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On a result of Sullivan and the mod  $p$   
decomposition of Lie groups.

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## §0. Introduction.

Let  $p$  be a prime. A simply connected CW complex  $X$  is called "mod  $p$  decomposable into  $r$  spaces" if there exist simply connected CW complexes  $X_i$ ,  $1 \leq i \leq r$ , with  $H^*(X_i; \mathbb{Z}_p) \neq 0$ , and there exists a  $p$ -equivalence  $f: \prod_{i=1}^r X_i \rightarrow X$ . A mod  $p$  decomposition is called irreducible if each  $X_i$  is not mod  $p$  decomposable.

In the present note, we shall consider the mod  $p$  decomposition of  $SU(n)$  and other simple Lie groups. For  $n \leq 2p$  or  $n = \infty$ , the mod  $p$  decomposition of  $SU(n)$  has been given by J. P. Serre [4], M. Mimura-H. Toda [6] and F. P. Peterson [8]. Then our result is as follows: Let  $G$  be a compact simply connected simple Lie group. Suppose that  $H^*(G)$  has no  $p$ -torsion. Hence  $H^*(G; \mathbb{Z}_p) = \Lambda(x_{n_1}, \dots, x_{n_e})$  is the exterior algebra with  $\deg x_{n_i} = 2n_i - 1$ . Let  $r(G)$  be the number of  $n_i$ 's which are distinct mod  $p-1$ .

Main Theorem. Let  $G$  be as above and suppose that  $H^*(G)$  has no  $p$ -torsion. Then  $G$  is mod  $p$  decomposable irreducibly into  $r(G)$  spaces if  $G \neq \text{Spin}(4n)$ .  $\text{Spin}(4n)$  is mod decomposable irreducibly into  $r(G)+1$  spaces for odd  $p$ .

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## §1. Localization of CW-complexes.

In this section, we review some results of [3]. Let  $P$  be a subset of all prime numbers. Let  $Q_p$  be the ring of the fractions whose denominator, in the lowest term, is prime to  $p$  for any  $p \in P$ . The void set will be denoted by  $(0)$ , and hence

$Q \cong Q_{(0)}$  is the field of rational numbers. If  $P$  is void, then a  $P$ -equivalence is called a 0-equivalence.

Let  $\mathcal{C}_1$  (resp.  $\mathcal{H}\mathcal{C}_1$ ) be the homotopy category of 1-connected (resp. 1-connected with finitely generated homology groups in each dimension) CW-complexes. Then we have

Theorem 1.1 (Theorem 2.4 and 2.5 in [3]). Let  $P$  be a subset of all prime numbers. Then there exists a functor  $L_P: \mathcal{H}\mathcal{C}_1 \rightarrow \mathcal{C}_1$  (we denote  $L_P(x)$  and  $L_P(f)$  briefly by  $X_P$  and  $f_P$ ) and a natural inclusion  $j_X: X \rightarrow X_P$  satisfying the following conditions.

(i)  $f: X \rightarrow Y$  in  $\mathcal{H}\mathcal{C}_1$  is a  $P$ -equivalence if and only if  $f_P: X_P \rightarrow Y_P$  is a homotopy equivalence.

(ii)  $\pi_*(X_P) \cong \pi_*(X) \otimes Q_P$  and  $(j_X)_*: \pi_*(X) \rightarrow \pi_*(X_P)$  coincides  $1 \otimes j: \pi_*(X) \otimes Z \rightarrow \pi_*(X) \otimes Q_P$ , where  $j$  is the canonical injection.

(iii)  $H_*(X_P) \cong H_*(X) \otimes Q_P$  and  $(j_X)_* = 1 \otimes j: H_*(X) \otimes Z \rightarrow H_*(X) \otimes Q_P$ .

For the proof, see [3]. But roughly speaking, the construction of  $X_P$  is as follows: For a space  $X$ , we associate a direct system  $\{X \xrightarrow{f_\lambda} X_\lambda\}$ , where  $f_\lambda$  varies all  $P$ -equivalences. Then we can define an appropriate linearly ordered cofinal subsystem  $\{X_n, f_n\}$  with  $X_0 = X$ , called a  $P$ -sequence in [3]. Then  $X_P$  is defined by the telescope construction of Adams [1].

Remark D. Sullivan has defined the localization functor for more general category, by use of the Postnikov system.

Now we call a countable CW-complex  $X$  finite  $P$ -local if  $H_*(X)$  is a finitely generated  $Q_P$ -module.

Theorem 1.2. Let  $\mathbb{P}_1$  and  $\mathbb{P}_2$  be such that  $\mathbb{P}_1 \cap \mathbb{P}_2 = (0)$  and  $\mathbb{P}_1 \cup \mathbb{P}_2 = \{\text{all primes}\}$ . Let  $X(\mathbb{P}_i)$   $i=1,2$ , be finite  $\mathbb{P}_i$ -local complexes and let  $X(0)$  be a finite 0-local complex. Assume that we are given 0-equivalences  $g_i: X(\mathbb{P}_i) \rightarrow X(0)$ . Put  $X = X(\mathbb{P}_1) \times_{X(0)} X(\mathbb{P}_2)$ , the pull-back of  $X(\mathbb{P}_i)$  over  $X(0)$ . Then  $X$  has a homotopy type of a finite CW complex and  $X_{\mathbb{P}_i} \cong X(\mathbb{P}_i)$ ,  $i=1,2$ .

For a proof, see [3].

Theorem 1.3. Let  $Z_p$  denote  $Z/pZ$  if  $p$  is a prime and  $Q$  if  $p=0$ . Then  $j_X^*: H^*(X_{\mathbb{P}}:Z_{\mathbb{P}}) \rightarrow H^*(X:Z_p)$  is isomorphic if  $p \in \mathbb{P}$  or  $p=0$ . If  $p \notin \mathbb{P}$ , then  $\tilde{H}^*(X_{\mathbb{P}}:Z_{\mathbb{P}}) \cong 0$ .

Proof. Note that  $\text{Hom}(Q_p, Z_p) \cong Z_p$  if  $p \in \mathbb{P}$  or  $p=0$ . and  $\text{Hom}(Q_p, Z_p) \cong 0$  if  $p \notin \mathbb{P}$ . Then the theorem immediately follows from the isomorphisms  $H^*(X_{\mathbb{P}}:Z_{\mathbb{P}}) \cong \text{Hom}(H_*(X_{\mathbb{P}}:Z_{\mathbb{P}}), Z_p) \cong \text{Hom}(H_*(X:Z_p) \otimes_{Q_p} Z_p, Z_p) \cong \text{Hom}(H_*(X:Z_p), \text{Hom}(Q_p, Z_p))$ .

§2. A result of D. Sullivan.

Theorem 2.1. (Sullivan). Let  $n$  be an integer and let  $q$  be a prime such that  $q > n$ . Then there exists a map  $\psi^q: BU(n) \rightarrow BU(n)$  such that  $(\psi^q)^* c_i = q^i c_i$ , where  $c_i \in H^{2i}(BU(n): \mathbb{Z})$  is the Chern class.

For a proof, see Chapter 5, of [10].

Now we shall state some easy consequences of Theorem 2.1. First consider the map  $\Omega\psi^q: U(n) \rightarrow U(n)$ . As is well-known  $H^*(U(n): \mathbb{Z}) \cong \Lambda(h_1, \dots, h_n)$  is the exterior algebra generated by the universal transgressive generators  $h_i$ . Since  $(\psi^q)^* x$

$= q^m x$  for any  $x \in H^{2m}(BU(n):Z)$  by Theorem 2.1, we have clearly  $(\Omega\Psi^q)^* h_i = q^i h_i$ .

Let  $k_r : U(n) \longrightarrow U(n)$  be the map defined by  $k_r(x) = x^{-q^r}$  for  $x \in U(n)$ . As is easily checked,  $k_r^*(h_i) = -q^r h_i$ . We consider the map for  $n < q$  and any  $r$

$$\lambda_{q,r} = \Omega\Psi^q + k_r : U(n) \longrightarrow U(n)$$

where the symbol  $+$  indicates the sum defined by the multiplication of  $U(n)$ .

Proposition 2.2.  $(\lambda_{q,r})^*(h_i) = (q^i - q^r)h_i$ .

Proof.  $\lambda_{q,r}$  is defined as the composition

$$U(n) \xrightarrow{\Delta} U(n) \times U(n) \xrightarrow{\Omega\Psi^q \times k_r} U(n) \times U(n) \xrightarrow{\mu} U(n)$$

where  $\Delta$  is the diagonal map and  $\mu$  is the multiplication.

Then  $(\lambda_{q,r})^*(h_i) = \Delta^* \circ (\Omega\Psi^q \times k_r)^* \circ \mu^*(h_i) = \Delta^* \circ (\Omega\Psi^q \times k_r)^*(h_i \otimes 1 + 1 \otimes h_i) = \Delta^*(q^i h_i \otimes 1 + 1 \otimes (-q^r)h_i) = (q^i - q^r)h_i$ , since  $h_i$  is primitive.

Now notice that we can define a map  $\lambda_{q,r}$  on  $SU(n)$  satisfying the property of Proposition 2.2. For we have the canonical homeomorphism  $SU(n) \times S^1 \cong U(n)$ .

§3. Mod  $p$  decomposition of  $SU(n)$ .

In this section, provided with the map  $\lambda_{q,r}$ , we do the similar arguments in §9 of [3].

Lemma 3.1. Let  $n$  be a positive integer and let  $p$  be a prime. Then there exists a prime  $q$  such that  $q > n$  and

$q$  is a primitive root mod  $p$ .

Proof. Let  $k$  be a primitive root mod  $p$ . Then so is  $k+pt$  for any positive integer  $t$ . Since  $(k,p)=1$ , there exist infinitely many prime numbers of the form  $k+pt$ , by the classical theorem of Dirichlet. This proves the lemma.

Lemma 3.2. Let  $q$  be a primitive root mod  $p$ . Then  $q^i - q^r \equiv 0(p)$  if and only if  $i-r \equiv 0(p-1)$ .

Proof. It is enough to show that  $q^m - 1 \equiv 0(p)$  if and only if  $m \equiv 0(p-1)$ . But this is just the definition of the primitive root.

Proposition 3.3. Let  $n$  and  $p$  be as above. Then for each  $m$  such as  $2 \leq m \leq p$  and  $m \leq n$ , there exists a 1-connected finite CW-complex  $X_m(n)$  and there exists a map  $f_m: SU(n) \rightarrow X_m(n)$  satisfying

$$(i) \quad H^*(X_m(n) : \mathbb{Z}_p) \cong \Lambda(x_m, x_{m+p-1}, \dots, x_{m+s(p-1)})$$

where  $\deg x_i = 2i-1$  and  $s = \lfloor \frac{n-m}{p-1} \rfloor$  is the largest integer  $\leq \frac{n-m}{p-1}$ .

$$(ii) \quad f_m^*(x_i) = h_i$$

Proof. Let  $q$  be a prime as in Lemma 3.1. Then by Theorem 2.1 and the remark at the end of §2, we can define  $\lambda_{q,r}: SU(n) \rightarrow SU(n)$  satisfying the property of Proposition 2.2. It is clear that  $\lambda_{q,r}$  is a 0-equivalence if  $r$  is large enough. Put  $g_m = \lambda_{q, (p-1)N+m}$ ,  $N$  large enough and  $2 \leq m \leq p$ .

Then by Proposition 2.2,  $g_m^* : H^*(SU(n):Z_p) \longrightarrow H^*(SU(n):Z_p)$  satisfies  $g_m^* h_i = (q^i - q^m) h_i$  and by Lemma 3.2,  $q^i - q^m \equiv 0(p)$  if and only if  $i - m \equiv 0(p-1)$ .

Now  $SU(n)$  is  $p$ -universal since it is an  $H$ -space, see Theorem 1.7 of [7]. Then by Theorem 5.3 of [3], there exists a  $p$ -sequence  $\{X_i, h_i\}$  of  $SU(n)$  such that  $X_i = SU(n)$  for any  $i$ , namely  $SU(n) \xrightarrow{h_1} SU(n) \xrightarrow{h_2} SU(n) \longrightarrow \dots$ . Then by inserting  $g_i$  into the above sequence, infinitely many times in any manner for each  $i \neq m$ , we have a sequence and the telescope  $Y_m$ . Note that this sequence is considered as a "sub" sequence of a 0-sequence of  $SU(n)$ . Then by taking the telescopes, we have the maps for  $2 \leq m \leq p$ , [3],

$$SU(n)_{(p)} \xrightarrow{a_m} Y_m \xrightarrow{b_m} SU(n)_{(0)}.$$

It is clear that  $Y_m$  is finite  $p$ -local. Then by Theorem 1.2, the pull-back  $X_m(n) = Y_m \times_{SU(n)_{(0)}} SU(n)_p$ ,  $P = \{\text{all primes } \neq p\}$ , is a 1-connected CW complex and there exists a map  $f_m : SU(n) \longrightarrow X_m(n)$  such that  $(f_m)_{(p)} \simeq a_m$ . Then by the property of  $g_i^*$  and by the definition of  $Y_m$ , we have  $H^*(Y_m : Z_p) \simeq \Lambda(y_m, y_{m+(p-1)}, \dots, y_{m+s(p-1)})$ , where  $\deg y_i = 2i-1$  and  $s = \lfloor \frac{n-m}{p-1} \rfloor$  and  $a_m^*(y_i) = h_i$ . Then by Theorem 1.2, we can see evidently that  $H^*(X_m : Z_p)$  and  $f_m^*$  have the required properties. Q. E. D.

$$\text{Now put } f = \prod_{m=2}^p f_m : SU(n) \longrightarrow \prod_{m=2}^p X_m(n) \text{ with the}$$

convention  $X_m(n) = *$  and  $f_m$  is the trivial map if  $m > n$ . It is clear by the above proposition that  $f$  is a  $p$ -equivalence. Since  $SU(n)$  is  $p$ -universal for any  $p$ , there exists a converse

$p$ -equivalence

$$g : \prod_{m=2}^p X_m(n) \longrightarrow SU(n).$$

Consider the composition  $g'_m : X_m(n) \xrightarrow{i_m} \prod_{i=2}^p X_i(n) \xrightarrow{g} SU(n) \xrightarrow{\pi} SU(n)/SU(m-1)$ , where  $i_m$  and  $\pi$  are obvious maps.

Clearly  $g'_m : H^*(SU(n)/SU(m-1); \mathbb{Z}_p) \longrightarrow H^*(X_m(n); \mathbb{Z}_p)$  is epimorphic. Thus  $X_m(n)$  is the space  $B_{m-1}^k(p)$  defined in [6], where  $k = \lfloor \frac{n-m}{p-1} \rfloor + 1$ . Thus we have obtained:

Theorem 3.4. Let  $p$  be a prime and let  $m$ ,  $1 \leq m < p$ , be an integer. Then for any integer  $k$ , there exists a space  $B_m^k(p)$  in [6] and there exists a  $p$ -equivalence

$$g : \prod_{m=1}^{p-1} B_m^{k_{n,m}}(p) \longrightarrow SU(n)$$

where  $k_{n,m} = \lfloor \frac{n-m-1}{p-1} \rfloor$ .

Corollary 3.5 Let  $p$  and  $m$  be as above. Then for any integer  $k$ ,  $B_m^k(p)$  is an H-space mod  $p$ .

#### §4. Mod $p$ decomposition of the other classical groups.

In this section,  $p$  denotes always an odd prime. Let  $F \xrightarrow{i} E \xrightarrow{p} B$  be a fibration. A map  $s: B \rightarrow E$  is called a cross-section mod  $p$  if  $p \circ s$  is a  $p$ -equivalence. If  $E$  is an H-space mod  $p$  and if  $p: E \rightarrow B$  admits a cross-section mod  $p$ . Let  $\mu$  be the multiplication of  $E$ . Then if  $F$ ,  $E$  and  $B$  are 1-connected finite CW complexes, then  $F \times B \xrightarrow{i \times s} E \times E \xrightarrow{\mu} E$  gives a  $p$ -equivalence by the Serre's class theory.

Consider the canonical bundles associated classical groups,

$$\mathrm{Sp}(n) \longrightarrow \mathrm{SU}(2n) \longrightarrow \mathrm{SU}(2n)/\mathrm{Sp}(n)$$

$$\mathrm{Spin}(2n+1) \longrightarrow \mathrm{SU}(2n+1) \longrightarrow \mathrm{SU}(2n+1)/\mathrm{Spin}(2n+1)$$

$$\mathrm{Spin}(2n-1) \longrightarrow \mathrm{Spin}(2n) \longrightarrow S^{2n-1}.$$

B. Harris [2] has shown that such bundles have cross-section mod  $p$  for odd  $p$ . Hence we have  $p$ -equivalences

$$\mathrm{Sp}(n) \times (\mathrm{SU}(2n)/\mathrm{Sp}(n)) \underset{p}{\sim} \mathrm{SU}(2n),$$

$$\mathrm{Spin}(2n+1) \times (\mathrm{SU}(2n+1)/\mathrm{Spin}(2n+1)) \underset{p}{\sim} \mathrm{SU}(2n+1) \quad \text{and}$$

$$\mathrm{Spin}(2n-1) \times S^{2n-1} \underset{p}{\sim} \mathrm{Spin}(2n).$$

It is also known [4] that  $\mathrm{Sp}(n) \underset{p}{\sim} \mathrm{Spin}(2n+1)$ .

Theorem 4.1. Let  $p$  be an odd prime. Let  $k_{a,b} = \lfloor \frac{2(a-b)}{p-1} \rfloor + 1$ . Then there exist the following  $p$ -equivalences

$$\mathrm{Sp}(n) \underset{p}{\sim} \mathrm{Spin}(2n+1) \underset{p}{\sim} \prod_{m=1}^{\frac{p-1}{2}} B_{2m}^{k_{n,m}}(p),$$

$$\mathrm{Spin}(2n) \underset{p}{\sim} S^{2n-1} \times \prod_{m=1}^{\frac{p-1}{2}} B_{2m}^{k_{n-1,m}}(p),$$

$$\mathrm{SU}(2n)/\mathrm{Sp}(n) \underset{p}{\sim} \prod_{m=1}^{\frac{p-1}{2}} B_{2m}^{k_{n-1,m}}(p),$$

$$\mathrm{SU}(2n+1)/\mathrm{Sp}(2n+1) \underset{p}{\sim} \prod_{m=1}^{\frac{p-1}{2}} B_{2m}^{k_{n,m}}(p).$$

Proof is straightforward from Theorem 3.4 and will be left to the reader.

## §5. Proof of the Main Theorem.

Theorem 5.1.  $SU(n)$  has no mod  $p$  decomposition into  $p$  spaces. Let  $p$  be odd, then  $Sp(n)$  and  $Spin(2n+1)$  have no mod  $p$  decomposition into  $\frac{p+1}{2}$  spaces.  $Sp(n)$  has no mod 2 decomposition into 2 spaces.

Proof. Assume that  $SU(n)$  is mod  $p$  decomposable into  $p$  spaces, i.e.,  $\prod_{i=1}^p X_i \xrightarrow{f} SU(n)$ . It is easy to see that  $H^*(X_i:Z_p)$  is an exterior algebra and hence there exists a number  $t$  such that the degree of the lowest generator of  $H^*(X_t:Z_p)$  is greater than  $2p+1$ . Let  $x$  be such a generator and let  $k = \deg x$ . Then clearly the mod  $p$  Hurewicz homomorphism  $h: \pi_k(\prod X_i) \otimes Z_p \rightarrow H_k(\prod X_i:Z_p)$  is non trivial. Hence so is  $h: \pi_k(SU(n)) \otimes Z_p \rightarrow H_k(SU(n):Z_p)$ . But since  $k \geq 2p+1$ , this is a contradiction. For  $Sp(n)$  and  $Spin(2n+1)$ , the proof is quite similar. q.e.d.

Proof of Main Theorem. By Theorem 5.1, apparently the mod  $p$  decompositions of the classical Lie groups given in Theorem 3.4 and 4.1 are irreducible. Also it is easy to see that the number of spaces in the decomposition is  $r(G)$  by the definition of  $B_m^k(p)$ . For exceptional Lie groups, Main Theorem follows from Theorem 4.2 [6] and Theorem 6.1 [7]. This completes the proof.

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