\psi \in Y} \Bigg[\inf_{x \in A_x(T)} \| \pi x - \pi\, y(T) \|$$

$$- \frac{1}{2\epsilon''} \int_0^T \| \dot{y}(t) - g\Big(y(t), \ v(t), t\Big) \|^2 dt \Bigg] . \tag{57}$$

Applying [2, Theorem 3. 1] again, we obtain

$$\lim_{\epsilon'' \to 0} \sup_{\psi \in Y} \Bigg[\inf_{x \in A_x(T)} \| \pi x - \pi y(T) \|$$

$$- \frac{1}{2\epsilon''} \int_0^T \| \dot{y}(t) - g\big(y(t), v(t), t\big) \|^2 dt \Bigg]$$

$$= \sup_{y \in A_y(T)} \inf_{x \in A_x(T)} \| \pi x - \pi y \| . \tag{58}$$

Relations (57) and (58) imply that

$$\lim_{\epsilon \to 0} \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi)$$

$$\leqq \sup_{y \in A_y(T)} \inf_{x \in A_x(T)} \| \pi x - \pi y \| . \tag{59}$$

From (53) and (59), we finally obtain

$$\lim_{\epsilon \to 0} \sup_{\psi \in Y} \inf_{\phi \in X} h_T(\epsilon ; \phi ; \psi)$$

$$= \sup_{y \in A_y(T)} \inf_{x \in A_x(T)} \| \pi x - \pi y \| . \qquad \text{(Q. E. D.)} \tag{60}$$

Theorem 3 shows that the epsilon problem approximates the original pursuit and evasion problem as closely as desired and provides an approximating sequence of controls of pursuer and evader that approximates the optimum. Furthermore, Balakrishnan [3] showed in the particular example of optimal control problem that the solution may be relatively insensitive to how small ε has to be.

If for sufficiently small $\varepsilon' > 0$ and $\varepsilon'' > 0$, and for a suitable value of $T > 0$, the value of $h_T(\varepsilon' , \varepsilon'')$ is smaller than δ, then the pursuit may be regarded as attained.

In the case where the convexity condition (Assumption 3) does not hold, by introducing the relaxed or generalized controls as in [3], [10], [11], we can show that there exist the relaxed controls which attain the sup-inf of the epsilon problem (19).

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