

超幾何分布の分布関数の
Wise の近似式の延長

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まえがき

超幾何分布の分布関数を求めたいとき、標本の大きさが大きくなると、その計算はかきりわけらわしくなるので、その近似式をつくっておきたい。

筆者^{(1), (2)} (1953) はロットの大きさを N としたときの N^{-1} の級数展開によって、検取検査の設計のため⁽³⁾ 計算方法を求めたが、
M. E. Wise^{(2), (3)} (1954) は $M = N - (n-1)/2$ ^(標本の大きさ n とし) として M^{-2} の ~~級数~~ 級数展開によって近似式を求めようと試みた。Wise は M の項までを示しているが、 M の項を用いるとさらによくなるであろうと思つて、求めた。

Wise は巧み⁽⁴⁾ 変数変換後の境界積分によって求めている。筆者は $\ln(N!/(N-n)!)$ の M^{-1} の 級数漸近 を求め、これをつかって計算を進

めら。

1. $\ln(N! / (N-n)!) の漸近展開$

$M = N - (n-1)/2$ とし, ガンマ関数の対数の Stirling の級数展開を用いて, 一般的形式の漸近展開が得られた。 B_{2t-1} は Bernoulli 数である。

$$\begin{aligned} \ln(N! / (N-n)!) &= \ln \Gamma(N+1) - \ln \Gamma(N+1-n) \\ &= \ln \Gamma(M + (n+1)/2) - \ln \Gamma(M - (n-1)/2) \\ &= (M + n/2) \ln(M + (n+1)/2) - (M - n/2) \ln(M - (n-1)/2) \\ &\quad - (M + (n+1)/2) + (M - (n-1)/2) \\ &\quad + \sum_{t=1}^{\infty} (-1)^{t-1} \frac{B_{2t-1}}{2t(2t-1)} \left[\left(M + \frac{n+1}{2}\right)^{-2t+1} - \left(M - \frac{n-1}{2}\right)^{-2t+1} \right] \\ &= n \ln M \end{aligned}$$

$$- \sum_{t=1}^{\infty} \frac{1}{(2M)^{2t}} \left[n^{2t+1} \frac{1}{2t(2t+1)} - \sum_{s=1}^t n^{2t+1-2s} \binom{2t-1}{2s-2} \left\{ \frac{1}{2s} + \sum_{r=1}^s (-1)^r \frac{B_{2r-1}}{2r(2r-1)} 2^{2r} \binom{2s-1}{2r-2} \right\} \right]$$

$$- \sum_{t=1}^{\infty} \frac{1}{(2M)^{2t+1}} \sum_{s=1}^t n^{2t+1-2s} \binom{2t}{2s-1} \left\{ \frac{1}{2s+1} + \sum_{r=1}^s (-1)^r \frac{B_{2r-1}}{2r(2r-1)} 2^{2r} \binom{2s-1}{2r-2} \right\}$$

... (1.1)

$$- \text{方 } y \operatorname{cosech} y = [2y / (e^{2y} - 1)] \cdot e^y = y + z$$

$$1 + \sum_{s=1}^{\infty} (-1)^s B_{2s-1} \frac{y^{2s} 2(2^{2s-1} - 1)}{(2s)!}$$

$$= \left\{ 1 - \frac{2y}{2} + \sum_{t=1}^{\infty} (-1)^{t-1} \frac{B_{2t-1}}{2} \frac{(2y)^{2t}}{(2t)!} \right\} \cdot \left\{ 1 + \sum_{t=1}^{\infty} \frac{y^t}{t!} \right\}$$

$$= 1 + \frac{y}{1!} (1-1) + \sum_{s=2}^{\infty} \frac{y^s}{s!} \left[1 - \binom{s}{1} + \sum_{t=1}^{\lfloor s/2 \rfloor} (-1)^{t-1} B_{2t-1} 2^{2t} \binom{s}{2t} \right]$$

であるから,

$$\frac{y^{2s+1}}{(2s+1)!} \text{ の係数} = 0 = 1 - \binom{2s+1}{1} + \sum_{t=1}^s (-1)^{t-1} B_{2t-1} 2^{2t} \binom{2s+1}{2t}$$

$$\therefore \frac{1}{2s+1} + \sum_{t=1}^s (-1)^t \frac{B_{2t-1}}{2t(2t-1)} 2^{2t} \binom{2s-1}{2t-2} = 0$$

$$\frac{y^{2s}}{(2s)!} \text{ の係数} = (-1)^s B_{2s-1} 2(2^{2s-1} - 1)$$

$$= 1 - \binom{2s}{1} + \sum_{t=1}^s (-1)^{t-1} B_{2t-1} 2^{2t} \binom{2s}{2t}$$

$$= -2s(2s-1) \left\{ \frac{1}{2s} + \sum_{t=1}^s (-1)^t \frac{B_{2t-1}}{2t(2t-1)} 2^{2t} \binom{2s-2}{2t-2} \right\}$$

$$\therefore - \left\{ \frac{1}{2s} + \sum_{t=1}^s (-1)^t \frac{B_{2t-1}}{2t(2t-1)} 2^{2t} \binom{2s-2}{2t-2} \right\}$$

$$= (-1)^s \frac{B_{2s-1}}{2s(2s-1)} 2(2^{2s-1} - 1)$$

∴ 左辺は (1.1) に τ を

$$\ln(N! / (N-n)!) = n \ln M$$

$$= \sum_{\tau=1}^{\infty} \frac{1}{(2M)^{2\tau}} \left\{ \frac{M^{2\tau+1}}{2\tau(2\tau+1)} + \sum_{s=1}^{\tau} M^{2\tau+1-2s} \binom{2\tau-1}{2s-2} (-1)^s \frac{B_{2s-1}}{2s(2s-1)} 2(2^{2s-1} - 1) \right\}$$

... (1.2)

∴ 左によつて $(2M)^{-2\tau}$ ののぞむ項まで書き下せる。 τ の小さい方から書けば、次の通り。

$$\tau=1: n(n^2-1)/(2 \cdot 3) = (n^{(3)} + 3n^{(2)})/6$$

$$\begin{aligned} \tau=2: (3n^5 - 10n^3 + 7n)/(2^2 \cdot 3 \cdot 5) \\ = (3n^{(5)} + 30n^{(4)} + 65n^{(3)} + 15n^{(2)})/60 \\ = n(n^2-1)(3n^2-7)/60 \end{aligned}$$

$$\begin{aligned} \tau=3: (3n^7 - 21n^5 + 49n^3 - 31n)/(2 \cdot 3^2 \cdot 7) \\ = (3n^{(7)} + 63n^{(6)} + 399n^{(5)} + 840n^{(4)} + 427n^{(3)} + 21n^{(2)})/126 \\ = n(n^2-1)(3n^4 - 18n^2 + 31)/126 \end{aligned}$$

$$\begin{aligned} \tau=4: (5n^9 - 60n^7 + 294n^5 - 620n^3 + 381n)/(2^3 \cdot 3^2 \cdot 5) \\ = (5n^{(9)} + 180n^{(8)} + 2250n^{(7)} + 11970n^{(6)} + 26649n^{(5)} \\ + 20790n^{(4)} + 3795n^{(3)} + 45n^{(2)})/360 \\ = n(n^2-1)(5n^6 - 55n^4 + 239n^2 - 381)/360 \end{aligned}$$

$$\begin{aligned} \tau=5: (3n^{11} - 55n^9 + 462n^7 - 2046n^5 + 4191n^3 - 2555n)/(2 \cdot 3 \cdot 5 \cdot 11) \\ = (3n^{(11)} + 165n^{(10)} + 3410n^{(9)} + 33660n^{(8)} + 167013n^{(7)} + 402633n^{(6)} \\ + 420519n^{(5)} + 151140n^{(4)} + 11231n^{(3)} + 33n^{(2)})/330 \\ = n(n^2-1)(3n^8 - 52n^6 + 410n^4 - 1636n^2 + 2555)/330 \end{aligned}$$

こゝで $n^{(m)} = n(n-1)(n-2)\cdots(n-m+1)$ である。

2. 計算の準備

ロットの大きさを N , 標本の大きさを n , 母集団不良率 p_0 , 標本のなかの不良品箇数が c と超えない確率 P はよく知られた通り

$$P = \sum_{d=0}^c \frac{n! (N-n)! (Np_0)! (N-Np_0)!}{d! (n-d)! N! (Np_0-d)! (N-Np_0-n+d)!} \dots (2.1)$$

$\ln(N!/(N-n)!)$ は $M = N - (n-1)/2$ を用いて, (1.2) によつて $(2M)^{-2\tau}$ の漸近級数に展開できることとを示しておいたが, (2.1) は $(Np_0)!/(Np_0-d)!$, $(N-Np_0)!/(N-Np_0-n+d)!$ があるから, これも同じように (1.2) によつて漸近級数に展開できるであらう. その上で P の初項が標本の大きさ n , 母集団不良率 p で, 標本のなかの不良品数が c を超えない二項分布の確率であらわされる, $(2M)^{-1}$ の項が 0 となるように p を決定する.

$$M_p = Np_0 - \frac{d-1}{2}, \quad M_q = Nq_0 - \frac{n-d-1}{2} \dots (2.2)$$

$$p_0 = p(1 + \alpha/(2M)), \quad 1-p_0=q_0 = q - p\alpha/(2M) \dots (2.3)$$

とおくと

$$M_p = M \left(1 + \frac{n-1}{2M}\right) p \left(1 + \frac{\alpha}{2M}\right) - \frac{d-1}{2}$$

$$= Mp \left\{ 1 + \frac{n-1 - (d-1)p^{-1}}{2M} + \frac{\alpha}{2M} \left(1 + \frac{n-1}{2M}\right) \right\} \dots (2.4a)$$

$$M_q = Mq \left\{ 1 + \frac{n-1 - (n-d-1)q^{-1}}{2M} - \frac{p}{q} \frac{\alpha}{2M} \left(1 + \frac{n-1}{2M}\right) \right\} \dots (2.4b)$$

こゝに

$$x_1 = x(1 + (n-1)/(2M)) \dots (2.5)$$

とおくと,

$$M_p = Mp \left\{ 1 + [n-1 - (d-1)p^{-1} + x_1] / (2M) \right\} \dots (2.6p)$$

$$M_q = Mq \left\{ 1 + [n-1 - (n-d-1)q^{-1} - x_1 p q^{-1}] / (2M) \right\} \dots (2.6q)$$

(1.2) 式の右辺第 1 項に よつて

$$-n \ln M + d \ln M_p + (n-d) \ln M_q$$

$$= d \ln p + (n-d) \ln q + d \ln \left\{ 1 + [n-1 - (d-1)p^{-1} + x_1] / (2M) \right\}$$

$$+ (n-d) \ln \left\{ 1 + [n-1 - (n-d-1)q^{-1} - x_1 p q^{-1}] / (2M) \right\}$$

--- (2.7)

(1.2) 式の右辺第 2 項は $(2M)^{-2} z^n$ はじまる

から, (2.1) $z^n (2M)^{-1}$ の係数は $\frac{1}{2M}$ に対しては (2.7) の

の z^n である. (2.7) によつて

$$\frac{1}{2M} \text{の係数} = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ d^{(1)} [n-1 - (d-1)p^{-1} + x_1] \right. \\ \left. + (n-d)^{(1)} [n-1 - (n-d-1)q^{-1} - x_1 p q^{-1}] \right\}$$

$$= n^{(1)} \left\{ (n-1 + x_1) p B(c-1, n-1, p) + (n-1 - \frac{x_1 p}{q}) q B(c, n-1, p) \right\}$$

$$- n^{(2)} \left\{ p B(c-2, n-2, p) + q B(c, n-2, p) \right\} \dots (2.8)$$

こゝに $B(c, n, p) = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} z^n$ である. なお

$$b(d, n, p) = \binom{n}{d} p^d q^{n-d}$$

とし、特に α を $\alpha < n$ とし $B(c, n), b(d, n)$ のように記す。

(2.8) を整理すると

$$\begin{aligned} \frac{1}{2M} \text{の係数} &= n^{(1)} \{ [(n-1)q - \alpha, p] b(c, n-1, p) \\ &\quad + ((n-1+\alpha)p + [(n-1)q - \alpha, p]) B(c-1, n-1, p) \} \\ &\quad - n^{(2)} \{ p B(c-2, n-2, p) + q B(c, n-2, p) \} \\ &= n b(c, n-1, p) [(n-1)q - \alpha, p] \\ &\quad + n^{(2)} \{ B(c-1, n-1, p) - p B(c-2, n-2, p) - q B(c, n-2, p) \} \\ &= n b(c, n-1, p) [(n-1)q - \alpha, p] - n^{(2)} q b(c, n-2, p) \\ &= n b(c, n-1, p) [(n-1)q - \alpha, p - (n-1-c)] = 0 \end{aligned}$$

$$\therefore \alpha, p = (n-1)q - (n-1-c) = cq - (n-1-c)p \quad \dots (2.9)$$

$$\left. \begin{aligned} n-1 + \alpha_1 &= n-1 + \frac{cq - (n-1-c)p}{p} = \frac{c}{p} \\ n-1 - \frac{\alpha_1 p}{q} &= n-1 - \frac{cq - (n-1-c)p}{q} = \frac{n-1-c}{q} \end{aligned} \right\} \dots (2.10)$$

を次の関係をつかいてみる。

$$\begin{aligned} B(c, n, p) &= \sum_{d=0}^c \binom{n-1}{d} p^d q^{n-d} + \sum_{d=0}^{c-1} \binom{n-1}{d} p^{d+1} q^{n-1-d} \\ &= B(c, n-1, p) \cdot q + B(c-1, n-1, p) \cdot p \\ &= \sum_{i=0}^{\wedge} B(c-i, n-1, p) \binom{\wedge}{i} p^i q^{\wedge-i} \quad \dots (2.11) \end{aligned}$$

(2.10) に よつて, (2.6) は

$$\left. \begin{aligned} M_p &= Mp \left(1 - \frac{d-1-c}{2Mp}\right) \\ M_q &= Mq \left(1 - \frac{n-d-1-(n-1-c)}{2Mq}\right) \end{aligned} \right\} \dots (2.12)$$

(2.3), (2.5) に よつて

$$\begin{aligned} p_0 &= p \left\{ 1 + \frac{x_1}{2M} \left(1 + \frac{n-1}{2M}\right)^{-1} \right\} \\ &= p + \frac{c q - (n-1-c)p}{2N} = \left(pM + \frac{c}{2}\right) N^{-1} \dots (2.13) \end{aligned}$$

$$Mp = Np_0 - \frac{c}{2}, \quad Mq = Np_0 - \frac{n-1-c}{2} \dots (2.14)$$

Wise o' $h, 1-x, p$ として $h, 1-x, p$ の $\overbrace{h, 1-x, p}^{\text{よれよれ}}$ の $\overbrace{h, 1-x, p}^{\text{よれよれ}}$

z の $q, q_0, c+1$ あり.

(2.7) に よつて

$$\begin{aligned} &\left(1 - \frac{d-1-c}{2Mp}\right)^d \left(1 - \frac{n-d-1-(n-1-c)}{2Mq}\right)^{n-d} \\ &= \left\{ 1 + \sum_{\lambda=1}^d (-1)^\lambda \frac{d^{(\lambda)}}{\lambda!} \left(\frac{d-1-c}{2Mp}\right)^\lambda \right\} \cdot \left\{ 1 + \sum_{\lambda=1}^{n-d} (-1)^\lambda \frac{(n-d)^{(\lambda)}}{\lambda!} \left(\frac{n-d-1-(n-1-c)}{2Mq}\right)^\lambda \right\} \\ &= 1 + \sum_{\lambda=1}^n (-1)^\lambda \frac{1}{\lambda! (2M)^\lambda} G_\lambda \dots (2.15.1) \end{aligned}$$

$$G_\lambda = \sum_{i=0}^{\lambda} \binom{\lambda}{i} d^{(\lambda-i)} (n-d)^{(i)} \left(\frac{d-1-c}{p}\right)^{\lambda-i} \left(\frac{n-d-1-(n-1-c)}{q}\right)^i$$

$$(G_0 = 1 \text{ と } \delta <) \dots (2.15.2)$$

(1.2) の右辺中各項 ε 書き直し

δ

$$- \sum_{\tau=1}^{\infty} \frac{1}{(2\tau)!(2M)^{2\tau}} f_{2\tau}(n)$$

と書くと, (2.1) は次のようになる. (Tは, $f_{2\tau}(n)$ は附録1のF)にTはる)

$$P = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{A=1}^m (-1)^A \frac{1}{A!(2M)^A} G_A \right\} \cdot$$

$$\cdot \exp \left[\sum_{\tau=1}^{\infty} \frac{1}{(2\tau)!(2M)^{2\tau}} \left\{ f_{2\tau}(n) - \frac{1}{p^{2\tau}} f_{2\tau}(d) \left(1 - \frac{d-1-c}{2Mp}\right)^{-2\tau} - \frac{1}{q^{2\tau}} f_{2\tau}(n-d) \left(1 - \frac{n-d-1-(n-1-c)}{2Mq}\right)^{-2\tau} \right\} \right]$$

..... (2.16)

(2.16) を整理する為

$$\phi_m = \sum_{\tau=1}^{\lfloor \frac{m}{2} \rfloor} m^{(m-2\tau)} \binom{m-1}{m-2\tau} \left\{ \frac{1}{p^{2\tau}} f_{2\tau}(d) \cdot (d-1-c)^{m-2\tau} + \frac{1}{q^{2\tau}} f_{2\tau}(n-d) \cdot (n-d-1-(n-1-c))^{m-2\tau} \right\}$$

..... (2.17)

$$F_{2m} = \sum_{\alpha_2=1}^m \sum_{2m}^{\alpha} \frac{(2m)!}{\alpha_2! (2!)^{\alpha_2} \alpha_4! (4!)^{\alpha_4} \dots} (f_2(n))^{\alpha_2} (f_4(n))^{\alpha_4} \dots$$

..... (2.18)

$$\sum_{2m}^{\alpha} \text{は} \left\{ \begin{array}{l} \alpha = \alpha_2 + \alpha_4 + \dots \quad ; \alpha_2, \alpha_4, \dots \geq 0 \\ 2m = 2\alpha_2 + 4\alpha_4 + \dots \end{array} \right\}$$

これがある $\alpha_2, \alpha_4, \dots$ のある組み合わせ

の総和の意味

$$\Phi_m = \sum_{\alpha=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^\alpha \sum_m^{\alpha} \frac{m!}{\alpha_2! (2!)^{\alpha_2} \alpha_3! (3!)^{\alpha_3} \alpha_4! (4!)^{\alpha_4} \dots}$$

$$\cdot \phi_2^{\alpha_2} \phi_3^{\alpha_3} \phi_4^{\alpha_4} \dots \dots (2.19)$$

$$\sum_{m \geq 0} \alpha \quad \left\{ \begin{array}{l} \alpha = \alpha_2 + \alpha_3 + \alpha_4 + \dots \quad ; \quad \alpha_2, \alpha_3, \alpha_4, \dots \geq 0 \\ m = 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \dots \end{array} \right.$$

ここで $\alpha_2, \alpha_3, \alpha_4, \dots$ のあらゆる組合

せの総和

F_{2m} の値は付録 2 の (F) に依る.

(2.17), (2.18), (2.19) を用いると (2.16) は

$$\begin{aligned} P &= \sum_{d=0}^{\infty} \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{\lambda=1}^n (-1)^\lambda \frac{1}{\lambda! (2M)^\lambda} G_\lambda \right\} \\ &\cdot \left\{ 1 + \sum_{m=1}^{\infty} \left[\frac{1}{(2m)! (2M)^{2m}} \sum_{s=0}^m \frac{(2m)!}{(2s)! (2m-2s)!} F_{2m-2s} \Phi_{2s} \right. \right. \\ &\quad \left. \left. + \frac{1}{(2m+1)! (2M)^{2m+1}} \sum_{s=1}^m \frac{(2m+1)!}{(2s+1)! (2m-2s)!} F_{2m-2s} \Phi_{2s+1} \right] \right\} \\ &= \sum_{d=0}^{\infty} \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{(2m)! (2M)^{2m}} \left[F_{2m} \right. \right. \\ &\quad \left. \left. + \sum_{t=1}^m F_{2m-2t} \binom{2m}{2t} \left(\sum_{s=0}^t \binom{2t}{2s} \Phi_{2s} G_{2t-2s} \right. \right. \right. \\ &\quad \left. \left. \left. - \sum_{s=1}^{t-1} \binom{2t}{2s+1} \Phi_{2s+1} G_{2t-1-2s} \right) \right] \right. \\ &\quad \left. + \sum_{m=1}^{\infty} \frac{1}{(2m+1)! (2M)^{2m+1}} \left[-\binom{2m+1}{1} F_{2m} G_1 \right. \right. \\ &\quad \left. \left. + \sum_{t=1}^m F_{2m-2t} \binom{2m+1}{2t+1} \left(\sum_{s=1}^t \binom{2t+1}{2s+1} \Phi_{2s+1} G_{2t-2s} \right. \right. \right. \\ &\quad \left. \left. \left. - \sum_{s=0}^t \binom{2t+1}{2s} \Phi_{2s} G_{2t+1-2s} \right) \right] \right\} \end{aligned}$$

$\left\{ \right\}$ の \dots (2.10)

と示す. (2.10) の示す \dots とは $(p, q), (d, n-d), (c, \bar{c})$ に
ついて \dots 構造を \dots としてある.

3. $(2M)^{-(2m+1)}$ の項について

(2.10) の $(2M)^{-(2m+1)}$ の項について, \dots の

と示す \dots とは, m の小さいとき \dots に計算した.

系係数の第1項は

$$\begin{aligned}
 & \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} G_1 \\
 &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left[\frac{d(d-1-c)}{p} + \frac{(n-d)(n-d-1-(n-1-c))}{q} \right] \\
 &= m^{(2)} \sum_{d=0}^{c-2} \binom{n-2}{d} p^{d+1} q^{n-2-d} - m^{(1)} \sum_{d=0}^{c-1} \binom{n-1}{d} p^d q^{n-1-d-c} \\
 & \quad + m^{(2)} \sum_{d=0}^c \binom{n-2}{d} p^d q^{n-1-d} - m^{(1)} \sum_{d=0}^c \binom{n-1}{d} p^d q^{n-1-d} \cdot \frac{1}{(n-1-c)} \\
 &= m^{(2)} B(c-2, n-2) p + m^{(2)} B(c, n-2) q \\
 & \quad - m B(c-1, n-1) c - m B(c, n-1) (n-1-c) \\
 &= -m b(c, n-1) \cdot (n-1-c) + m^{(2)} [B(c-2, n-2) p + B(c, n-2) q \\
 & \quad - B(c-1, n-1)] \\
 &= -m b(c, n-1) \cdot (n-1-c) + m^{(2)} b(c, n-2) q \\
 &= m b(c, n-1) \{ -(n-1-c) + (n-1-c) \} = 0 \quad \dots (3.1)
 \end{aligned}$$

次に $F_{2m-2c} \binom{2m+1}{2c+1}$ の係数 ④ に ついては

$$\begin{aligned}
 A_{2c+1} &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left[\sum_{s=1}^c \binom{2c+1}{2s+1} \Phi_{2s+1} G_{2c-2s} \right. \\
 & \quad \left. - \sum_{s=0}^c \binom{2c+1}{2s} \Phi_{2s} G_{2c+1-2s} \right] \\
 & \quad \dots (3.2)
 \end{aligned}$$

と おく と

$$A_3 = \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} [\Phi_3 - \binom{3}{2} \Phi_2 G_1 - G_3] \quad \dots (3.3)$$

今便宜上 $\bar{c} = (n-1-c)$ とおき, $(p, q), (d, n-d),$

(c, c̄) を同時に取リ換えたもの E 加え 3 意味とあらわして + ~ と記しておく

$$\begin{aligned} & \Phi_3 - 3\Phi_2 \Gamma_1 - \Gamma_3 \\ &= -6 \left\{ \frac{1}{p^3} f_2(d) \cdot (d-1-c) + \sim \right\} + 3 \left\{ \frac{1}{p^2} f_2(d) + \sim \right\} \left\{ \frac{d(d-1-c)}{p} + \sim \right\} \\ & \quad - \left\{ \frac{d^{(3)}(d-1-c)^3}{p^3} + 3 \frac{d^{(2)}(d-1-c)^2(n-d)(n-d-1-\bar{c})}{p^2 q} + \sim \right\} \\ &= \frac{1}{p^3} [-d^{(6)} + d^{(5)}(c_3-8) - d^{(4)}(c^{(2)}_3 - c_{11}+12) + d^{(3)}(c^{(3)}_3 - c^{(2)}_3 + c_3)] \\ & \quad + \frac{1}{p^2 q} [-d^{(4)}_3 + d^{(3)}(c_6-8) - d^{(2)}(c^{(2)}_3 - c_3)] \cdot [(n-d)^{(2)} - (n-d)\bar{c}] \\ & \quad + \sim \end{aligned}$$

と 3 から, (3.3) = λ 4 2

$$\begin{aligned} A_3 &= -n^{(6)} B(c, n-6) q^3 + n^{(5)} B(c, n-5) q^2 [\bar{c}_3 - 8] \\ & \quad - n^{(4)} B(c, n-4) q [\bar{c}^{(2)}_3 - \bar{c}_{11} + 12] + n^{(3)} B(c, n-3) [\bar{c}^{(3)}_3 - \bar{c}^{(2)}_3 + \bar{c}_3] \\ & \quad - n^{(4)} B(c-2, n-6) p q^2_3 + n^{(5)} B(c-2, n-5) p q [\bar{c}_6 - 8] - n^{(4)} B(c-2, n-4) p \cdot \\ & \quad \quad \quad \cdot [\bar{c}^{(2)}_3 - \bar{c}_3] \\ & \quad + c \left\{ n^{(5)} B(c-1, n-5) q^2_3 - n^{(4)} B(c-1, n-4) q [\bar{c}_6 - 8] + n^{(3)} B(c-1, n-3) [\bar{c}^{(2)}_3 - \bar{c}_3] \right\} \\ & \quad - n^{(6)} B(c-4, n-6) p^2 q_3 + n^{(5)} B(c-3, n-5) p q [\bar{c}_6 - 8] - n^{(4)} B(c-2, n-4) q [c^{(2)}_3 - c_3] \\ & \quad + \bar{c} \left\{ n^{(5)} B(c-4, n-5) p^2_3 - n^{(4)} B(c-3, n-4) p [\bar{c}_6 - 8] + n^{(3)} B(c-2, n-3) [c^{(2)}_3 - c_3] \right\} \\ & \quad - n^{(6)} B(c-6, n-6) p^3 + n^{(5)} B(c-5, n-5) p^2 [c_3 - 8] \\ & \quad - n^{(4)} B(c-4, n-4) p [c^{(2)}_3 - c_{11} + 12] + n^{(3)} B(c-3, n-3) [c^{(3)}_3 - c^{(2)}_3 + c_3] \\ & \quad \quad \quad \dots (3.4) \end{aligned}$$

∴ c B(c, n-6) q と の I Q = 2 1 2

$$- n^{(6)} b(c-1, n-6) q^3 + n^{(5)} b(c-1, n-5) q^2 [\bar{c}_3 - 8]$$

$$\begin{aligned}
 & -n^{(4)}b(c-1, n-4)q [\bar{c}^{(2)}_3 - \bar{c}_{11+12}] + n^{(3)}b(c-1, n-3) [\bar{c}^{(3)}_3 - \bar{c}^{(2)}_3 + \bar{c}_3] \\
 & = nb(c, n-1) \frac{c^{(\lambda)}}{p^\lambda q^{2-\lambda}} \left\{ -\bar{c}^{(5-\lambda)} + \bar{c}^{(4-\lambda)} [\bar{c}_3 - 8] - \bar{c}^{(3-\lambda)} [\bar{c}^{(2)}_3 - \bar{c}_{11+12}] \right. \\
 & \quad \left. + \bar{c}^{(2-\lambda)} [\bar{c}^{(2)}_3 - \bar{c}^{(2)}_3 + \bar{c}_3] \right\} \\
 & = nb(c, n-1) \frac{c^{(\lambda)}}{p^\lambda q^{2-\lambda}} \left\{ \bar{c}^{(3-\lambda)} \binom{\lambda}{1} - \bar{c}^{(2-\lambda)} \left[\binom{\lambda}{3} 6 + \binom{\lambda}{2} 6 - \binom{\lambda}{1} 3 \right] \right\} \\
 & \quad (0 \leq \lambda \leq 2) \qquad \dots (3.5)
 \end{aligned}$$

$\lambda=0$ のときは $0 < \tau_3$ 。

次に

$$\begin{aligned}
 & -n^{(6)}b(c-1-\lambda, n-6) p q^2 \bar{c}_3 + n^{(5)}b(c-1-\lambda, n-5) p q [\bar{c}_6 - 8] \\
 & \quad - n^{(4)}b(c-1-\lambda, n-4) p [\bar{c}^{(2)}_3 - \bar{c}_3] \\
 & + c \left\{ n^{(5)}b(c-1, n-5) q^2 \bar{c}_3 - n^{(4)}b(c-1, n-4) q [\bar{c}_6 - 8] + n^{(3)}b(c-1, n-3) [\bar{c}^{(2)}_3 - \bar{c}_3] \right\} \\
 & = nb(c, n-1) \frac{-c^{(1+\lambda)} + c \cdot c^{(\lambda)}}{p^\lambda q^{2-\lambda}} \left\{ \bar{c}^{(4-\lambda)}_3 - \bar{c}^{(3-\lambda)} [\bar{c}_6 - 8] + \bar{c}^{(2-\lambda)} [\bar{c}^{(2)}_3 - \bar{c}_3] \right\} \\
 & = nb(c, n-1) \frac{c^{(\lambda)} \lambda}{p^\lambda q^{2-\lambda}} \left\{ -\bar{c}^{(3-\lambda)} + \bar{c}^{(2-\lambda)} \left[\binom{\lambda-1}{2} 6 + \binom{\lambda-1}{1} 3 - 3 \right] \right\} \\
 & \quad (1 \leq \lambda \leq 2) \\
 & = nb(c, n-1) \frac{c^{(\lambda)}}{p^\lambda q^{2-\lambda}} \left\{ -\bar{c}^{(3-\lambda)} \binom{\lambda}{1} + \bar{c}^{(2-\lambda)} \left[\binom{\lambda}{3} 18 + \binom{\lambda}{2} 6 - \binom{\lambda}{1} 3 \right] \right\} \\
 & \quad \dots (3.6)
 \end{aligned}$$

次の $\lambda=1$ と $\lambda=2$ のとき、 τ_3 に関する λ の式 (3.6) を用い

$$nb(c, n-1) \frac{\bar{c}^{(\lambda)}}{p^{2-\lambda} q^\lambda} \left\{ -c^{(3-\lambda)} \binom{\lambda}{1} + c^{(2-\lambda)} \left[\binom{\lambda}{3} 18 + \binom{\lambda}{2} 6 - \binom{\lambda}{1} 3 \right] \right\}$$

を $\lambda=0$ のとき 0 とする。

(3.5) と (3.6) とを $n \rightarrow 3$ と

$$nb(c, n-1) \frac{c^{(\lambda)}}{p^\lambda q^{2-\lambda}} \bar{c}^{(2-\lambda)} \binom{\lambda}{3} 12 = 0 \quad (1 \leq \lambda \leq 2)$$

と等しい、(3.4) は \bar{c} の \bar{c} に \bar{c} である。

$$\begin{aligned}
 A_3 &= -n^{(6)}B(c-3, n-6)q^3 + n^{(5)}B(c-3, n-5)q^2[\bar{c}3-8] - n^{(4)}B(c-3, n-4)q[\bar{c}^{(2)}_3 - \bar{c}^{(1)}_{11} + 12] \\
 &\quad + n^{(3)}B(c-3, n-3)[\bar{c}^{(3)}_3 - \bar{c}^{(2)}_3 + \bar{c}3] \\
 &\quad - n^{(6)}B(c-4, n-6)pq^3 + n^{(5)}B(c-4, n-5)pq[\bar{c}6-8] - n^{(4)}B(c-4, n-4)p[\bar{c}^{(2)}_3 - \bar{c}3] \\
 &\quad + c\{n^{(5)}B(c-3, n-5)q^23 - n^{(4)}B(c-3, n-4)q[\bar{c}6-8] + n^{(3)}B(c-3, n-3)[\bar{c}^{(2)}_3 - \bar{c}3]\} \\
 &\quad - n^{(6)}B(c-5, n-6)p^2q^3 + n^{(5)}B(c-4, n-5)pq[c6-8] - n^{(4)}B(c-3, n-4)q[c^{(2)}_3 - c3] \\
 &\quad + \bar{c}\{n^{(5)}B(c-5, n-5)p^23 - n^{(4)}B(c-4, n-4)p[c6-8] + n^{(3)}B(c-3, n-3)[c^{(2)}_3 - c3]\} \\
 &\quad - n^{(6)}B(c-6, n-6)p^3 + n^{(5)}B(c-5, n-5)p^2[c3-8] - n^{(4)}B(c-4, n-4)p[c^{(2)}_3 - c^{(1)}_{11} + 12] \\
 &\quad + n^{(3)}B(c-3, n-3)[c^{(3)}_3 - c^{(2)}_3 + c3] \\
 &= -n^{(6)}B(c-3, n-3) + n^{(5)}B(c-3, n-3)[(n-1)3-8] - n^{(4)}B(c-3, n-3)[(n-1)^{(2)}_3 - (n-1)_{11} + 12] \\
 &\quad + n^{(3)}B(c-3, n-3)[(n-1)^{(3)}_3 - (n-1)^{(2)}_3 + (n-1)3] \\
 &= B(c-3, n-3)\{-n^{(6)} + n^{(5)}[(n-5)3+4] - n^{(4)}[(n-4)^{(2)}_3 + (n-4)_{11} - 3] \\
 &\quad + n^{(3)}[(n-3)^{(3)}_3 + (n-3)^{(2)}_3 - (n-3)3]\} \\
 &= 0
 \end{aligned}$$

このようにして $A_5 = 0, A_7 = 0$ の計算も同様である。

4. $(2M)^{-2m}$ の $I\mathbb{Q}$ に ついて

(2.10) の $1/((2m)!(2M)^{2m})$ の $I\mathbb{Q}$ は K_{2m} とおくと

$$\begin{aligned}
 K_{2m} &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} [F_{2m} + \sum_{t=1}^m F_{2m-2t} \binom{2m}{2t} A_{2t}] \quad \dots (4.1) \\
 A_{2t} &= \left\{ \sum_{s=0}^t \binom{2t}{2s} \Phi_{2s} G_{2t-2s} - \sum_{s=1}^{t-1} \binom{2t}{2s+1} \Phi_{2s+1} G_{2t-1-2s} \right\} \quad \dots (4.2) \\
 &\quad \left\{ \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \right\}
 \end{aligned}$$

(4.1) の F_{2m} の係数は (2.11) によって

$$\sum_{d=0}^c \binom{n}{d} p^d q^{n-d} = B(c, n)$$

$$= B(c, n-2) q^2 + B(c-1, n-2) p q^2 \\ + B(c-2, n-2) p^2$$

$$= b(c, n-2) q^2 - b(c-1, n-2) p^2 + B(c-1, n-2)$$

$$= \frac{nb(c, n-1)}{n^{(2)}} A_{0,0} + B(c-1, n-2) \dots (4.3)$$

$$A_{0,0} = \bar{c} q - c p \dots (4.4)$$

同様に $\lambda \geq 1$ として

$$B(c-\lambda, n-2\lambda) = B(c-\lambda, n-2\lambda-2) q^2 \\ + B(c-\lambda-1, n-2\lambda-2) p q^2 + B(c-\lambda-2, n-2\lambda-2) p^2$$

$$= b(c-\lambda, n-2\lambda-2) q^2 - b(c-\lambda-1, n-2\lambda-2) p^2 \\ + B(c-\lambda-1, n-2\lambda-2)$$

$$= \frac{nb(c, n-1)}{n^{(2\lambda+2)}} A_{0,\lambda} + B(c-\lambda-1, n-2\lambda-2) \dots (4.5)$$

$$A_{0,\lambda} = \frac{c^{(\lambda)} \bar{c}^{(\lambda+1)}}{p^\lambda q^{\lambda-1}} - \frac{c^{(\lambda+1)} \bar{c}^{(\lambda)}}{p^{\lambda-1} q^\lambda} \dots (4.6)$$

次に A_2 を計算する

$$A_2 = \cancel{\Phi_2 + G_2} \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \{ \Phi_2 + G_2 \}$$

$$\begin{aligned}
&= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ - \frac{d^{(3)} + d^{(2)} 3}{3p^2} - \sim \right. \\
&\quad + \frac{d^{(4)} - d^{(3)}(c-3) + d^{(2)}(c^{(2)} - c + 1)}{p^2} + \sim \\
&\quad \left. + \frac{2}{pq} (d^{(2)} - dc) ((n-d)^{(2)} - (n-d)\bar{c}) \right\} \\
&= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ \frac{1}{p^2} [d^{(4)} - d^{(3)}(c-3) + d^{(2)}(c^{(2)} - c)] + \sim \right. \\
&\quad \left. + \frac{2}{pq} (d^{(2)} - dc) ((n-d)^{(2)} - (n-d)\bar{c}) \right\} \\
&= n^{(4)} B(c, n-4) q^2 - n^{(3)} B(c, n-3) q (\bar{c} - \frac{8}{3}) + n^{(2)} B(c, n-2) (\bar{c}^{(2)} - \bar{c}) \\
&\quad + n^{(4)} B(c-4, n-4) p^2 - n^{(3)} B(c-3, n-3) p (c - \frac{8}{3}) + n^{(2)} B(c-2, n-2) (c^{(2)} - c) \\
&\quad + n^{(4)} B(c-2, n-4) pq^2 - n^{(3)} B(c-1, n-3) q c^2 \\
&\quad - n^{(3)} B(c-2, n-3) p \bar{c}^2 + n^{(2)} B(c-1, n-2) c \bar{c}^2 \\
&= n^{(4)} [b(c, n-4) q^2 - b(c-3, n-4) p^2] + n^{(4)} [B(c-1, n-4) q^2 \\
&\quad + B(c-2, n-4) pq^2 + B(c-3, n-4) p^2] \\
&\quad - n^{(3)} [b(c, n-3) q (\bar{c} - \frac{8}{3}) - b(c-2, n-3) p (c - \frac{8}{3})] \\
&\quad - n^{(3)} [B(c-1, n-3) q ((n-1)^2 - \frac{8}{3}) + B(c-2, n-3) p ((n-1)^2 - \frac{8}{3})] \\
&\quad + n^{(2)} [b(c, n-2) (\bar{c}^{(2)} - \bar{c}) - b(c-1, n-2) (c^{(2)} - c)] \\
&\quad + n^{(2)} B(c-1, n-2) ((n-1)^{(2)} - (n-1)) \\
&= nb(c, n-1) A_{2,0} + B(c-1, n-2) \left\{ n^{(4)} - n^{(3)} ((n-3)^2 + \frac{4}{3}) \right. \\
&\quad \left. + n^{(2)} ((n-2)^{(2)} + (n-2) - 1) \right\} \\
&= nb(c, n-1) A_{2,0} - B(c-1, n-2) f_2(n) \quad \dots \quad (4.7)
\end{aligned}$$

$$\begin{aligned}
 A_{2,0} &= \frac{\bar{c}^{(3)}}{q} - \frac{c^{(3)}}{p} - \frac{\bar{c}^{(2)}}{q} \left(\bar{c} - \frac{2}{3} \right) + \frac{c^{(2)}}{p} \left(c - \frac{2}{3} \right) \\
 &\quad + \frac{\bar{c}}{q} (\bar{c}^{(2)} - \bar{c}) - \frac{c}{p} (c^{(2)} - c) \\
 &= -\frac{\bar{c}(\bar{c}+2)}{3q} + \frac{c(c+2)}{3p} \dots \dots (4.8)
 \end{aligned}$$

よって (4.1) に よって

$$\begin{aligned}
 K_{2m} &= mb(c, n-1) \left\{ \frac{F_{2m}}{n^{(2)}} A_{0,0} + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)}{n^{(4)}} A_{0,1} \right. \\
 &\quad \left. + \binom{2m}{2} F_{2m-2} A_{2,0} \right\}
 \end{aligned}$$

$$+ [F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)] B(c-2, n-4)$$

$$+ \sum_{t=2}^m F_{2m-2t} \binom{2m}{2t} A_{2t} \dots \dots (4.9)$$

よって

$$K_2 = mb(c, n-1) \left\{ \frac{n+1}{3} A_{0,0} + A_{2,0} \right\} \dots (4.10)$$

となる。

次に A_4 について計算する。

$$\begin{aligned}
 A_4 &= \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ \sum_{s=0}^2 \binom{4}{2s} \Phi_{2s} G_{4-2s} - \binom{4}{1} \Phi_3 G_1 \right\} \\
 &\dots (4.11)
 \end{aligned}$$

まず (4.11) の $\{ \}$ 内の d の x の項, $(n-d)$ の x の項をそれぞれ

$$\begin{aligned}
& \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ \frac{1}{p^d} \left[d^{(8)} - d^{(7)}(c4-16) + d^{(6)}(c^{(2)}6 - c38 + \frac{208}{3}) \right. \right. \\
& \quad - d^{(5)}(c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) \\
& \quad \left. \left. + d^{(4)}(c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \right] + n \right\} \\
& = n^{(8)} B(c-8, n-8) p^4 - n^{(7)} B(c-7, n-7) p^3 (c4-16) \\
& \quad + n^{(6)} B(c-6, n-6) p^2 (c^{(2)}6 - c38 + \frac{208}{3}) - n^{(5)} B(c-5, n-6) p (c4 - c^{(2)}28 + c76 - \frac{384}{5}) \\
& \quad + n^{(4)} B(c-4, n-4) (c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \\
& + n^{(8)} B(c, n-8) q^4 - n^{(7)} B(c, n-7) q^3 (\bar{c}4-16) \\
& \quad + n^{(6)} B(c, n-6) q^2 (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) - n^{(5)} B(c, n-6) q (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) \\
& \quad + n^{(4)} B(c, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)}6 + \bar{c}^{(2)}15 - \bar{c}15) \\
& = \sum_{\lambda=0}^1 \left\{ n^{(8)} b(c-\lambda, n-8) q^4 - n^{(7)} b(c-\lambda, n-7) q^3 (\bar{c}4-16) + n^{(6)} b(c-\lambda, n-6) q^2 (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) \right. \\
& \quad - n^{(5)} b(c-\lambda, n-5) q (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) + n^{(4)} b(c-\lambda, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)}6 + \bar{c}^{(2)}15 - \bar{c}15) \\
& \quad - n^{(8)} b(c-7+\lambda, n-8) p^4 + n^{(7)} b(c-6+\lambda, n-7) p^3 (c4-16) + n^{(6)} b(c-5+\lambda, n-6) p^2 (c^{(2)}6 - c38 + \frac{208}{3}) \\
& \quad \left. + n^{(5)} b(c-4+\lambda, n-5) p (c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) - n^{(4)} b(c-3+\lambda, n-4) (c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \right\} \\
& + n^{(8)} B(c-2, n-8) q^4 - n^{(7)} B(c-2, n-7) q^3 (\bar{c}4-16) + n^{(6)} B(c-2, n-6) q^2 (\bar{c}^{(2)}6 - \bar{c}38 + \frac{208}{3}) \\
& \quad - n^{(5)} B(c-2, n-5) q (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)}6 + \bar{c}^{(2)}15 - \bar{c}15) \\
& + n^{(8)} B(c-6, n-8) p^4 - n^{(7)} B(c-5, n-7) p^3 (c4-16) + n^{(6)} B(c-4, n-6) p^2 (c^{(2)}6 - c38 + \frac{208}{3}) \\
& \quad - n^{(5)} B(c-3, n-5) p (c^{(3)}4 - c^{(2)}28 + c76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (c^{(4)} - c^{(3)}6 + c^{(2)}15 - c15) \\
& = n b(c, n-1) \sum_{\lambda=0}^1 \left\{ \frac{c^{(\lambda)}}{p^\lambda q^{3-\lambda}} \left[\bar{c}^{(7-\lambda)} - \bar{c}^{(6-\lambda)} (\bar{c}4-16) + \bar{c}^{(5-\lambda)} (c^{(2)}6 - c38 + \frac{208}{3}) \right. \right. \\
& \quad - \bar{c}^{(4-\lambda)} (\bar{c}^{(3)}4 - \bar{c}^{(2)}28 + \bar{c}76 - \frac{384}{5}) \\
& \quad \left. \left. + \bar{c}^{(3-\lambda)} (\bar{c}^{(4)} - \bar{c}^{(3)}6 - \bar{c}^{(2)}15 + \bar{c}15) \right] \right. \\
& \quad \left. - \dots \right\} + n^{(8)} B(c-2, n-8) q^4 - \dots
\end{aligned}$$

$$\begin{aligned}
 &= n b(c, n-1) \sum_{\lambda=0}^1 \left\{ \frac{c^{(\lambda)}}{p^\lambda q^{3-\lambda}} \left[\bar{c}^{(5-\lambda)} \frac{1}{3} + \bar{c}^{(4-\lambda)} \left[-\binom{\lambda}{2} 4 - \binom{\lambda}{1} 2 + \frac{19}{5} \right] \right. \right. \\
 &\quad \left. \left. + \bar{c}^{(3-\lambda)} \left[\binom{\lambda}{4} 24 + \binom{\lambda}{3} 36 - \binom{\lambda}{2} 6 - \binom{\lambda}{1} 9 + 9 \right] \right] \right. \\
 &\quad \left. - \sim \right\} \\
 &+ n^{(8)} B(c-2, n-8) q^4 - n^{(9)} B(c-2, n-7) q^3 (\bar{c} 4 - 16) + n^{(6)} B(c-2, n-6) q^2 (\bar{c}^{(2)} 6 - \bar{c} 38 + \frac{208}{3}) \\
 &\quad - n^{(5)} B(c-2, n-5) q (\bar{c}^{(3)} 4 - \bar{c}^{(2)} 28 + \bar{c} 76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (\bar{c}^{(4)} - \bar{c}^{(3)} 6 + \bar{c}^{(2)} 15 - \bar{c} 15) \\
 &+ n^{(8)} B(c-6, n-8) p^4 - n^{(9)} B(c-5, n-7) p^3 (c 4 - 16) + n^{(6)} B(c-4, n-6) p^2 (c^{(2)} 6 - c 38 + \frac{208}{3}) \\
 &\quad - n^{(5)} B(c-3, n-5) p (c^{(3)} 4 - c^{(2)} 28 + c 76 - \frac{384}{5}) + n^{(4)} B(c-2, n-4) (c^{(4)} - c^{(3)} 6 + c^{(2)} 15 - c 15) \\
 &\quad \dots (4.12)
 \end{aligned}$$

次に (4.11) の $\{ \}$ 内の $[(n-d)^{(2)} - (n-d)\bar{c}]$ を因子とし
 左の因子は d の 2 の冪と \bar{c} の対称の冪とを基の

$$\begin{aligned}
 &\sum_{d=0}^5 \binom{n}{d} p^d q^{n-d} \left\{ \frac{4}{p^3 q} \left[d^{(6)} - d^{(5)} (c 3 - 8) + d^{(4)} (c^{(2)} 3 - c 11 + 12) - d^{(3)} (c^{(3)} - c^{(2)} 3 + c 3) \right] \right. \\
 &\quad \left. \cdot [(n-d)^{(2)} - (n-d)\bar{c}] + \sim \right\} \\
 &= 4 \left\{ n^{(8)} B(c-2, n-8) p q^3 - n^{(9)} B(c-2, n-7) p q^2 (\bar{c} 3 - 8) + n^{(6)} B(c-2, n-6) p q (\bar{c}^{(2)} 3 - \bar{c} 11 + 12) \right. \\
 &\quad \left. - n^{(5)} B(c-2, n-5) p (\bar{c}^{(3)} - \bar{c}^{(2)} 3 + \bar{c} 3) \right. \\
 &\quad \left. + n^{(8)} B(c-6, n-8) p^3 q - n^{(9)} B(c-5, n-7) p^2 q (c 3 - 8) + n^{(6)} B(c-4, n-6) p q (c^{(2)} 3 - c 11 + 12) \right. \\
 &\quad \left. - n^{(5)} B(c-3, n-5) q (c^{(3)} - c^{(2)} 3 + c 3) \right. \\
 &\quad \left. - c \left[n^{(9)} B(c-1, n-7) q^3 - n^{(6)} B(c-1, n-6) q^2 (\bar{c} 3 - 8) + n^{(5)} B(c-1, n-5) q (\bar{c}^{(2)} 3 - \bar{c} 11 + 12) \right. \right. \\
 &\quad \left. \left. - n^{(4)} B(c-1, n-4) (\bar{c}^{(3)} - \bar{c}^{(2)} 3 + \bar{c} 3) \right] \right. \\
 &\quad \left. - \bar{c} \left[n^{(9)} B(c-6, n-7) p^3 - n^{(6)} B(c-5, n-6) p^2 (c 3 - 8) + n^{(5)} B(c-4, n-5) p (c^{(2)} 3 - c 11 + 12) \right. \right. \\
 &\quad \left. \left. - n^{(4)} B(c-3, n-4) (c^{(3)} - c^{(2)} 3 + c 3) \right] \right\} \dots (4.13)
 \end{aligned}$$

右項の $c-\lambda$ を $- \circ \tau$ だけの中の方に書けると

$$\begin{aligned}
 & n^{(8)} b(c-2, n-8) p q^3 - n^{(7)} b(c-2, n-7) p q^2 (\bar{c}3-8) + n^{(6)} b(c-2, n-6) p q (\bar{c}^{(2)}3 - \bar{c}11+12) \\
 & - n^{(5)} b(c-2, n-5) p (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3) + \sim \\
 & - c [n^{(7)} b(c-1, n-7) q^3 - n^{(6)} b(c-1, n-6) q^2 (\bar{c}3-8) + n^{(5)} b(c-1, n-5) q (\bar{c}^{(2)}3 - \bar{c}11+12) \\
 & - n^{(4)} b(c-1, n-4) (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3)] - \sim \\
 & = n b(c, n-1) \left\{ \frac{c^{(2)} - c \cdot c}{p q^2} [\bar{c}^{(5)} - \bar{c}^{(4)} (\bar{c}3-8) + \bar{c}^{(3)} (\bar{c}^{(2)}3 - \bar{c}11+12) - \bar{c}^{(2)} (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3)] \right. \\
 & \quad \left. + \sim \right\} \\
 & = 0
 \end{aligned}$$

と \$\bar{c}\$ の \$\tau\$, (4.13) は \$\bar{c}\$ の \$\bar{c}\$ に \$\bar{c}\$ 3.

$$\begin{aligned}
 (4.13) = & 4 \left\{ n^{(8)} B(c-3, n-8) p q^3 - n^{(7)} B(c-3, n-7) p q^2 (\bar{c}3-8) \right. \\
 & + n^{(6)} B(c-3, n-6) p q (\bar{c}^{(2)}3 - \bar{c}11+12) - n^{(5)} B(c-3, n-5) p (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3) \\
 & + n^{(6)} B(c-5, n-8) p^3 q - n^{(7)} B(c-4, n-7) p^2 q (c3-8) \\
 & + n^{(6)} B(c-3, n-6) p q (c^{(2)}3 - c11+12) - n^{(5)} B(c-2, n-5) q (c^{(2)} - c^{(2)}3 + c3) \\
 & - c [n^{(7)} B(c-2, n-7) q^3 - n^{(6)} B(c-2, n-6) q^2 (\bar{c}3-8) + n^{(5)} B(c-2, n-5) q (\bar{c}^{(2)}3 - \bar{c}11+12) \\
 & - n^{(4)} B(c-2, n-4) (\bar{c}^{(3)}2 - \bar{c}^{(2)}3 + \bar{c}3)] \\
 & \left. - \bar{c} [n^{(6)} B(c-5, n-7) p^3 - n^{(6)} B(c-4, n-6) p^2 (\bar{c}3-8) + n^{(5)} B(c-3, n-5) p (c^{(2)}3 - c11+12) \right. \\
 & \quad \left. - n^{(4)} B(c-2, n-4) (c^{(3)}2 - c^{(2)}3 + c3) \right\} \dots (4.14)
 \end{aligned}$$

(4.11) の \$\bar{c}\$ の \$\bar{c}\$ の \$\bar{c}\$ に \$\bar{c}\$ 3 と

$$\begin{aligned}
 \sum_{d=0}^c \binom{n}{d} p^d q^{n-d} \left\{ - \frac{2}{p^2 q^2} [d^{(3)} + d^{(2)}3] [(n-d)^{(4)} - (n-d)^{(3)} (\bar{c}2-3) + (n-d)^{(2)} (\bar{c}^{(2)}2 - \bar{c}+1)] \right. \\
 - \sim \\
 + \frac{6}{p^2 q^2} [d^{(4)} - d^{(3)} (c2-3) + d^{(2)} (c^{(2)} - c+1)] [(n-d)^{(4)} - (n-d)^{(3)} (\bar{c}c-3) + (n-d)^{(2)} (\bar{c}^{(2)}2 - \bar{c}+1)] \\
 \left. + \frac{2}{3} \frac{1}{p^2 q^2} [d^{(3)} + d^{(2)}3] [(n-d)^{(3)} + (n-d)^{(2)}3] \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= -2 \left\{ n^{(7)} B(c-3, n-7) p q^2 - n^{(6)} B(c-3, n-6) p q (\bar{c}-3) + n^{(5)} B(c-3, n-5) p (\bar{c}^{(2)} \bar{c}+1) \right. \\
 &\quad \left. + n^{(7)} B(c-4, n-7) p^2 q - n^{(6)} B(c-3, n-6) p q (c-3) + n^{(5)} B(c-2, n-5) q (c^{(2)} c+1) \right\} \\
 &- 6 \left\{ n^{(6)} B(c-2, n-6) q^2 - n^{(5)} B(c-2, n-5) q (\bar{c}-3) + n^{(4)} B(c-2, n-4) (\bar{c}^{(2)} \bar{c}+1) \right. \\
 &\quad \left. + n^{(6)} B(c-4, n-6) p^2 - n^{(5)} B(c-3, n-5) p (c-3) + n^{(4)} B(c-2, n-4) (c^{(2)} c+1) \right\} \\
 &+ 6 \left\{ n^{(8)} B(c-4, n-8) p^2 q^2 - n^{(7)} B(c-3, n-7) p q^2 (c-3) - n^{(7)} B(c-4, n-7) p^2 q (\bar{c}-3) \right. \\
 &\quad \left. + n^{(6)} B(c-2, n-6) q^2 (c^{(2)} c+1) + n^{(6)} B(c-3, n-6) p q (c-3) (\bar{c}-3) + n^{(6)} B(c-4, n-6) p^2 (\bar{c}^{(2)} \bar{c}+1) \right. \\
 &\quad \left. - n^{(5)} B(c-2, n-5) q (c^{(2)} c+1) (\bar{c}-3) - n^{(5)} B(c-3, n-5) p (\bar{c}^{(2)} \bar{c}+1) (c-3) \right. \\
 &\quad \left. + n^{(4)} B(c-2, n-4) (c^{(2)} c+1) (\bar{c}^{(2)} \bar{c}+1) \right\} \\
 &+ n^{(6)} B(c-3, n-6) p q \frac{2}{3} + n^{(5)} B(c-2, n-5) q^2 + n^{(5)} B(c-3, n-5) p^2 \\
 &+ n^{(4)} B(c-2, n-4) 6 \dots (4.15)
 \end{aligned}$$

(4.12), (4.14), (4.15) E ու ձ

$$\begin{aligned}
 A_4 &= mb(c, n-1) A_{4,0} + B(c-2, n-4) \left\{ n^{(8)} - n^{(7)} [(n-1)4-16] \right. \\
 &\quad \left. + n^{(6)} [(n-1)^{(4)} 6 - (n-1)38 + \frac{208}{3}] - n^{(5)} [(n-1)^{(3)} 4 - (n-1)^{(2)} 28 + (n-1)76 - \frac{384}{5}] \right. \\
 &\quad \left. + n^{(4)} [(n-1)^{(4)} - (n-1)^{(3)} 6 + (n-1)^{(2)} 15 - (n-1)15] \right\}
 \end{aligned}$$

$$B(c-2, n-4) \text{ ու } \left\{ \frac{2}{3} \frac{2}{3} \frac{1}{3} \right\} n^{(6)} \frac{1}{3} + n^{(5)} \frac{19}{5} + n^{(4)} 9 = -f_4(n) + \frac{4!}{2!(2!)^2} (f_2(n))^2$$

... (4.16)

և 5)

$$A_4 = mb(c, n-1) A_{4,0} - B(c-2, n-4) \left\{ f_4(n) - \frac{4!}{2!(2!)^2} (f_2(n))^2 \right\}$$

... (4.17)

$$\begin{aligned}
 A_{4,0} &= \frac{1}{q^3} \left[\bar{c}^{(5)} \frac{1}{3} + \bar{c}^{(4)} \frac{19}{5} + \bar{c}^{(3)} 9 \right] - \frac{1}{p^3} \left[c^{(5)} \frac{1}{3} + c^{(4)} \frac{19}{5} + c^{(3)} 9 \right] \\
 &\quad + \frac{c}{p q^2} \left[\bar{c}^{(4)} \frac{1}{3} + \bar{c}^{(3)} \frac{9}{5} \right] - \frac{\bar{c}}{p^2 q} \left[c^{(4)} \frac{1}{3} + c^{(3)} \frac{9}{5} \right] \dots (4.18) \\
 &= ((\bar{c}^2 + \bar{c}2) (c - \frac{13}{5}) + \frac{24}{5}) \frac{1}{q^3} \left[\frac{\bar{c}^2 + \bar{c}2}{3q} + \frac{c\bar{c}}{3p} \right] - \sim
 \end{aligned}$$

よって (4.9) に よって

$$\begin{aligned}
 K_{2m} = mb(c, n-1) & \left\{ \frac{F_{2m}}{n^{(2)}} A_{0,0} + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)}{n^{(4)}} A_{0,1} \right. \\
 & + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n) - \binom{2m}{4} F_{2m-4} \left[f_4(n) - \frac{4!}{2!(2!)} \frac{(f_2(n))^2}{2} \right]}{n^{(6)}} A_{0,2} \\
 & \left. + \binom{2m}{2} F_{2m-2} A_{2,0} + \binom{2m}{4} F_{2m-4} A_{4,0} \right\} \\
 & + \left\{ F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n) - \binom{2m}{4} F_{2m-4} \left[f_4(n) - \frac{4!}{2!(2!)} \frac{(f_2(n))^2}{2} \right] \right\} \\
 & \quad \cdot B(c-3, n-6) \\
 & + \sum_{t=3}^m F_{2m-2t} \binom{2m}{2t} A_{2t} \quad \dots (4.19)
 \end{aligned}$$

よって K_4 は次のように計算される。

$$\begin{aligned}
 K_4 = mb(c, n-1) & \left\{ \frac{n+1}{3} \left[n(n^2-1) + (n^2-1) \frac{18}{5} - \frac{24}{5} \right] A_{0,0} \right. \\
 & - \frac{n+1}{3} \left[n+1 + \frac{2}{5} \right] A_{0,1} \\
 & \left. + 2n(n^2-1) A_{2,0} + A_{4,0} \right\} \quad \dots (4.20)
 \end{aligned}$$

(4.20) は筆者が別紙に本題のために (4) と一致して

いる。

むすび

できるだけ一般的に計算を進めたいと思っ

て工夫もして見たが、今はこの程度にとま
 った。なお筆者が前に書いたもの⁽⁴⁾とは記号も
 計算の進め方もちがっているので、それと
 混まるときは注意してほしい。

計算機による取値例については、戸田英雄
 電子君が行っている。

参考文献

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 and Inverse," *Biometrika*, vol. 41 (1954) pp. 317~329

(3) G. J. Lieberman and David B. Owen:
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(4) 山内：本文の同じ題目で近刊「応用統計
 学」に投稿中。

付録 1 $f_{2n}(n)$ の値

$$f_2 = (n+1)^{(3)} / 3$$

$$f_4 = \frac{3!}{5} \left\{ (n+2)^{(5)} + (n+1)^{(3)} \frac{5}{3} \right\}$$

$$f_6 = \frac{5!}{7} \left\{ (n+3)^{(7)} + (n+2)^{(5)} 7 + (n+1)^{(3)} \frac{7}{3} \right\}$$

$$f_8 = \frac{7!}{9} \left\{ (n+4)^{(9)} + (n+3)^{(7)} 9 \cdot 2 + (n+2)^{(5)} \frac{9 \cdot 21}{5} + (n+1)^{(3)} \frac{9}{3} \right\}$$

$$f_{10} = \frac{9!}{11} \left\{ (n+5)^{(11)} + (n+4)^{(9)} \frac{11 \cdot 10}{3} + (n+3)^{(7)} 11 \cdot 21 + (n+2)^{(5)} 11 \cdot 17 + (n+1)^{(3)} \frac{11}{3} \right\}$$

$$f_{12} = \frac{11!}{13} \left\{ (n+6)^{(13)} + (n+5)^{(11)} 13 \cdot 5 + (n+4)^{(9)} \frac{13 \cdot 11 \cdot 19}{3} + (n+3)^{(7)} \frac{13 \cdot 11 \cdot 128}{7} \right. \\ \left. + (n+2)^{(5)} \frac{13 \cdot 11 \cdot 31}{5} + (n+1)^{(3)} \frac{13}{3} \right\}$$

付録 2 F_{2m} の値

$$F_2 = (n+1)^{(3)} / 3$$

$$F_4 = (n+3)^{(6)} / 3 + (n+3)^{(5)} / 5$$

$$F_6 = (n+5)^{(9)} 5 / 3^2 + (n+5)^{(8)} + (n+5)^{(7)} / 7$$

$$F_8 = (n+7)^{(12)} 35 / 3^3 + (n+7)^{(11)} 14 / 3 + (n+7)^{(10)} 41 / 15 \\ + (n+7)^{(9)} / 9$$

$$F_{10} = (n+9)^{(15)} 35 / 3^2 + (n+9)^{(14)} 70 / 3 + (n+9)^{(13)} 31 \\ + (n+9)^{(12)} 23 / 3 + (n+9)^{(11)} / 11$$

$$F_{12} = (n+11)^{(18)} 11 \cdot 7 \cdot 5 / 3^3 + (n+11)^{(17)} 11 \cdot 7 \cdot 5 / 3 + (n+11)^{(16)} 11 \cdot 83 / 3 \\ + (n+11)^{(15)} 11 \cdot 268 / 15 + (n+11)^{(14)} 157 / 7 \\ + (n+11)^{(13)} / 13$$

超幾何分布に於ける山内の式

$$[\text{記号}] \quad p_d = \frac{\binom{Np_0}{d} \binom{N-Np_0}{n-d}}{\binom{N}{n}}, \quad (d=0, 1, \dots, n) \quad (1)$$

$$0 < Np_0 < N$$

$d=0, 1, \dots, n$ なる區間に於ては確率 p_0, p_1, \dots, p_n で出現する。

$$P = \sum_{d=0}^c p_d \equiv H(N, n, Np_0, c) \quad (2)$$

$$p_d \sim \binom{n}{d} p_0^d (1-p_0)^{n-d} \equiv b(d, n) \quad (3)$$

$$P \sim \sum_{d=0}^c \binom{n}{d} p_0^d (1-p_0)^{n-d} \equiv B(c, n, p_0) \quad (4)$$

[山内の近似式 (1953)]

$$P = B(c, n, p_0) - n \binom{n-1}{c} p_0^c (1-p_0)^{n-c-1} \cdot \left[\frac{A_1}{N} + \frac{A_2}{N^2} - \frac{A_1^3}{N^3 p_0 (1-p_0)} \right] \quad (5)$$

$$A_1 = - (c - (n-1)p_0) / 2$$

$$A_2 = A_1^2 \left[\frac{1}{6 p_0 (1-p_0)} + \frac{7}{6 p_0} \right] + A_1 \left[\frac{3c}{4 p_0} + \frac{1}{6(1-p_0)} + \frac{1}{6} \right]$$

$$+ \frac{c}{6} - \frac{c}{12 p_0}$$

実用的に $H(N, n, Np_0, c)$ を $B(c, n, p_0)$ で近似する時の目標は $\frac{1}{N}$ の

未満の値:

$$\frac{n}{N} \frac{c - (n-1)p_0}{2} \binom{n-1}{c} p_0^c (1-p_0)^{n-1-c}$$

が近似の尺度となる

[Wick - 山田の式 (1971)]

$$p = \frac{Np_0 - \frac{c}{2}}{N - \frac{n-1}{2}}, \quad \left(N - \frac{n-1}{2} = M \text{ とおく} \right)$$

なる p を代入して M^{-2} の級数展開が得られた。

$$P = B(c, n, p)$$

$$\begin{aligned}
 & + \frac{n \cdot b(c, n-1, p)}{2! (2M)^2} \left\{ \frac{n+1}{3} [(n-1-c)g - cp] - \frac{(n-1-c)^2 + 2(n-1-c)}{3g} + \frac{c^2 + 2c}{3p} \right\} \\
 & \begin{array}{l} \uparrow \\ \text{Wick} \\ \downarrow \\ \text{山田} \end{array} + \frac{n \cdot b(c, n-1, p)}{4! (2M)^4} \left\{ \frac{n+1}{3} [(n-1-c)g - cp] \cdot \left[n^2 - n + \frac{(n^2-1)18-24}{5} \right] \right. \\
 & \quad \left. + \left[-\frac{(n-1-c)^2 + 2(n-1-c)}{3g} + \frac{c^2 + 2c}{3p} \right] 2(n^3 - n) \right. \\
 & \quad \left. + (n+1 + \frac{2}{5})(n+1) \cdot \left[\frac{c(n-1-c)}{3g} \cdot (c-1) - \frac{c(n-1-c) \cdot (n-1-c-1)}{3p} \right] \right. \\
 & \quad \left. + \left(\left[(n-1-c)^2 + 2(n-1-c) \right] (n-1-c - \frac{13}{5}) + \frac{24}{5} \right) \frac{1}{g^2} \left[\frac{(n-1-c)^2 + 2(n-1-c)}{3g} + \frac{c(n-1-c)}{3p} \right] \right. \\
 & \quad \left. - \left([c^2 + 2c] (c - \frac{13}{5}) + \frac{24}{5} \right) \frac{1}{p^2} \left[\frac{c^2 + 2c}{3p} + \frac{c(n-1-c)}{3g} \right] \right\} \\
 & + \dots
 \end{aligned}$$

N	項数	$\exp(\ln \Gamma(N+1))$	N!	秒	$(\Gamma(N+1) - N!)/N!$
2	25	.2000001123195979D+01	.2000000000000000D+01	.00	.5615976740978831D-06
3	23	.6000000016052141D+01	.6000000000000000D+01	.00	.2615232423541415D-08
4	15	.239999999999564D+02	.2400000000000000D+02	.01	-.161783407711186D-12
5	13	.120000000000005D+03	.1200000000000000D+03	.01	.5391057970408699D-13
6	11	.719999999999956D+03	.7200000000000000D+03	.02	-.6112394541131097D-13
7	11	.503999999999949D+04	.5040000000000000D+04	.02	-.1033289162507302D-13
8	11	.40319999999993D+05	.4032000000000000D+05	.01	-.1741111664015328D-14
9	9	.3628800000000069D+06	.3628800000000000D+06	.01	.1895281893410150D-13
10	9	.362880000000024D+07	.3628800000000000D+07	.02	.6611681611764134D-14
11	9	.399168000000009D+08	.3991680000000000D+08	.02	.2235458357893051D-14
12	9	.479001600000002D+09	.4790016000000000D+09	.02	.38254027953019D-15
13	9	.622702019999996D+10	.6227020300000000D+10	.02	-.5928617880601521D-15
14	9	.871782911999998D+11	.8717829120000000D+11	.02	-.1238709588373926D-14
15	2	.130767368000002D+13	.1307674368000000D+13	.03	.1898013992363573D-14
16	9	.209227898800003D+14	.2092278988000000D+14	.03	.1669345644200852D-14
17	7	.355687428095990D+15	.3556874280960000D+15	.03	-.2837540109732523D-14
18	7	.6402373705727990D+16	.6402373705728000D+16	.03	.1502128420182006D-14
19	7	.1216451004088319D+18	.1216451004083320D+18	.03	-.831825290178001D-15
20	7	.2432902008176639D+19	.2432902008176640D+19	.04	-.542561926272061D-15
21	7	.5109094217170941D+20	.510909421717094D+20	.04	.5323839969242426D-15
22	7	.1124000727777607D+22	.112400072777608D+22	.03	-.6472878371160204D-15
23	7	.258520167388498D+23	.258520167388498D+23	.04	.8663512880336322D-15
24	7	.6204484017332387D+24	.620448401733239D+24	.04	-.1170345314729657D-14
25	7	.135112100433096D+26	.135112100433099D+26	.04	.1515888712635254D-14
26	7	.4032914611266049D+27	.403291461126605D+27	.05	-.184640644972072D-14
27	7	.108888945041833D+29	.108888945041835D+29	.04	.222909615191703D-14
28	7	.3048883446117130D+30	.304888344611713D+30	.04	-.267315234738445D-14
29	7	.8841761993739675D+31	.884176199373970D+31	.04	.3032204990471046D-14
30	7	.265252859812191D+33	.265252859812191D+33	.04	-.3461494721226616D-14
31	7	.8222838654177953D+34	.822283865417923D+34	.05	.364356064957852D-14
32	7	.263130836933694D+36	.263130836933693D+36	.05	.3452663155018681D-14
33	7	.8683317618811915D+37	.8683317618811887D+37	.05	.322535356444924D-14
34	7	.2952327990396050D+39	.295232799039804D+39	.05	.3038634621650236D-14
35	7	.1033314796638617D+41	.1033314796638614D+41	.08	.2715941532359694D-14
36	7	.319933267899021D+42	.319933267899017D+42	.07	.2486142049814794D-14
37	7	.1376373309122638D+44	.1376373309122635D+44	.06	.219689140070408D-14
38	7	.5230222174666021D+45	.5230222174666011D+45	.06	.1938309742910059D-14
39	7	.2039788208119748D+47	.2039788208119744D+47	.06	.1619199489273998D-14
40	7	.8159152832478988D+48	.815915283247897D+48	.07	.1273996839536251D-14
41	7	.3345252661316384D+50	.3345252661316361D+50	.07	.1030110948422173D-14
42	7	.1405006117752881D+52	.1405006117752880D+52	.08	.6194697303552833D-15
43	7	.6041526306337385D+53	.6041526306337384D+53	.07	.2867073888580512D-15
44	7	.2658271574788449D+55	.2658271574788449D+55	.08	-.5850405841969392D-16
45	7	.1196222208654801D+57	.1196222208654802D+57	.07	.37538683234512D-15
46	7	.5302622159812085D+58	.5302622159812089D+58	.07	-.8088677052610630D-15
47	7	.2386232415111679D+60	.2386232415111682D+60	.07	-.1134970925726716D-14
48	7	.124139155923605D+62	.124139155923607D+62	.08	-.148960566850679D-14
49	7	.6082818640342664D+63	.6082818640342664D+63	.06	.182890300435227D-14
50	7	.3041409320171331D+65	.3041409320171338D+65	.08	-.2319329052372524D-14
51	7	.155111875287378D+67	.155111875287382D+67	.09	-.2685810102148830D-14
52	7	.8065817517094363D+68	.8065817517094388D+68	.09	-.303966091350860D-14
53	7	.42748324060010D+70	.427483240600026D+70	.09	-.3515780523895119D-14
54	7	.2308436973392405D+72	.2308436973392414D+72	.09	-.4026354439165213D-14
55	7	.1269640335365822D+74	.1269640335365828D+74	.09	-.4332759317942727D-14

□-1 $\exp(\ln \Gamma(N+1))$ と $N!$ の比較

N	項数	$\ln \Gamma(N+1)$	$\ln N + \ln(N-1) + \dots + \ln 2$	秒	$(1) - (2)$	$\ln(N!)$	秒
2	25	.69314774215777710+00	.69314718055994540+00	.10	.51597831675730-06	.69314718055994540+00	.01
3	23	.17917594719035780+01	.17917594692280350+01	.08	.2675232979767590-08	.17917594692280350+01	.02
4	15	.31780538303477640+01	.31780538303479660+01	.07	-.18199851348210670-12	-.1780538303479660+01	.02
5	13	.478749174278201000+01	.47874917427820600+01	.07	.558651527665010690-13	.47874917427820600+01	.02
6	11	.65792512120100400+01	.65792512120101010+01	.06	-.61360638792501620-13	.65792512120101010+01	.02
7	11	.85251613610659400+01	.85251613610654400+01	.05	-.101933314483970-13	.85251613610654400+01	.02
8	11	.10604602902745250+02	.10604602902745250+02	.06	-.20192381260370030-13	.10604602902745250+02	.03
9	9	.12801827480081490+02	.12801827480081700+02	.05	.118596235662471370-13	.12801827480081700+02	.03
10	9	.15104412573075200+02	.15104412573075200+02	.05	.62103100430969690-14	.15104412573075200+02	.03
11	9	.17502307845873890+02	.17502307845873890+02	.06	.183186799086315080-14	.17502307845873890+02	.03
12	9	.19987214495661890+02	.19987214495661890+02	.06	.0000000000000000+00	.19987214495661890+02	.03
13	9	.22551638531234200+02	.22551638531234200+02	.05	-.91593399531575410-15	.22551638531234200+02	.04
14	9	.25191221182738680+02	.25191221182738680+02	.05	-.14988010832439610-14	.25191221182738680+02	.04
15	9	.27899271383840890+02	.27899271383840890+02	.05	.17486012637846220-14	.27899271383840890+02	.04
16	9	.30671860106080670+02	.30671860106080670+02	.06	.148016565685892600-14	.30671860106080670+02	.04
17	7	.33505073450136890+02	.33505073450136890+02	.05	-.31918911957973250-14	.33505073450136890+02	.04
18	7	.36395445208030500+02	.36395445208030500+02	.05	-.19151347174783950-14	.36395445208030500+02	.05
19	7	.39339884187199490+02	.39339884187199490+02	.05	-.113045120539345500-14	.39339884187199490+02	.04
20	7	.423352616460753480+02	.423352616460753480+02	.04	-.10547118733938990-14	.423352616460753480+02	.04
21	7	.45380138898476910+02	.45380138898476910+02	.05	-.10547118733938990-14	.45380138898476910+02	.04
22	7	.48471181351352200+02	.48471181351352200+02	.05	-.11657341758564140-14	.48471181351352200+02	.05
23	7	.51606675567764370+02	.51606675567764370+02	.05	-.13600232051658170-14	.51606675567764370+02	.05
24	7	.54784729398112320+02	.54784729398112320+02	.05	-.16098233857064770-14	.54784729398112320+02	.05
25	7	.58003605222980320+02	.58003605222980320+02	.05	-.19151347174783950-14	.58003605222980320+02	.06
26	7	.61261701761002000+02	.61261701761002000+02	.05	-.2322690436346840-14	.61261701761002000+02	.05
27	7	.64557538270063300+02	.64557538270063300+02	.05	-.2442490654753440-14	.64557538270063300+02	.06
28	7	.67889743137181530+02	.67889743137181530+02	.05	-.2773557561562890-14	.67889743137181530+02	.06
29	7	.71257038967168010+02	.71257038967168010+02	.05	-.30531133177191800-14	.71257038967168010+02	.06
30	7	.74658236488301600+02	.74658236488301600+02	.06	-.33306690738754700-14	.74658236488301600+02	.06
31	7	.78092223533153100+02	.78092223533153100+02	.05	-.39412917374193060-14	.78092223533153100+02	.06
32	7	.81557959561150400+02	.81557959561150400+02	.05	-.36082248300317590-14	.81557959561150400+02	.06
33	7	.85054467017581520+02	.85054467017581520+02	.05	-.32751579226451200-14	.85054467017581520+02	.07
34	7	.88590827521976800+02	.88590827521976800+02	.05	-.29976021666879230-14	.88590827521976800+02	.06
35	7	.92136175603687400+02	.92136175603687400+02	.05	-.27200464103316340-14	.92136175603687400+02	.08
36	7	.95719694542143210+02	.95719694542143210+02	.05	-.23869795029440870-14	.95719694542143210+02	.07
37	7	.99330612454781430+02	.99330612454781430+02	.05	-.19984014443252820-14	.99330612454781430+02	.07
38	7	.10298198614512800+03	.10298198614512800+03	.05	.116653345369377350-14	.10298198614512800+03	.07
39	7	.10663176026064350+03	.10663176026064350+03	.05	.11322676295501880-14	.10663176026064350+03	.08
40	7	.11032639714757400+03	.11032639714757400+03	.05	-.99920072216264080-15	.11032639714757400+03	.08
41	7	.11403421178143470+03	.11403421178143470+03	.05	.71264496600635180-15	.11403421178143470+03	.08
42	7	.117718136974510+03	.117718136974510+03	.05	.38857805861880480-15	.117718136974510+03	.09
43	7	.12153308151543600+03	.12153308151543600+03	.05	.0000000000000000+00	.12153308151543600+03	.08
44	7	.12531727114935590+03	.12531727114935590+03	.05	-.1277557561562890-15	.12531727114935590+03	.09
45	7	.12912393363912720+03	.12912393363912720+03	.05	-.366613381477509380-15	.12912393363912720+03	.09
46	7	.13295257503561330+03	.13295257503561330+03	.05	-.99920072216264080-15	.13295257503561330+03	.09
47	7	.13680272263752640+03	.13680272263752640+03	.05	-.13322676295501880-14	.13680272263752640+03	.10
48	7	.14063739236423350+03	.14063739236423350+03	.05	-.155432344752190-14	.14063739236423350+03	.09
49	7	.14445574394634490+03	.14445574394634490+03	.05	-.18873791418627660-14	.14445574394634490+03	.10
50	7	.14827716695177300+03	.14827716695177300+03	.05	-.2220446049293130-14	.14827716695177300+03	.10
51	7	.15210959258446740+03	.15210959258446740+03	.05	-.2535129566378660-14	.15210959258446740+03	.09
52	7	.15636083030307080+03	.15636083030307080+03	.05	-.2175557561562890-14	.15636083030307080+03	.10
53	7	.16033112821663090+03	.16033112821663090+03	.05	-.32196467174129540-14	.16033112821663090+03	.10
54	7	.16432011263193200+03	.16432011263193200+03	.05	-.3527136780005010-14	.16432011263193200+03	.11
55	7	.16832744514627200+03	.16832744514627200+03	.05	-.3714750083755330-14	.16832744514627200+03	.10
56	7	.172345781351120000+03	.172345781351120000+03	.05	-.41603219411507000-14	.172345781351120000+03	.10

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57	1.7639528494069973D+03	.05	1.7639528494069974D+03	.70	-7432298997980381111D-14	1.683274454484277D+03	1.0
58	1.804562914173438D+03	.05	1.804562914173438D+03	.71	-7773599003888173D-14	1.683274454484277D+03	1.0
59	1.84338288614495D+03	.05	1.84338288614495D+03	.72	-488481448350689D-14	1.683274454484277D+03	1.0
60	1.88281734236716D+03	.04	1.88281734236716D+03	.74	-4996003610813204D-14	1.683274454484277D+03	1.0
61	1.927390472878449D+03	.05	1.927390472878449D+03	.75	-5440092820663267D-14	1.683274454484277D+03	1.0
62	1.96866181672890D+03	.05	1.96866181672890D+03	.76	-5862137425588298D-14	1.683274454484277D+03	1.0
63	2.010093163992815D+03	.04	2.010093163992815D+03	.77	-9325873406851315D-14	1.683274454484277D+03	1.0
64	2.051681984826412D+03	.05	2.051681984826412D+03	.78	-9214851108388799D-14	1.683274454484277D+03	1.0
65	2.09342586752368D+03	.04	2.09342586752368D+03	.80	-8770761894538737D-14	1.683274454484277D+03	1.0
66	2.135322414945633D+03	.04	2.135322414945633D+03	.81	-8437694987151190D-14	1.683274454484277D+03	1.0
67	2.17369341139542D+03	.04	2.17369341139542D+03	.83	-8216550382226158D-14	1.683274454484277D+03	1.0
68	2.219564418191303D+03	.05	2.219564418191303D+03	.83	-799360577301127D-14	1.683274454484277D+03	1.0
69	2.261905483237276D+03	.04	2.261905483237276D+03	.84	-7660538869913580D-14	1.683274454484277D+03	1.0
70	2.304390435657770D+03	.04	2.304390435657770D+03	.86	-7438494264988549D-14	1.683274454484277D+03	1.0
71	2.347017234428183D+03	.05	2.347017234428183D+03	.87	-721684966063518D-14	1.683274454484277D+03	1.0
72	2.389783895618343D+03	.05	2.389783895618343D+03	.88	-68838275267591D-14	1.683274454484277D+03	1.0
73	2.432688490029827D+03	.05	2.432688490029827D+03	.89	-6661338147750939D-14	1.683274454484277D+03	1.0
74	2.475729170961869D+03	.04	2.475729170961869D+03	.91	-621724893790087D-14	1.683274454484277D+03	1.0
75	2.518904022097232D+03	.04	2.518904022097232D+03	.93	-5995204332975845D-14	1.683274454484277D+03	1.0
76	2.56221135550095D+03	.05	2.56221135550095D+03	.93	-55511512125783D-14	1.683274454484277D+03	1.0
77	2.605649409718632D+03	.04	2.605649409718632D+03	.95	-5292907051820075D-14	1.683274454484277D+03	1.0
78	2.649216497985528D+03	.04	2.649216497985528D+03	.96	-4884981308350689D-14	1.683274454484277D+03	1.0
79	2.69210976510198D+03	.04	2.69210976510198D+03	.97	-4884981308350689D-14	1.683274454484277D+03	1.0
80	2.736731242856327D+03	.05	2.736731242856327D+03	.98	-42188474937532D-14	1.683274454484277D+03	1.0
81	2.780675734403661D+03	.04	2.780675734403661D+03	1.00	-3996802888650564D-14	1.683274454484277D+03	1.0
82	2.824742926876304D+03	.04	2.824742926876304D+03	1.01	-3996802888650564D-14	1.683274454484277D+03	1.0
83	2.868931332954270D+03	.04	2.868931332954270D+03	1.02	-377475828372532D-14	1.683274454484277D+03	1.0
84	2.913239500942703D+03	.05	2.913239500942703D+03	1.03	-3108624468950438D-14	1.683274454484277D+03	1.0
85	2.957666013507606D+03	.04	2.957666013507606D+03	1.05	-3108624468950438D-14	1.683274454484277D+03	1.0
86	3.002209486470141D+03	.04	3.002209486470141D+03	1.06	-32664522525100316D-14	1.683274454484277D+03	1.0
87	3.04686567656687D+03	.04	3.04686567656687D+03	1.07	-2220446049250313D-14	1.683274454484277D+03	1.0
88	3.091641955801469D+03	.05	3.091641955801469D+03	1.08	-2220446049250313D-14	1.683274454484277D+03	1.0
89	3.136528299498791D+03	.04	3.136528299498791D+03	1.10	-1776356839400250D-14	1.683274454484277D+03	1.0
90	3.181526396202093D+03	.04	3.181526396202093D+03	1.11	-1554312234475219D-14	1.683274454484277D+03	1.0
91	3.226634931267262D+03	.04	3.226634931267262D+03	1.12	-1133227629550186D-14	1.683274454484277D+03	1.0
92	3.271852877037752D+03	.04	3.271852877037752D+03	1.13	-1110223024625157D-14	1.683274454484277D+03	1.0
93	3.317178871969285D+03	.04	3.317178871969285D+03	1.15	-16661338147750939D-15	1.683274454484277D+03	1.0
94	3.362611819791985D+03	.05	3.362611819791985D+03	1.16	-14440892098500626D-15	1.683274454484277D+03	1.0
95	3.408150588707990D+03	.04	3.408150588707990D+03	1.17	-12220446049250313D-15	1.683274454484277D+03	1.0
96	3.453794070622669D+03	.04	3.453794070622669D+03	1.19	-12220446049250313D-15	1.683274454484277D+03	1.0
97	3.499541180407702D+03	.05	3.499541180407702D+03	1.19	-12220446049250313D-15	1.683274454484277D+03	1.0
98	3.545390855194408D+03	.05	3.545390855194408D+03	1.20	-14440892098500626D-15	1.683274454484277D+03	1.0
99	3.591342053695754D+03	.05	3.591342053695754D+03	1.22	-16881784197001252D-15	1.683274454484277D+03	1.0
100	3.6373937555555635D+03	.04	3.6373937555555635D+03	1.23	-1110223024625157D-14	1.683274454484277D+03	1.0

図-2 $\ln(\Gamma(N+1))$ と $\ln N!$ の比較

N= 100 S= 40 C= 16

NPO	P1	P2	P3
10.0	.999996	.999996	.999995
15.0	.999990	.999990	.999990
20.0	.9999747	1.000023	.999918
25.0	.9981362	.9989400	.9988530
30.0	.9723940	.9770684	.9770343
35.0	.8490185	.8572767	.8575598
40.0	.5808146	.5833520	.5834712
45.0	.2761124	.2698345	.2695708
50.0	.0832474	.0764569	.0763262

TIME= .7000

NPO	PE	ER
10.0	.999980	.000015
15.0	.999995	-.000005
20.0	.9999944	-.000025
25.0	.9988522	.000008
30.0	.9770260	.000083
35.0	.8575714	-.000016
40.0	.5834769	-.000057
45.0	.2695580	.0000128
50.0	.0763256	.0000006

TIME= 1.0700

N= 200 S= 80 C= 32

NPO	P1	P2	P3
20.0	.9999997	.9999997	.9999996
30.0	.9999986	.9999986	.9999986
40.0	.9999979	.9999979	.9999978
50.0	.9999593	.9999890	.9999810
60.0	.9947863	.9962339	.9961442
70.0	.9054632	.9129043	.9130719
80.0	.5574017	.5591999	.5592844
90.0	.1627297	.1551904	.1549481
100.0	.0179172	.0149796	.0150384

TIME= .8400

NPO	PE	ER
20.0	.9999987	.0000010
30.0	.9999955	.0000031
40.0	.9999948	.0000031
50.0	.9999845	-.0000035
60.0	.9961402	.0000040
70.0	.9130753	-.0000034
80.0	.5592884	-.0000040
90.0	.1549392	.0000089
100.0	.0150432	-.0000048

TIME= 1.1400