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LINEAR BOUNDED PHASE COORDINATE CONTROL PROBLEMS UNDER CERTAIN REGULARITY AND NORMALITY CONDITIONS

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Abstract. In this paper, function space bounded phase coordinate control problems are considered by a functional analysis approach. Concepts of regularity and normality are defined and under these conditions, existence and uniqueness of solutions are discussed. Complete characterization of the solution is given in terms of a hyperplane. Furthermore, the relation of the normality condition to a finction space version of the Bang-Bang steering principle is pointed out.

1. Introduction.

In recent years, considerable attention has been focused upon the method of functional analysis in the study of optimal control problems which are, in many cases, describable in terms of the optimization of functionals on Banach spaces. This functional analysis method, though applicable to a wide range of problems, seems to be best suited for the investigation of optimization problems arising from linear control systems, since linearity plays an essential role in functional analysis.

In the articles [9] and [10], W.A.Porter formulated Neustadt's minimum effort control problem [8] in Banach space and presented the complete analysis of the abstract problem by using techniques of functional analysis. Also, in [12], a related Banach space minimization problem was considered.

In the present paper, we shall formulate and solve the abstract version of the corresponding bounded phase control problems. Specifically, let X, Y and Z be real Banach spaces. Let $S:X\to Y$ and $T:X\to Z$ be bounded linear transformations.

Problem I With T onto, $\xi\in Y$ and $\eta\in Z$, find an element, called an optimal solution, (if one exists) $u\in X$ satisfying the constraints $\eta=Tu$ and $\|\xi-Su\|\leqq\varepsilon$ $(\varepsilon>0)$ which minimizes $\|u\|$.

Problem II With S and T into, find an element (if one exists) $u\in\rho U_X=\{u \mid \|u\|\leq\rho,\ u\in X\}$ satisfying $\|\xi-Su\|\leq\varepsilon$ which minimizes $\|\eta-Tu\|$.

2. Some preliminaries.

Let us introduce notations and conventions adhered to throughout the paper. Let B be a real Banach space. Let C and D be two sets in B. By the vector sum $C+D$ is meant $C+D=\{c+d \mid c\in C, d\in D\}$, by $int(C)$ the interior of C, by ∂C the boundary of C and by $C\times D$ the rectangular set, i.e., $C\times D=\{(c,d) \mid c\in C, d\in D\}$. Let B' be the conjugate of B. For each $\phi\in B'$, suppose that there exists a vector $x\in U_B$, a closed unit ball in B, such that $\langle x,\phi\rangle=\|\phi\|$. Here, $\langle x,\phi\rangle$ denotes the value of a linear functional $\phi\in B'$ at a point $x\in X$. The set of all such vectors x in U_B is called an extremal of ϕ and is denoted by $\bar{\Phi}$ (see [9]). For convenience, we sometimes identify a suitable element $x\in\bar{\Phi}$ with the set $\bar{\Phi}$. It will be obvious from the context whether $\bar{\Phi}$ indicates the member or the set. If, for example, $\phi=0$, then $\bar{\Phi}$ denotes a suitable element in U_B, or the set U_B itself. Note that if $\phi\neq 0$, then $\bar{\Phi}\subset\partial U_B$.

A convex body is a convex set having a non-empty interior. A convex body K in a Banach space B is called smooth if at each of its boundary points, there is a unique hyperplane of support of K. Also, a convex body K in B is called rotund if K contains no straight-line segments in its boundary (see [14]). A Banach space B is called smooth or rotund according as its unit ball is smooth or rotund. Note that there exists at most one extremal $\bar{\Phi}$ of $\phi(\neq 0)\in B'$ if B is rotund.

3. Minimum effort control problem with bounded phase coordinate.

In this section, we shall consider Problem I in which T is assumed to be an onto mapping. The methods used in this and the next section are closely related mainly to [11] and others [4],[7].

Let $\hat{S}$ be a linear mapping of X into $Y\times Z$ defined by $\hat{S}:u\to(Su,Tu)$, where $Y\times Z$ denotes a product Banach space equipped with the usual product topology. Let $\hat{S}(U_X)$ denote the image of the unit ball U_X under $\hat{S}$. Motivated by the geometrical interpretation of Problem I, we shall examine the properties of the set $\{\alpha\hat{S}(U_X)+(\varepsilon U_Y\times\{0\})\}=C_\varepsilon(\alpha,0)$ for $\alpha>0$ (C.f. Porter [9]). Let us begin by introducing the following definition.

Definition: We shall say that a pair (ξ,η) is regular if there exists at least one element $u\in X$ satisfying the constraint $\eta=Tu$ and the strict inequality $\|\xi-Su\|<\varepsilon$.

Note that if $\hat{S}$ has dense range, an arbitrary pair (ξ,η) in $Y\times Z$ is regular.

Lemma 3.1. *The set $C_\varepsilon(\alpha,0)$ is a convex body.*

Proof. The lemma is an easy consequence of the assumption that T is an onto mapping and the interior mapping principle (see [3], pp.55).

Lemma 3.2. *Suppose that $(\xi,\eta)\in\partial C_\varepsilon(\alpha,0)$ is a regular pair. Then any hyperplane $(\phi_1,\phi_2)(\neq0)\in(Y\times Z)'$ of support of $C_\varepsilon(\alpha,0)$ at (ξ,η) satisfies*

(1) $$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\alpha\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|, \tag{3.1}$$

(2) $$\|S'\phi_1+T'\phi_2\|\neq0, \tag{3.2}$$

where S' denotes the conjugate of S.

Proof. By Lemma 3.1 and the Hahn-Banach theorem ([3], pp.58), such a hyperplane stated in the lemma exists:

$$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\langle\alpha(Su,Tu)+\varepsilon(y,0),(\phi_1,\phi_2)\rangle,\quad \text{for all } u\in U_X,\ y\in U_Y.$$

Hence taking the supremum of the right side yields (3.1). To see (3.2), suppose contrary that $S'\phi_1+T'\phi_2=0$. Then we necessarily have $\phi_1\neq0$ and, for all $u\in T^{-1}(\eta)\equiv\{u\,|\,\eta=Tu,\ u\in X\}$,

$$\|\xi-Su\|\,\|\phi_1\|\geqq\langle\xi-Su,\phi_1\rangle=\langle\xi,\phi_1\rangle+\langle u,T'\phi_2\rangle=\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\varepsilon\|\phi_1\|. \tag{3.3}$$

Hence

$$\|\xi-Su\|\geqq\varepsilon,\qquad \text{for all } u\in T^{-1}(\eta), \tag{3.4}$$

which contradicts the regularity of the pair (ξ,η).

The following lemma lists one property of the set $C_\varepsilon(\alpha,0)$.

Lemma 3.3. *Let $(\xi,\eta)\in\partial C_\varepsilon(\alpha,0)$ be a regular pair. Then for all $u\in X$ satisfying $\eta=Tu$ and $\|\xi-Su\|\leq\varepsilon$, we have*

$$\|u\|\geqq\alpha. \tag{3.5}$$

Proof. Let (ϕ_1,ϕ_2) be the hyperplane in Lemma 3.2. Then for all $u\in T^{-1}(\eta)$, we have

$$\|u\|\|S'\phi_1+T'\phi_2\|+\|\xi-Su\|\|\phi_1\|\geqq\langle u,S'\phi_1+T'\phi_2\rangle+\langle\xi-Su,\phi_1\rangle$$
$$=\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\alpha\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|. \tag{3.6}$$

Hence

$$(\|u\|-\alpha)\|S'\phi_1+T'\phi_2\|\geqq(\varepsilon-\|\xi-Su\|)\|\phi_1\|,\quad\text{for all } u\in T^{-1}(\eta). \tag{3.7}$$

Since we have $\|S'\phi_1+T'\phi_2\|\neq 0$, this proves the lemma.

Lemma 3.4. *Suppose that (ξ,η) is regular and that $(\xi,\eta)\in\partial C_\varepsilon(\alpha,0)$. Then for all $\hat\alpha>\alpha$, we have*

$$(\xi,\eta)\in int\{C_\varepsilon(\hat\alpha,0)\}\subset C_\varepsilon(\hat\alpha,0).$$

Proof. We first note that the assumption $(\xi,\eta)\in\partial C_\varepsilon(\alpha,0)$ and continuity of the linear form $\langle\ ,\ \rangle$ imply

$$\langle(\xi,\eta),(\psi_1,\psi_2)\rangle\leqq\alpha\|S'\psi_1+T'\psi_2\|+\varepsilon\|\psi_1\|,\quad\text{for all } (\psi_1,\psi_2)\in(Y\times Z)'. \tag{3.8}$$

Suppose now that the conclusion of the lemma is false and that there exists an $\hat\alpha>\alpha$ such that $(\xi,\eta)\notin int\{C_\varepsilon(\hat\alpha,0)\}$. Then a separating hyperplane $(\phi_1,\phi_2)(\neq 0)\in(Y\times Z)'$ exists:

$$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\hat\alpha\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|, \tag{3.9}$$

which, combined with the result of Lemma 3.2, contradicts (3.8).

Combining Lemma 3.3 and 3.4, we have the following theorem.

Theorem 3.1. *Problem I has a solution for each regular pair $(\xi,\eta)\in\partial C_\varepsilon(\alpha,0)$ if and only if $(\xi,\eta)\in C_\varepsilon(\alpha,0)$.*

Theorem 3.1 indicates that existence of solutions to Problem I depends upon whether or not the set $C_\varepsilon(\alpha,0)$ is closed in $Y\times Z$. We now state sufficient conditions to guarantee this situation.

Corollary. (C.f. [11]) Suppose that either of the following holds:

(A_1) X is a reflexive Banach space.

(A_2) Each Banach space is the conjugate of another normed space, i.e., normed spaces X_1, Y_1 and Z_1 exist such that $X=X_1'$, $Y=Y_1'$ and $Z=Z_1'$, respectively.

Then Problem I has a solution for every regular pair.

Proof. For each regular pair (ξ,η), let α_0 denote the infimum over the set of all real numbers $\alpha\geq 0$ such that $\{(\xi,\eta)\in C_\varepsilon(\alpha,0)\}$. It then follows easily that $(\xi,\eta)\in\partial C_\varepsilon(\alpha_0,0)$. Hence, it is sufficient to show that $C_\varepsilon(\alpha,0)$ $(\alpha\geq 0)$ is closed in $Y\times Z$. We shall do this by assuming (A_1). The case (A_2) may be treated similarly. Note first that $\hat{S}(U_X)$ is weakly compact as the continuous image of the weakly compact set U_X when Banach spaces X and $Y\times Z$ are equipped with their weak topologies (see [9]). Now, it is known ([3], pp.414) that if A and K are closed subsets of an additive topological group G, with K compact, then $A+K$ is closed. Since $U_Y\times\{0\}$ is convex, closed, hence weakly closed in $Y\times Z$ ([3], pp. 422), it follows that $C_\varepsilon(\alpha,0)$ is weakly closed, whence closed in $Y\times Z$.

The following lemma characterizes the regular pair (ξ,η) in the dual space.

Lemma 3.5. *A pair (ξ,η) is regular if and only if*

$$<(\xi,\eta),(\phi_1,\phi_2)> < \varepsilon\|\phi_1\| \tag{3.10}$$

holds for all $(\phi_1,\phi_2)(\neq 0)\in(Y\times Z)'$ satisfying $S'\phi_1+T'\phi_2=0$.

Proof. "Only if part". If (ξ,η) is a regular pair, then there exists an element $u\in X$ such that $\|\xi-Su\|<\varepsilon$ and $\eta=Tu$. Hence it follows from Lemma 3.4 that for $\alpha\geq\|u\|$, $(\xi,\eta)\in int\{C_\varepsilon(\alpha,0)\}$. Therefore, for all $(\psi_1,\psi_2)(\neq 0)\in(Y\times Z)'$, we have

$$<(\xi,\eta),(\psi_1,\psi_2)> < \alpha\|S'\psi_1+T'\psi_2\|+\varepsilon\|\psi_1\|, \tag{3.11}$$

from which (3.10) follows. "If part". Suppose that (ξ,η) is not a regular pair, i.e., for all u satisfying $\eta=Tu$, we have $\|\xi-Su\|\geq\varepsilon$. It then follows easily that (ξ,η) cannot be an interior point of $\{\hat{S}(X)+(\varepsilon U_Y\times\{0\})\}$. Hence there exists a separating hyperplane $(\phi_1,\phi_2)\in(Y\times Z)'$ such that

$$<(\xi,\eta),(\phi_1,\phi_2)>\geq<u,S'\phi_1+T'\phi_2>+\varepsilon\|\phi_1\| \text{, for all } u\in X, \tag{3.12}$$

which, in turn, implies $S'\phi_1+T'\phi_2=0$ and $<(\xi,\eta),(\phi_1,\phi_2)>\geq\varepsilon\|\phi_1\|$. But this contradicts (3.10).

We now state the main result in this section.

Theorem 3.2. *Suppose that (ξ,η) is a regular pair, and that either (A_1) or (A_2) in the corollary to Theorem 3.1 holds. Then an optimal solution u_0 of Problem I exists and is necessarily of the form:*

$$u_0=\frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|}\,(\overline{S'\phi_1+T'\phi_2}), \tag{3.13}$$

where $(\phi_1,\phi_2)\in(Y\times Z)'$ *of norm 1 solves either of the following:*

(1)
$$\begin{cases}\xi=\dfrac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|}\,S(\overline{S'\phi_1+T'\phi_2})+\varepsilon\overline{\phi}_1 & \text{(3.14-a)}\\ \eta=\dfrac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|}\,T(\overline{S'\phi_1+T'\phi_2}), & \text{(3.14-b)}\end{cases}$$

(2)
$$\max_{\|S'\phi_1+T'\phi_2\|\neq0}\left\{\frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|}\right\} \tag{3.15}$$

Conversely, if (ϕ_1,ϕ_2) *of norm 1 solves either of the above conditions, then the suitable element* $u_0\in\{(\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|)/\|S'\phi_1+T'\phi_2\|\}(\overline{S'\phi_1+T'\phi_2})$ *is optimal. Furthermore, if X is rotund, the solution is unique.*

Proof. Suppose that $u_0(\neq0)$ is an optimal solution, and we show (3.13)-(3.15). u_0 thus satisfies $\|\xi-Su_0\|\leqq\varepsilon$ and $\eta=Tu_0$. It further follows that $(\xi,\eta)\in\partial C_\varepsilon(\alpha_0,0)$, where we put $\|u_0\|=\alpha_0$. Let (ϕ_1,ϕ_2) be a hyperplane of support of $C_\varepsilon(\alpha_0,0)$ at (ξ,η). We then have, by Lemma 3.2, $S'\phi_1+T'\phi_2\neq0$ and

$$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\alpha_0\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|. \tag{3.16}$$

On the other hand, we have

$$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle=\langle\xi-Su_0,\phi_1\rangle+\langle u_0,S'\phi_1+T'\phi_2\rangle\leqq\alpha_0\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|. \tag{3.17}$$

Hence we conclude that

$$u_0=\alpha_0(\overline{S'\phi_1+T'\phi_2}), \tag{3.18}$$

$$\xi-Su_0=\varepsilon\overline{\phi}_1, \tag{3.19}$$

$$\alpha_0=\frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|}. \tag{3.20}$$

These relations yield (3.13) and (3.14). To see (3.15), note that by (3.8),

$$\langle(\xi,\eta),(\psi_1,\psi_2)\rangle\leqq\alpha_0\|S'\psi_1+T'\psi_2\|+\varepsilon\|\psi_1\|,\quad \text{for all } (\psi_1,\psi_2)\in(Y\times Z)'.$$

Hence for all $S'\psi_1+T'\psi_2\neq0$, we have

$$\alpha_0\geqq\frac{\langle(\xi,\eta),(\psi_1,\psi_2)\rangle-\varepsilon\|\psi_1\|}{\|S'\psi_1+T'\psi_2\|} \tag{3.21}$$

which, in view of (3.20), yields (3.15).

Conversely, suppose that (ϕ_1,ϕ_2) of norm 1 solves either of conditions (1) and (2). Let us first consider the case (1). Set $\alpha_0=\langle(\xi,\eta),(\phi_1,\phi_2)\rangle/\|S'\phi_1+T'\phi_2\|$. It then follows from (3.14) and the equality

$$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle=\alpha_0\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|=\sup_{\|u\|\leq 1,\|y\|\leq 1}\{\langle\alpha_0(Su,Tu)+(\varepsilon y,0),(\phi_1,\phi_2)\rangle\} \tag{3.22}$$

that $(\xi,\eta)\in C_\varepsilon(\alpha_0,0)\cap\partial C_\varepsilon(\alpha_0,0)$. Hence by Lemma 3.3, $u_0=\alpha_0\overline{(S'\phi_1+T'\phi_2)}$ is an optimal solution. Next, consider the latter case (2). In this case, we have, with α_0 defined as before,

$$\langle(\xi,\eta),(\psi_1,\psi_2)\rangle\leqq\alpha_0\|S'\psi_1+T'\psi_2\|+\varepsilon\|\psi_1\|, \tag{3.23}$$

for all $(\psi_1,\psi_2)\in(Y\times Z)'$ satisfying $S'\psi_1+T'\psi_2\neq 0$. But if $S'\psi_1+T'\psi_2=0$, we have, by Lemma 3.5, $\langle(\xi,\eta),(\psi_1,\psi_2)\rangle<\varepsilon\|\psi_1\|$. Hence (3.23) holds for all $(\psi_1,\psi_2)\in(Y\times Z)'$. This, in turn, implies $(\xi,\eta)\in C_\varepsilon(\alpha_0,0)\cap\partial C_\varepsilon(\alpha_0,0)$, with (ϕ_1,ϕ_2) defining a hyperplane of support of $C_\varepsilon(\alpha_0,0)$ at (ξ,η). Let $u_0\in\alpha_0U_X$ and $y_0\in\varepsilon U_Y$ be any preimage of (ξ,η) so that $(\xi,\eta)=\hat{S}u_0+(y_0,0)$. It then follows from Lemma 3.3 and (3.18) that u_0 is an optimal solution and $u_0\in\alpha_0\overline{(S'\phi_1+T'\phi_2)}$.

Finally, it remains to be shown that u_0 is unique if X is rotund. To this end, let u_0 and u_1 be two solutions. Then $\|u_0\|=\|u_1\|=\alpha_0$ and from (3.16) and (3.17), we have

$$\langle u_0,S'\phi_1+T'\phi_2\rangle=\langle u_1,S'\phi_1+T'\phi_2\rangle=\alpha_0\|S'\phi_1+T'\phi_2\|. \tag{3.24}$$

In other words, the hyperplane $S'\phi_1+T'\phi_2\neq 0$ supports α_0U_X at u_0 and u_1. This implies $u_0=u_1$ by rotundity of X.

Corollary 1. *Suppose that (ξ,η) is a regular pair. Then the following duality relation holds:*

$$\sup_{\|S'\phi_1+T'\phi_2\|\neq 0}\left\{\frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|}\right\}=\inf\{\|u\|\mid \|\xi-Su\|\leq\varepsilon,\ \eta=Tu,\ u\in X\}.$$

Corollary 2. *(ϕ_1,ϕ_2) defines a hyperplane of support of $C_\varepsilon(\alpha,0)$ at (ξ,η) if and only if the vector (ϕ_1,ϕ_2) solves either of the following:*

$$(1) \quad \begin{cases} \xi=\alpha S(\overline{S'\phi_1+T'\phi_2})+\varepsilon\bar{\phi}_1 \\ \eta=\alpha T(\overline{S'\phi_1+T'\phi_2}), \end{cases}$$

$$(2) \quad \max_{\|S'\phi_1+T'\phi_2\|\neq 0} \left\{ \frac{<(\xi,\eta),(\phi_1,\phi_2)>-\varepsilon\|\phi_1\|}{\|S'\phi_1+T'\phi_2\|} \right\} =\alpha.$$

Corollary 3. *Suppose that* $(\xi,\eta)\in\partial C_\varepsilon(\alpha,0)$ *is a regular pair, and that Banach spaces* X *and* Y *are both smooth. Then there are at most two hyperplanes* $(0,\phi_2)$ *and* $(\phi_1(\neq 0),\phi_2)$ *of support of* $C_\varepsilon(\alpha,0)$ *at* (ξ,η).

Proof. In proving the theorem, we have shown that ϕ_1 and $S'\phi_1+T'\phi_2(\neq 0)$ define support hyperplanes of εU_Y and $\alpha_0 U_X$ at $\xi-Su_0$ and u_0, respectively. Hence, by noting that T' is one to one, the stated result follows.

Corollary 4. *Suppose that* (ξ,η) *is a regular pair. Then the unique solution to the Hilbert space version of Problem I is given by*

$$u_0=\begin{cases} T^*(TT^*)^{-1}\eta, & \text{if } \|ST^*(TT^*)^{-1}\eta-\xi\|\leq\varepsilon, \\ (\lambda I+S^*S)^{-1}S^*\xi \\ \quad -(\lambda I+S^*S)^{-1}T^*\{T(\lambda I+S^*S)^{-1}T^*\}^{-1}\{T(\lambda I+S^*S)^{-1}S^*\xi-\eta\}, & \text{if } \|ST^*(TT^*)^{-1}\eta-\xi\|>\varepsilon, \end{cases}$$

where T^* *denotes the adjoint of* T, *and* $\lambda>0$ *is a constant uniquely determined by* $\|Su_0-\xi\|=\varepsilon$.

Proof. Note that Hilbert spaces are rotund and smooth, so that there exists a unique extremal $\bar{\phi}$ given by $\bar{\phi}=\phi/\|\phi\|$. This corollary follows from (3.13), (3.14), Corollary 3 and the next lemma (C.f. [6]).

Lemma 3.6. *Let* (ξ,η) *be a regular pair and suppose that the inequality*

$$\inf_{\substack{\|\xi-Su\|\leq\varepsilon \\ \eta=Tu}} \|u\|=\alpha_0 > \inf_{\eta=Tu} \|u\|=\alpha_1 \tag{3.25}$$

holds. Then the hyperplane $(\phi_1,\phi_2)(\neq 0)\in(Y\times Z)'$ *of support of* $C_\varepsilon(\alpha_0,0)$ *at* (ξ,η) *satisfies* $\phi_1\neq 0$.

Proof. Suppose contrary that $\phi_1=0$. Then from (3.1), we have

$$<\eta,\phi_2>\geqq\alpha_0\|T'\phi_2\|. \tag{3.26}$$

That is, $\eta\notin int\{\alpha_0 T(U_X)\}$. But this, in turn, contradicts (3.25)(C.f.[9]).

4. Minimization problem with bounded phase coordinate.

In the preceding section, the function space version of the minimum effort control problem with bounded phase coordinate was studied. Use of the set $C_\varepsilon(\alpha,0)$ directly led to the main results: existence theorem, necessary and sufficient conditions, uniqueness theorem for optimality. Attention now turns to the investigation for Problem II. We shall consider, in the present setting, the set $C_\varepsilon(\rho,\alpha)=\{\rho\hat{S}(U_X)+(\varepsilon U_Y\times\alpha U_Z)\}$ $(\varepsilon>0,\ \alpha>0)$. Most of the arguments we develop can parallel those of the preceding section.

Definition 4.1. We shall say that $\xi\in Y$ is (ε,ρ)-regular (with respect to S) if there exists at least one element $u\in\rho U_X$ which satisfies $\|\xi-Su\|<\varepsilon$.

Lemma 4.1. *Let ξ be an (ε,ρ)-regular element and suppose that $(\xi,\eta)\in\partial C_\varepsilon(\rho,\alpha)$. Then any hyperplane $(\phi_1,\phi_2)(\neq 0)$ of support of $C_\varepsilon(\rho,\alpha)$ at (ξ,η) satisfies*

(1) $\quad \langle(\xi,\eta),(\phi_1,\phi_2)\rangle\geqq\rho\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|+\alpha\|\phi_2\|,$ (4.1)

(2) $\quad \|\phi_2\|\neq 0.$ (4.2)

Proof. It is easy to see (4.1). Hence we shall show (4.2) by contradiction. Suppose that $\phi_2=0$. Then we have $\phi_1\neq 0$ and, for all $u\in X$,

$$\|u\|\|S'\phi_1\|+\|\xi-Su\|\|\phi_1\|\geqq\langle\xi,\phi_1\rangle\geqq\rho\|S'\phi_1\|+\varepsilon\|\phi_1\|.$$

Hence

$$(\|\xi-Su\|-\varepsilon)\|\phi_1\|\geqq(\rho-\|u\|)\|S'\phi_1\|\geqq 0,\qquad \text{for all } u\in\rho U_X.$$

This is contradictory to the assumption.

Corollary (C.f.[2]). *ξ is an (ε,ρ)-regular element if and only if*

$$\langle\xi,\phi\rangle<\rho\|S'\phi\|+\varepsilon\|\phi\|,\qquad \text{for all } \phi(\neq 0)\in Y'.$$

Throughout the section, we shall assume that ξ is an (ε,ρ)-regular element with respect to S.

Lemma 4.2. *Suppose that $(\xi,\eta)\in\partial C_\varepsilon(\rho,\alpha)$. Then for all $u\in\rho U_X$ satisfying $\|\xi-Su\|\leqq\varepsilon$, we have*

$$\|\eta-Tu\|\geqq\alpha. \tag{4.3}$$

Proof. With (ϕ_1,ϕ_2) defined in the previous lemma, we have, for all $u\in X$,

$$\|u\|\|S'\phi_1+T'\phi_2\|+\|\xi-Su\|\|\phi_1\|+\|\eta-Tu\|\|\phi_2\|\geqq(\xi,\eta)\geqq\rho\|S'\phi_1+T'\phi_2\|+\varepsilon\|\phi_1\|+\alpha\|\phi_2\|.$$

Hence

$$(\|\eta-Tu\|-\alpha)\|\phi_2\|\geqq(\rho-\|u\|)\|S'\phi_1+T'\phi_2\|+(\varepsilon-\|\xi-Su\|)\|\phi_1\|, \tag{4.4}$$

which, combined with Lemma 4.1, proves the lemma.

The following results are analogous to those in Lemma 3.4 and Theorem 3.1.

Lemma 4.3. *Suppose that* $(\xi,\eta)\in\partial C_\varepsilon(\rho,\alpha)$. *Then for all* $\hat{\alpha}>\alpha$, *we have*

$$(\xi,\eta)\in int\{C_\varepsilon(\rho,\hat{\alpha})\}\subset C_\varepsilon(\rho,\hat{\alpha}).$$

Theorem 4.1. *Problem II has a solution for each* $(\xi,\eta)\in\partial C_\varepsilon(\rho,\alpha)$ *if and only if* $(\xi,\eta)\in C_\varepsilon^*(\rho,\alpha)$.

In order to completely characterize the optimal solution in terms of the hyperplane, we need the following definitions.

Definition 4.2. We shall say that $\eta\in Z$ is normal (with respect to $(\hat{S},\xi,\varepsilon,\rho)$) if either

$$\inf_{\|u\|\leq\rho}\{\|\eta-Tu\|\}>\inf_{u\in X}\{\|\eta-Tu\|\} \tag{4.5}$$

or

$$\inf_{\|\xi-Su\|\leq\varepsilon}\{\|\eta-Tu\|\}>\inf_{u\in X}\{\|\eta-Tu\|\} \tag{4.6}$$

holds.

Definition 4.3. We shall say that $\eta\in Z$ is (ε,ρ)-normal (with respect to $(\hat{S},\xi)$) if

$$\inf_{\substack{\|u\|\leq\rho\\ \|\xi-Su\|\leq\varepsilon}}\{\|\eta-Tu\|\}>\inf_{\|\xi-Su\|\leq\varepsilon}\{\|\eta-Tu\|\} \tag{4.7}$$

holds.

Lemma 4.4. *Suppose that* (ϕ_1,ϕ_2) *supports* $C_\varepsilon(\rho,\alpha)$ *at* (ξ,η). *Then we have (C.f. LaSalle* [5], *Schmaedeke and Russell* [10])

(1) $\|S'\phi_1+T'\phi_2\|+\|\phi_1\|\neq 0$, *for each normal element* $\eta\in Z$,

(2) $\|S'\phi_1+T'\phi_2\|\neq 0$, *for each* (ε,ρ)*-normal element* $\eta\in Z$.

Proof. Proof is similar to that of Lemma 4.1.

We now summarize:

Theorem 4.2. *Assume that either* (A_1) *or* (A_2) *stated in the corollary to Theorem 3.1 holds. Then there exists a solution to Problem II for each* (ε,ρ)*-regular element* ξ. *Suppose, further, that* η *is a normal element. Then* u_0 *is an optimal solution if and only if* u_0 *takes the form*

$$u_0=\rho\,\overline{(S'\phi_1+T'\phi_2)}, \tag{4.8}$$

where (ϕ_1,ϕ_2) *of norm 1 is determined by either of the following*

(1)
$$\xi=\rho S\overline{(S'\phi_1+T'\phi_2)}+\varepsilon\overline{\phi}_1, \tag{4.9-a}$$
$$\eta=\rho T\overline{(S'\phi_1+T'\phi_2)}+\{(\langle\xi,\phi_1\rangle+\langle\eta,\phi_2\rangle-\rho\|S'\phi_1+T'\phi_2\|-\varepsilon\|\phi_1\|)/\|\phi_2\|\}\overline{\phi}_2, \tag{4.9-b}$$

(2) $\max_{\|(\phi_1,\phi_2(\neq 0))\|=1}\left\{\dfrac{<(\xi,\eta),(\phi_1,\phi_2)>-\rho\|S'\phi_1+T'\phi_2\|-\varepsilon\|\phi_1\|}{\|\phi_2\|}\right\}$. (4.10)

In this case, either $u_0\in\partial\{\rho U_X\}$ *or* $(\xi-Su_0)\in\partial\{\varepsilon U_Y\}$ *holds. Moreover, if* η *is an* (ε,ρ)*-normal element and if* X *is rotund, then* $u_0\in\partial\{\rho U_X\}$ *is unique.*

Application to a minimum effort problem.

As an application of the theory developed in this section, we shall consider a minimum effort control problem with amplitude constraints.

Let us suppose that a dynamical system is described by the linear differential equation:

$$dx(t)/dt=Ax(t)+Bu(t),$$

where $x(t)$ is an $n\times 1$ state vector, $u(t)$ is an $r\times 1$ control vector, and A,B are constant matries of appropriate dimensions. A control vector $u(t)$ which satisfies $|u_j(t)|\leq\rho$ $(j=1,\cdots,r)$ will be called admissible. The problem we shall consider is to find an admissible control vector $u(t)$ which drives the system from a given initial state $x(t_0)=x_0$ to an ε-neighborhood of the target state x^d, i.e., $\max_{1\leq j\leq n}|x_j(t_1)-x_j^d|=\|x(t_1)-x^d\|\leq\varepsilon$, while minimizing the fuel functional

$$I(u)=\int_{t_0}^{t_1}\sum_{j=1}^{r}|u_j(t)|\,dt,$$

where t_0 and t_1 are fixed intial and final times, respectively.

At the outset, we shall make the following (ε,ρ)-regularity assumption.

(A) There exists at least one admissible control $u(t)$ which enforces the system so that $\|x(t_1)-x^d\|<\varepsilon$.

To formulate the problem in function spaces, let us introduce some standard notations:

$L_p(r,[t_0,t_1])$: The space of (equivalence classes of) r-dimensional vector valued functions, deinfed and integrable (in the sense of Lebesgue) on the interval $[t_0,t_1]$ equipped with the norm

$$\|f\|=\left\{\int_{t_0}^{t_1}\sum_{j=1}^{r}|f_j(t)|^p dt\right\}^{1/p},\qquad f=(f_1,\cdots,f_r),\quad (1\leq p\leq+\infty)$$

where, for $p=+\infty$, the norm represents the essential supremum of f.

$l_\infty(n)$: The n-dimensional vector space equipped with the norm

$$\|x\|=\max_{1\leq j\leq n}|x_j|.$$

$C(r,[t_0,t_1])$: The space of r-dimensional vector valued continuous functions defined on $[t_0,t_1]$ equipped with the norm

$$\|f\|=\max_{t_0\le t\le t_1}\max_{1\le j\le r}|f_j(t)|,$$

$NBV(r,[t_0,t_1])$: The space of r-dimensional vector valued (normalized) functions of bounded variation on $[t_0,t_1]$ equipped with the norm

$$\|f\|=\sum_{j=1}^{r}v(f_j,[t_0,t_1]),\qquad f(t_0)=0,$$

$v(f_j,[t_0,t_1])$ denoting the total variation of f_j on $[t_0,t_1]$ (see [3], pp. 241).

We then specify the basic function spaces and linear operators as follows.

$$X=L_\infty(r,[t_0,t_1]),\qquad Y=L_1(r,[t_0,t_1]),\qquad Z=l_\infty(n),$$

$$T(X\to Y):\quad Tu=-u \text{ (the natural embedding of } X \text{ into } Y),$$

$$S(X\to Z):\quad Su=\int_{t_0}^{t_1}e^{A(t_1-s)}Bu(s)ds.$$

By taking $\xi=x^d-e^{A(t_1-t_0)}x_0$ and $\eta=0$, the problem at hand is seen to be described in terms of Problem II. Note first that, since $L_1(r,[t_0,t_1])$ can continuously and isometrically be embedded into $NBV(r,[t_0,t_1])$, and since $L_\infty(r,[t_0,t_1])$ and $NBV(r,[t_0,t_1])$ can be identified, respectively, with the duals of $L_1(r,[t_0,t_1])$ and $C(r,[t_0,t_1])$, the assumption (A_2) in the corollary to Theorem 3.1 is, in this case, satisfied. Thus, by applying Theorem 4.1, we have

$$u_0=\rho(\overline{S'\phi_1-\phi_2})=\{(<\xi,\phi_1>-\rho\|S'\phi_1-\phi_2\|-\varepsilon\|\phi_1\|)/\|\phi_2\|\}\overline{\phi}_2. \tag{4.11}$$

Here $(S'\phi_1)(t)=B^*e^{A^*(t_1-t)}\phi_1$ is an analytic function, and the extremals $(\overline{S'\phi_1-\phi_2})\in L_\infty(r,[t_0,t_1])$ and $\overline{\phi}_2\in L_\infty(r,[t_0,t_1])$ are given, respectively, by (C.f.[11])

$$(\overline{S'\phi_1-\phi_2})_j(t)=\begin{cases}sign[(S'\phi_1-\phi_2)_j(t)], & t\in A_j=\{t\in[t_0,t_1]\,|\,(S'\phi_1-\phi_2)_j(t)\ne0\},\\ |(\overline{S'\phi_1-\phi_2})_j(t)|\le1, & t\in A_j^c \text{ (the complement of } A_j),\end{cases}$$

$$(\overline{\phi_2})_j(t)=\begin{cases}(\overline{\phi_2})_j(t)\ge0, & t\in B_j^+=\{t\in[t_0,t_1]\,|\,(\phi_2)_j(t)=\|\phi_2\|\},\\ (\overline{\phi_2})_j(t)\le0, & t\in B_j^-=\{t\in[t_0,t_1]\,|\,(\phi_2)_j(t)=-\|\phi_2\|\},\\ 0, & t\in(B_j^+\cup B_j^-)^c,\end{cases}$$

where $j=1,\cdots,r$ and

$$\int_{t_0}^{t_1}\sum_{j=1}^{r}|(\phi_2)_j(t)|\,dt=1 .$$

Hence, by (4.11), the optimal solution can be characterized in a more explicit form:

$$(u_0)_j(t)=\begin{cases}\rho\, sign[(S'\phi_1-\phi_2)_j(t)], & t\in A_j\subset(B_j^+\cup B_j^-),\\ 0\leqq(u_0)_j(t)\leqq\rho, & t\in A_j^c\cap B_j^+,\\ -\rho\leqq(u_0)_j(t)\leqq 0, & t\in A_j^c\cap B_j^-,\\ 0, & t\in(B_j^+\cup B_j^-)^c.\end{cases} \qquad (4.12)$$

We shall show that, if the matrix A is non-singular, then $mes[A_j^c\cap(B_j^+\cup B_j^-)]$ the measure of the set $A_j^c\cap(B_j^+\cup B_j^-)$ is zero, and hence the controls $(u_0)_j(t)$ $(j=1,\cdots,r)$ are uniquely determined by (4.12) (C.f.[1], [5]).

To see this, suppose contrary that $mes[A_j^c\cap(B_j^+\cup B_j^-)]$ is positive for some j $(1\leqq j\leqq r)$. Then, by appealing to analyticity of the function $(S'\phi_1)_j(t)$, we have

$$|(S'\phi_1)_j(t)|=\|\phi_2\|, \qquad \text{for all } t\in[t_0,t_1].$$

On the other hand, it can easily be deduced that, A being non-singular, the function $(S'\phi_1)_j(t)$ is equal to a constant if and only if $(S'\phi_1)_j(t)=0$ for all $t\in[t_0,t_1]$. But this contradicts $\|\phi_2\|\neq 0$ by Lemma 4.1.

Remark. In order that $S'\phi_1-\phi_2\neq 0$ for every hyperplane $(\phi_1,\phi_2)(\neq 0)$ of support of $C_\varepsilon(\rho,\alpha)$ at $(\xi,0)$, it is necessary and sufficient that

$$\min_{\substack{\|u\|\leq\rho\\ \|\xi-Su\|\leq\varepsilon}}\|u\| > \inf_{\|\xi-Su\|\leq\varepsilon}\|u\|.$$

Hence, it follows that if A is non-singular, the fuel functional $I(u)$ can be made smaller by enlarging the admissible class of control functions so as to include impulses, and in this case the optimal control u_0 may consist of impulse functions.

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On Certain Necessary and Sufficient Conditions for Singular or Bang-Bang Controls

N. Minamide and K. Nakamura

ABSTRACT

In this short monograph, necessary and sufficient conditions for the existence of Bang-Bang controls to the Final Value Problem and the Time Optimal Problem are studied. It is shown that the j-th component $u_j(t)$ of the optimal control $u(t)$ ($|u_i(t)| \leq 1$, $i=1,\cdots,r$) is of Bang-Bang type if and only if the release of the amplitude constraint imposed on the j-th component brings in the better index of performance than otherwise.

1. Introduction.

In solving optimal control problems, we sometimes encounter situations in which Pontryagin's "Maximum Principle" may provide no information for determining optimal controls. These situations are referred to as "singular ones" and the corresponding solutions as "singular controls". The sigular solutions have recently received significant attention [3-11]. Most of the papers are concerned with optimization problems in which the system equations and the index of performance are linear with respect to the control inputs. For this class of problems, the optimal control turns out to be either of Bang-Bang type or of singular type, as Pontryagin's M.P. indicates. D.H.Jacobson[10,11] obtained new necessary conditions for singular solutions to be optimal by using Differential Dynamic Programming methods. J.P.McDanell and W.F.Powers [8] proposed new Jacobi-type necessary and sufficient conditions for the second variation.

In the study of the time optimal control problem, J.P.Lasalle defined in [1] the concept of "Normal Systems" and guaranteed the Bang-Bang optimal control for these systems. Athans and Falb [2] also paid particular attention to the discussion on the existence of singular controls for the similar kinds of control problems. The purpose of the present monograph is to give further investigation for existence or non-existence of singular controls to a particular kind of control problems, i.e., the final value problem and the time optimal problem.

2. Problem statement.

Let the dynamical system be described by the following differential equation:

$$dx(t)/dt=Ax(t)+Bu(t), \quad x(t_0)=x_0, \tag{2.1}$$

where $x(t)$ is an $n\times 1$ state vector, $u(t)$ is an $r\times 1$ control vector and A,B are constant matrices of appropriate dimensions, respectively. A (measurable) control vector function u(t) is called admissible if each component $u_j(t)$ satisfies

$$|u_j(t)|\leq 1, \quad (j=1,\cdots,r). \tag{2.2}$$

We shall consider the following two problems;

FVP(Final Value Problem): Given the fixed final time t_f and the final desired state x^d, find an admissible control u(t) which minimizes

$$I=\left\{ \sum_{j=1}^{n} |x_j(t_f)-x_j^d|^p\right\}^{1/p}=\|x(t_f)-x^d\|. \qquad (1\leq p\leq +\infty)$$

TOP(Time Optimal Problem): Find an admissible control u(t) which enforces the system (2.1) from $x(t_0)=x_0$ to the origin in the minimum time $t^*\geq t_0$.

3. Investigation to FVP.

We shall first study FVP and show how to deal with the problem. We shall prepare some lemmas and definitions.

Definition 3.1. We shall denote by R(t) the reachable set (from the origin) at time t:

$$R(t)=\left\{x(t)\,\middle|\, x(t)=\int_0^t \Phi(t-\tau)Bu(\tau)d\tau,\ |u_j(t)|\leq 1\ (j=1,\cdots,r)\right\},$$

where $\Phi(t)=e^{At}$ is the transition function of the system (2.1).

Definition 3.2. We shall denote by $R_i(t)$ the reachable set at time t with i-th component released from the amplitude constraint:

$$R_i(t)=\left\{x(t)\,\middle|\, x(t)=\int_0^t \Phi(t-\tau)Bu(\tau)d\tau,\ |u_j(t)|\leq 1\ (j\neq i),\ |u_i(t)|<+\infty\right\}$$

Lemma 3.1. Both $R(t)$ and $R_1(t)$ are closed convex sets in $\ell(n;p)$, where $\ell(n;p)$ denotes the n-dimensional vector space equipped with the norm $\|x\|=(\sum_{i=1}^{n}|x_i|^p)^{1/p}$.

Proof. The closure of $R(t)$ and the convexity of $R(t)$ and $R_1(t)$ can be shown as in [1]. To see the closure of $R_1(t)$, note that any subspace in a finite dimensional space is closed, and the vector sum of A and B, with A closed and B compact, is also closed.

We further consider the following two sets;

$$R(t,\alpha)=R(t)+\alpha U(\ell(n;p))=\{y \mid y=y_1+y_2,\ y_1\in R(t),\ \|y_2\|\leq\alpha\},$$

$$R_i(t,\alpha)=R_i(t)+\alpha U(\ell(n;p)).$$

Lemma 3.2. Both $R(t,\alpha)$ and $R_1(t,\alpha)$ are closed convex sets in $\ell(n;p)$.

Proof. Proof is similar to that of Lemma 3.1.

Lemma 3.3. Suppose that $(x^d-\Phi(t_f,t_0)x_0)\in\partial R(t_f,\alpha)$. Then for all admissible control $u(t)$, we have

$$\|x(t_f)-x^d\|\geq\alpha.$$

Proof. Since $R(t_f,\alpha)$ is closed convex body, and $x^d-\Phi(t_f,t_0)x_0$ lies on the boundary of $R(t_f,\alpha)$, there exists a hyperplane such that

$$\langle x^d-\Phi(t_f,t_0)x_0,\phi\rangle\geq\langle R(t_f,\alpha),\phi\rangle, \tag{3.1}$$

where $\langle x,\phi\rangle=\sum_{i=1}^{n}x_i\phi_i$ denotes the bi-linear form and

$$\langle R(t_f,\alpha),\phi\rangle=\sup_{y\in R(t_f,\alpha)}\langle y,\phi\rangle=\langle R(t_f)+\alpha U(\ell(n;p)),\phi\rangle=\int_{t_0}^{t_f}\|B^*\Phi^*(t_f,s)\phi\|ds+\alpha\|\phi\|.$$

On the other hand, we have

$$\int_{t_0}^{t_f}\sum_{j=1}^{r}|u_j(t)|\cdot|b_j^*\Phi^*(t_f,t)\phi|dt+\sum_{j=1}^{n}|x_j(t_f)-x_j^d|\cdot|\phi_j|$$
$$\geq\int_{t_0}^{t_f}\sum_{j=1}^{r}u_j(t)b_j^*\Phi^*(t_f,t)\phi dt+\langle x^d-\Phi(t_f,t_0)x_0,\phi\rangle-\int_{t_0}^{t_f}\langle\Phi(t_f,t)Bu(t),\phi\rangle dt$$
$$=\langle x^d-\Phi(t_f,t_0)x_0,\phi\rangle\geq\int_{t_0}^{t_f}\|B^*\Phi^*(t_f,t)\phi\|dt+\alpha\|\phi\|. \tag{3.2}$$

Hence,

$$(\|x(t_f)-x^d\|-\alpha)\|\phi\|\geq\int_{t_0}^{t_f}\sum_{j=1}^{r}|b_j^*\Phi^*(t_f,s)\phi|ds-\int_{t_0}^{t_f}\sum_{j=1}^{r}|u_j(t)|\cdot|b_j^*\Phi^*(t_f,t)\phi|dt\geq 0, \tag{3.3}$$

which proves the lemma.

Lemma 3.4. $u_0(t)$ defines an optimal solution to FVP if and only if $u_0(t)$ takes the form:

$$(u_0)_j(t)=\operatorname{sign}[b_j^*\Phi^*(t_f,t)\phi] \tag{3.4}$$

where ϕ satisfies either of the following

$$(1)\quad x_j^d-(\Phi(t_f,t_0)x_0)_j=\int_{t_0}^{t_f}\{\Phi(t_f,s)B\,\mathrm{sign}[B^*\Phi^*(t_f,s)\phi]\}_j\,ds$$
$$+\{\langle x^d-\Phi(t_f,t_0)x_0,\phi\rangle-\int_{t_0}^{t_f}\|B^*\Phi^*(t_f,s)\phi\|ds\}\|\phi\|^{q-2}|\phi_j|\,\mathrm{sign}[\phi_j],$$
$$(j=1,\cdots,n)\quad\cdots(3.5)$$

$$(2)\quad \max_{\|\phi\|\neq 0}\left\{\frac{\langle x^d-\Phi(t_f,t_0)x_0,\phi\rangle-\int_{t_0}^{t_f}\|B^*\Phi^*(t_f,s)\phi\|ds}{\|\phi\|}\right\}(=\alpha_0).\quad(3.6)$$

Proof. Let α_0 denote the infimum of α such that $\{x^d-\Phi(t_f,t_0)x_0\}\in R(t_f,\alpha)$. It then easily follows from Lemma 3.1 that $\{x^d-\Phi(t_f,t_0)x_0\}\in\partial R(t_f,\alpha)$. Let an admissible control $u_0(t)$ and a vector in a unit ball in $\ell(n;p)$ y_0 be such that

$$x^d-\Phi(t_f,t_0)x_0=\int_{t_0}^{t_f}\Phi(t_f,t)Bu_0(s)ds+\alpha_0 y_0. \qquad (3.7)$$

Lemma 3.3 then shows that $u_0(t)$ is an optimal solution to FVP. The conclusions of Lemma 3.4 now follow from Eqs. (3.1), (3.7) and the easily established fact: $(x^d-(t_f,t_0)x_0)\in R(t_f,\alpha_0)$ implies and is implied by

$$\langle x^d-\Phi(t_f,t_0)x_0,\varphi\rangle\leq\langle R(t_f,\alpha_0),\varphi\rangle,\quad \text{for all } \varphi\in U_r, \qquad (3.8)$$

where equality holds for some $\varphi(\neq 0)\in U(\ell(n;p))$ if and only if $\{x^d-(t_f,t_0)x_0\}\in\partial R(t_f,\alpha_0)$.

It is, at this point, to be observed that, if $b_j^*\Phi^*(t_f,t)\phi\neq 0$ except on the set of measure-zero, $(u_0)_j(t)$ can be uniquely determined by the condition $(u_0)_j(s)=\mathrm{sign}[b_j^*\Phi^*(t_f,s)\phi]$.

Definition 3.3. We shall say that the j-th component of the optimal solution $u_0(t)$ is of Bang-Bang control with respect to a hyperplane ϕ if $b_j^*\Phi^*(t_f,t)\phi\neq 0$ holds except on the set of measure-zero.

Now, we are persenting one of our main results.

Theorem 3.1.(C.f.[13]) In order for the j-th component of the optimal solution $u_0(t)$ to be of Bang-Bang control with respect to any hyperplane of support of $R(t,\alpha)$, it is necessary and sufficient that

$$\inf_{\substack{|u_i|\leq 1\\(i=1,\cdots,j-1,j+1,\cdots,r)}}\|x^d-x(t_f)\|<\inf_{\substack{|u_j(t)|\leq 1\\(j=1,\cdots,r)}}\|x^d-x(t_f)\|\equiv\alpha. \qquad (3.9)$$

In this case, the j-th component $(u_0)_j(t)$ is unique.

Proof. Necessity: Proof proceeds by showing contradiction. Suppose, contrary, that the reverse inequality in (3.9) holds. It then follows easily that

$$(x^d - \Phi(t_f, t_0)x_0) \in \partial R_j(t_f, \alpha).$$

Since $R_j(t_f, \alpha)$ is a convex body, there exists a hyperplane of support of $R_j(t_f,\alpha)$ at $(x^d - \Phi(t_f, t_0)x_0)$ such that

$$\langle x^d - \Phi(t_f, t_0)x_0, \varphi\rangle \geqq \langle R_j(t_f,\alpha), \varphi\rangle.$$

By appealing to the definition of the set $R_j(t_f, \alpha)$, we can easily arrive at the conclution:

$$b_j^* \Phi^*(t_f, t)\varphi = 0, \qquad \text{for all } t \in [t_0, t_f],$$

contradicting the assumption of the theorem.

Sufficiency: Again, we shall show by contradiction. Suppose that there exists a supporting hyperplane $\phi \in U(\ell(n;p))$ such that $[b_j^* \Phi^*(t_f, t)\phi] = 0$ on the set of positive-measure. Since $[b_j^* \Phi^*(t_f,t)\phi]$ is an analytic function of t, this assumption implies

$$b_j^* \Phi^*(t_f, t)\phi = 0, \quad \text{for all } t \in [t_0, t_f].$$

Then, for all $u_j(t)$ satisfying $\operatorname{ess\,sup}_{t_0 \leqq t \leqq t_f} |u_j(t)| < +\infty$ $(j = 1, \cdots, r)$, we have

$$\|x(t_f) - x^d\| \cdot \|\phi\| + \int_{t_0}^{t_f} \sum_{j=1} |u_j(t)| \cdot |b_j^* \Phi^*(t_f, t)\phi| dt$$

$$\geqq \langle x^d - \Phi(t_f, t_0)x_0, \phi\rangle \geqq \int_{t_0}^{t_f} \sum_{i \neq j} |b_i^* \Phi^*(t_f, t)\phi| dt + \alpha \|\phi\|,$$

i.e., for all $|u_i(t)| \leqq 1$ $(i \neq j)$,

$$(\|x(t_f) - x^d\| - \alpha)\|\phi\| \geqq \int_{t_0}^{t_f} \sum_{i \neq j} |b_i^* \Phi^*(t_f, t)\phi| dt - \int_{t_0}^{t_f} \sum_{i \neq j} |u_i(t)| \cdot |b_i^* \Phi^*(t_f, t)\phi| dt \geqq 0,$$

which contradicts the assumption. The proof of the uniqueness may be done as in [1].

4. Investigation to TOP.

In the previous section, we showed the necessary and sufficient condition for Bang-Bang controls, or in other words, for non-existence of singular controls. This result is now extended to TOP in this section.

We first remark that finding a control $u(\cdot)$ which transfers the system (2.1) from $x(t_0) = x_0$ to the origin at time t

$$0 = x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

is equivalent to finding u(·) such that

$$-e^{-At_0}x_0=\int_{t_0}^{t}e^{-A\tau}Bu(\tau)d\tau.$$

Hence we consider the following two sets that correspond to R(t) and $R_1(t)$ in the previous section, respectively.

Definition 4.1. $A(t)=\{y \mid y=\int_{t_0}^{t}e^{-As}Bu(s)ds,\ |u_j|\leq 1\ (j=1,\cdots,r)\}.$

Definition 4.2.

$$A_j(t)=\{y \mid y=\int_{t_0}^{t}e^{-As}Bu(s)ds,\ |u_i|\leq 1\ (i\neq j),\ |u_j|<+\infty\}.$$

Definition 4.3.

$$B_t=\{y \mid y=\int_{t_0}^{t}e^{-As}Bu(s)ds,\quad \operatorname*{ess\,sup}_{t_0\leq s\leq t}|u_j(s)|<+\infty\ (j=1,\cdots,r)\}.$$

Lemma 4.1. Define, as usual, the distance of two sets A,B in $\ell(n;p)$ by

$$d(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}\{\|a-b\|\},\ \sup_{b\in B}\inf_{a\in A}\{\|a-b\|\}\}. \quad (4.1)$$

A(t) and $A_j(t)$ then are continuous set functions of t with respect to the topology induced by (4.1).

Proof. Proof may be done as in [1], and hence omitted.

Lemma 4.2. For $t>t_0$ and s>0, we have

$$B_t=B_{t+s}=R(A|B),$$

where R(A|B) denotes the range of $(A|B)\equiv\{B \mid AB \mid \cdots \mid A^{n-1}B\}$.

Proof. See, for example, the reference [12, Chapter 2].

Lemma 4.3. Let C denote the matrix whose columns are constructed from all the independent column vectors of (A|B). We then have;

(a) R(C)=R(A|B),

(b) rank(C)=dim(R(C*))=dim(R(A|B)),

(c) $N(C)=\{x \mid Cx=0\}=\{0\}$.

Proof. Easy proof is omitted.

Lemma 4.4. Suppose that $e^{-At_0}x_0\in R(A|B)$. Then u(s) solves the equation

$$-C^*e^{-At_0}x_0=C^*\int_{t_0}^{t}e^{-As}Bu(s)ds \quad (4.2)$$

if and only if it solves

$$-e^{-At_0}x_0=\int_{t_0}^{t}e^{-As}Bu(s)ds. \quad (4.3)$$

Proof. The proof of "if part" being obvious, we shall show "only if part". Let $u(s)$ solve Eq.(4.2). We then have, from Lemma 4.2 and the hypothesis,

$$\{e^{-At_0}x_0+\int_{t_0}^{t} e^{-As}Bu(s)ds\} \in \mathcal{R}(C)\cap \mathcal{N}(C^*)=\{0\},$$

which proves the lemma.

Lemma 4.5. Let t^* denote the minimum time of TOP, i.e.,

$$-e^{-At_0}x_0=\int_{t_0}^{t^*} e^{-As}Bu^*(s)ds, \text{ for some } |u_j^*(s)|\leq 1 \ (j=1,\cdots,r).$$

Then,

$$-C^* e^{-At_0}x_0 \in \partial\{C^*(A(t^*))\}.$$

Proof. It can be shown, by using Lemma (4.1), that t^* is the minimum time if and only if (C.f.[1])

$$-e^{-At_0}x_0 \in \partial\{A(t)\}. \tag{4.4}$$

Since C^* is onto, and consequently maps open sets into open sets, we easily see, by vitue of Lemma 4.1, Lemma 4.4 and Eq.(4.4), that

$$-C^* e^{-At_0}x_0 \in \partial\{C^*(A(t^*))\}.$$

We are now ready to state the main result.

Theorem 4.1. Suppose that $-e^{-At_0}x_0 \in R(A|B)$. In order for the j-th component of the time optimal solution $u_0(t)$ to be of Bang-Bang control with respect to each non-trivial support hyperplane belonging to $R(A|B)=R(C)$, it is necessary and sufficient that

$$\inf\{t \mid -e^{-At_0}x_0 \in A(t)\}=t^* > \inf\{t \mid -e^{-At_0}x_0 \in A_j(t)\}\equiv \bar{t}. \tag{4.5}$$

Moreover, in this case, the j-th component $(u_0)_j(t)$ is unique.

Proof. The necessity of the condition (4.5) can be shown similarly as in Theorem 2.1. We shall hence show the sufficiency. Let $\varphi=C\phi\in R(C)$ be a supporting hyperplane of the set $A(t^*)$ at $-e^{-At_0}x_0$:

$$\langle -e^{-At_0}x_0, C\phi\rangle \geq \int_{t_0}^{t^*} \|B^* e^{-A^*s}C\phi\| ds. \tag{4.6}$$

We shall then show that $b_j^* e^{-A^*s}C\phi \neq 0$ for all $s\in[t_0,t^*]$ except on the set of measure-zero. To see this, suppose contrary that $b_j^* e^{-A^*s}C\phi=0$ on the set of positive-measure. Eq.(4.6) then shows that ϕ defines a support hyperplane of $C^*(A_j(t^*))$ as well as of $C^*(A(t^*))$ at $-e^{-At_0}x_0$.

Let $\varepsilon>0$ be any positive number such that $t^*>\bar{t}+\varepsilon>\bar{t}$. We then have, by the assumption of the theorem, $-e^{-At_0}x_0 \in A_j(t+\varepsilon)\subset A_j(t^*)$. Since ϕ supports $C^*(A_j(t^*))$ at $-C^*e^{-At_0}x_0$, it must be that ϕ also supports $C^*(A_j(t+\varepsilon))$ at $-C^*e^{-At_0}x_0$:

$$\langle -C^* e^{-At_0}x_0, \phi\rangle \geq \langle C^*(A_j(\bar{t}+\varepsilon)), \phi\rangle = \int_{t_0}^{\bar{t}+\varepsilon} \| B^* e^{-A^*s} C\phi \| ds. \tag{4.7}$$

The reverse inequalities in (4.6) and (4.7) are obvious, so that we have

$$\int_{\bar{t}+\varepsilon}^{t^*} \| B^* e^{-A^*s} C\phi \| ds = 0 \tag{4.8}$$

Eq.(4.8) implies

$$B^* e^{-A^*s} C\phi = 0, \quad \text{for all} \quad s \in (-\infty, +\infty) \tag{4.9}$$

Succesively differentiating Eq.(4.9) (n-1)-times and setting s=0 yield

$$\left.\begin{array}{c} B^* C\phi = 0 \\ (AB)^* C\phi = 0 \\ \vdots \\ (A^{n-1}B)^* C\phi = 0 \end{array}\right\} \tag{4.10}$$

From Eq.(4.10), we conclude that $C\phi=0$, whence $\phi=0$, contradicting $C\phi=\phi\neq 0$, by hypothesis. This contradiction establishes the desired result $b_j^* e^{-A^*s} C\phi = 0$, for all $t\in[t_0,t_f]$ except on the set of measure-zero.

5. Conclusion.

Wehave investigated the necessary and sufficient conditons for the Bang-Bang optimal controls. As conclusion, it can generally be suggested that singularity for these problems essentially arises from the lack of the uniqueness of solutions. Hence in order to meet such singular cases, it is advisable to set the second criterion functional and make the solution unique.To take an example from the final value problem, suppose that $b_j^* e^{-A^*s}\phi = 0$, $j=1,\cdots,r_1$ $(r_1\leq r)$, where ϕ is a support hyperplane. In this case, admissible solutions $u_j(t)$ $(j=1,\cdots,r_1)$ which satisfy

$$\int_{t_0}^{t_f} \{\Phi(t_f,s)B\,u(s)\}_j\, ds = (x^d - \Phi(t_f,t_0)x_0)_j - \alpha_0 \|\phi\|^{q-1} |\phi_j|^{q-1} \operatorname{sign}[\phi_j], \ (j=1,\cdots,r_1)$$

turn out to be optimal, as will be seen from Eq.(3.6) in §3. Hence the second criterion functional

$$I_2 = \int_{t_0}^{t_1} \sum_{j=1}^{r_1} |u_j(t)| dt$$

is, for example, suggested.

Finally, we note that our results contain those obtained by Lasalle [1] and Athans and Falb [2] as special cases and that the extension of the results to the time varying systems is not difficult.

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