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<td>IYANAGA, KENICHI</td>
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CERTAIN DOUBLE COSET SPACES OF ALGEBRAIC GROUPS AND
RATIONAL BOUNDARY COMPONENTS OF SYMMETRIC BOUNDED DOMAINS

Kenichi IYANAGA

I

In part I we consider the problem of determining the order of double cosets $\gamma G/P$, where $G$ is a certain $k$-algebraic group, $P$ is its $k$-parabolic subgroup and $\gamma$ is its arithmetic subgroup. A detailed discussion on the subject is found in [5].

Let $k$ be an algebraic number field of finite degree, and $K$ be either a quadratic extension of $k$ or $k$ itself, and $\sigma$ the involution of $K$ stabilizing each element of $k$. Let $V$ be a finite dimensional vector space over $K$ supplied with a non-degenerate $k$-bilinear form $F: V \times V \to K$ such that $F(ax, by) = a^eF(x, y)b$ for $a, b \in K$, $x, y \in V$ and that $F(x, y)^\sigma = eF(y, x)$, $e = \pm 1$.

We set $G = \{ g \in GL(V); \quad F(g(x), g(y)) = F(x, y), \quad x, y \in V \}$ and $G^1 = G \cap SL(V)$. Then the groups $G$ and $G^1$ are $k$-algebraic groups.

Suppose that there exists a proper non-zero subspace $W$ of $V$ such that $F(w, w') = 0$ for all $w, w' \in W$ (i.e. $W$ is a totally isotropic subspace of $V$).

We set $G_W = \{ g \in G; \quad g(W) = W \}$. This is a maximal $k$-parabolic subgroup of $G$.

Let $O_K$ be the ring of integers in $K$ and let $L$ be an $O_K$-lattice in $V$.

We set $G_L = \{ g \in G; \quad g(L) = L \}$. This is an arithmetic subgroup of $G$.

Similarly, we get a maximal $k$-parabolic subgroup $G_W^1$ and an arithmetic subgroup $G_L^1$ of $G^1$.

Now, given any subgroup $H$ of $G$ and $O_K$-submodules $X, Y$ of $V$, we write $X \sim_H Y$ if and only if there exists an element $h$ of $H$ such that $h(X) = Y$. /
We denote the set of $\mathcal{O}_K$-submodules $Y$ such that $X \sim Y$ by $(X)_H$. Then, the double coset space $G_L \backslash G/G_w$ is in a bijective correspondence with either one of the sets $(W)_G \sim (G_L)_G$, or $(L)_G \sim (G_L)_G$. Thus the problem of determining the order $|G_L \backslash G/G_w|$ is reduced to a certain classification problem of lattices. The determination of the order $|G_L^* \backslash G/G_w|$ is, to a great extent, reduced to the determination of $|G_L \backslash G/G_w|$.

Associated to the lattice $L$ we have a fractional ideal $\mathcal{M}_0(L)$ generated by $F(x,y)$ for $x,y \in L$. The lattice $L$ is called a $(\mathcal{M}_0(L))$-modular if $L = \{ x \in V; F(x,L) \subset \mathcal{M}_0(L) \}$.

Then we have the following decomposition theorem:

Let $L$ be an $\mathcal{J}$-modular lattice in $V$. Then there exist $\mathcal{O}_K$-ideals $\mathcal{O}_1, \ldots, \mathcal{O}_S$, a basis $\{ w_1, \ldots, w_S \}$ of $W$, and elements $w_1', \ldots, w_S'$ of $V$ such that

$$L = \sum_{i=1}^S (\mathcal{O}_i - \mathcal{O}_2 + \mathcal{O}_1 w_1') + L',$$

where $\mathcal{O}_1 > \mathcal{O}_2 > \cdots > \mathcal{O}_S$.

$w_i \in L, F(w_i, w_j') = \delta_{ij}, F(w_i', w_j') = m_{ij}$ for all $i,j$.

In the above, when $m_{ij} = 0$ for all $i$ (e.g. when $e = -1$), it is easy to determine the order $G_L \backslash G/G_w$. When $e = 1$, it becomes necessary to investigate the properties of the submodule $S(\mathcal{O}_k') = \{ N(x) + Tr(y); x, y \in \mathcal{O}_k \}$ of $\mathcal{O}_k'$, and submodule $S(L,W,\mathcal{O}) = \{ F(ax,ax) + Tr(b); a \in \mathcal{O}, x \in L', b \in K - \mathcal{J} \}$ of the module $S(L,\mathcal{O}) = \{ F(ax,ax) + Tr(b); a \in \mathcal{O}, x \in L, b \in K - \mathcal{J} \}$ for $\mathcal{O}_k$-ideals $\mathcal{O}$. It can be shown that if $K$ is a quadratic extension of $k$, then $S(\mathcal{O}_k') = \mathcal{O}_k'$, and that the order $|S(L,\mathcal{O})/S(L,W,\mathcal{O})|$ is generally independent of the choice of the ideal $\mathcal{O}$; we denote the order by $s(L,W)$.

The order $|G_L \backslash G/G_w|$ for an $\mathcal{J}$-modular lattice $L$ can be evaluated in terms of $h(K)$ (= the class number of $K$), $h(L')$ (= $G$-class number of $L'$), $s(L,W)$ etc. Specifically, we have the following estimation:

1) When $K = k$ and $e = -1$, then $|G_L \backslash G/G_w| = h(k)$.
2) If \( S(\mathcal{C}_K) = \mathcal{C}_K \), and \( s(K, W) = 1 \) for all \( M \) belonging to the same \( G \)-genus as \( L \), then \( |G_L \backslash G/G_W| \leq h(K) h(L) \), and if, moreover, all \( J \)-modular lattices in \( V \) are \( G \)-equivalent, then \( |G_L \backslash G/G_W| = h(K) h(L') \).

The latter case occurs, for example, in the following situations:

1) \( K = k \), \( \dim V \) is odd, \( S(\mathcal{C}_K) = \mathcal{C}_K \), \( h(K) = 1 \),
2) \( K \) is a quadratic extension of \( k \), \( \dim_K V \) is odd, and every ideal class in \( K \) is represented by a \( \sigma \)-invariant ideal.

EXAMPLES:

1) \( k = Q, K = Q(\sqrt{-1}) \), \( \dim_K V \) is odd and \( V \) has a basis \( \{ v_1, \ldots, v_n \} \) such that \( (F(v_i, v_j)) = \text{diag}(1_p, -1_q) \), and \( L = \sum K v_i \). In this case,

\[
|G_L \backslash G/G_W| = h(L') \leq |G_L \backslash G^{1}/G^{1}_W| \leq 2 h(L'),
\]

\( h(L') = 1 \) when \( W^+/W \) is indefinite ([4]), or the rank of \( L' < 5 \) (4)

\[
\begin{cases}
> 1 & \text{when the rank of } L' \geq 5, \\
= 2 & \text{when the rank of } L' = 5, \\
= 4 & \text{when the rank of } L' = 7.
\end{cases}
\]

2) \( k = Q, K = Q(\sqrt{-p}), p \equiv 3 \mod 4 \), \( \dim_K V \) is odd, and \( V \) has a basis \( \{ v_1, \ldots, v_n \} \) such that \( (F(v_i, v_j)) = \text{diag}(1_p, -1_q) \), and \( L = \sum \mathcal{C}_K v_i \). Then

\[
|G_L \backslash G/G_W| = |G_L^{1} \backslash G^{1}_W| = h(K).
\]

II

We assume that \( G^1 \) is simply connected (hence, \( G^1 \) is either \( SU(V, H) \) or \( Sp(V, A) \)). We assume further that the Lie group \( (\mathcal{R}_K/\mathcal{C}_K(G^1))_R \) admits a maximal compact subgroup \( \mathcal{H} \) such that \( L = (\mathcal{R}_K/\mathcal{C}_K(G^1))_R/\mathcal{H} \) has the structure of a symmetric bounded domain (hence, \( k \) is totally real, and \( K \) is either \( k \) itself or a totally imaginary quadratic extension of \( k \)).
In this case, the subspace $W$ corresponds to a rational boundary component $B(W)$ of $\overline{D}$, and conversely, for any rational boundary component of $\overline{D}$ there exists a totally isotropic subspace $W'$ of $V$ such that the boundary component may be written as $B(W')$ (cf. [1]); the dimension of such a subspace $W'$ is determined by the given boundary component which we shall call the type of the boundary component. Let $\mathcal{B}(W)$ be the set of rational boundary components of $\overline{D}$ having the same type as $B(W)$. $\mathcal{B}(W)$ is a $G^1$-orbit space. The double coset space $G_L^1 \backslash G^1 / G^1_W$ is in a bijective correspondence with the set of $G_L^1$-orbits among $\mathcal{B}(W)$.

III

We make a remark concerning our previous work in [2] and [3].

Let $D^* = D \cup \{\text{rational boundary components of } D\}$ supplied with Satake topology, and let $V^* = G_L^1 \backslash D^*$. Then $V^*$ has the structure of a projective variety.

Consider a functor sending the category of Hermitian vector spaces $(V,H)$ to the category of alternating vector spaces $(V',A)$, where $V' = \mathcal{O}_{K/k} V$ and $A$ is the "imaginary part" of $H$. This functor naturally induces a rational homomorphism sending $G^1 = SU(V,H)$ into $G^1 = Sp(V',A)$; lattices $L$ in $V$ naturally correspond to lattices $L'$ in $V'$.

When $L$ is modular and $\mathcal{O}_{K}^*(L)$ is an ideal in $k$, then the corresponding lattice $L'$ is maximal in $V'$. When, in general, $L$ is $\mathcal{O}$-modular, the elementary divisors of $L'$ may be explicitly described in terms of $\mathcal{O}$ if (2) is a prime ideal in $k$ (cf. [6]).

Let $D$, $D'$ be the symmetric bounded domains corresponding to $G^1$, $G'$. Assume that $(\mathcal{O}_{K/k}^*)^*(K) \subset K'$, then $\mathcal{O}$ induces a holomorphic imbedding of $D$ into $D'$ (cf. [7]); this $\mathcal{O}$ further induces a morphism of the variety.
$V^*$ into $V^*$ (We have $\mathfrak{p}(G_L^1) \subset G_L^1$.)

We may ask here, when automorphic forms on $D$ with respect to $G_L$ may be extendable to automorphic forms on $D'$ with respect to $G_L'$? The above I, II may be helpful to consider this problem.

In particular, the field of rational functions $C(V^*)$, which is identified with the field of automorphic functions on $D$ with respect to $G_L$, may be identified with a subfield of $C(M(V^*))$, and their relations may be described in terms of certain Galois cohomology group (cf. [2], [3]).

Especially, when $k = Q, K = Q(\sqrt{-p}), p \equiv 3 \mod 4, p > 3, \dim_K V$ is odd then $C(V^*) = C(\mathfrak{p}(V^*))$.

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