数理解析研究所講究録
138巻 1972年 99-110



# Markov processes in duality

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## §1. Introduction

Let $X$ be a locally compact seperable Hausdorff space and $m$ be a positive Radon measure on $X$. Consider standard Markov processes $M = (X_t, P_x)$ and $\hat{M} = (\hat{X}_t, \hat{P}_x)$ which are in duality with respect to $m$ in the sense that the equality

(1. 1) $$(f, T_t g) = (\hat{T}_t f, g) \qquad t > 0 ,$$

holds for any non-negative Borel functions $f$ and $g$ on $X$. Here $T_t$ (resp. $\hat{T}_t$) is the semi-group associated with $M$ (resp. $\hat{M}$) and $(f, g)$ is the integral $\int_X f(x)g(x)m(dx)$. The quantities relative to the dual process $M$ are denoted with $\wedge$ and designated by the prefix co- . Notice that the present duality is much weaker than that of Blumenthal-Getoor [ 1 ; VI] and we do not assume the absolute continuity of resolvents or transition probabilities at all.

A set $A \subset X$ is said to be <u>almost polar</u> if there is a Borel set $B$ such that $A \subset B$ and

(1. 2) $P_x(\sigma_B < +\infty) = 0$ for m-a.e. $x \in X$ ,

where $\sigma_B$ is the hitting time $\inf\{t>0;\ X_t \in B\}$. "Quasi-everywhere" or "q. e." will mean "except on an almost polar set".
Recently the notion of almost polarity was employed independently by S. Port and C. Stone [3] for additive processes with $m$ being the

Haar measure and by the auther [2] for general m-symmetric Markov processes whose associated Dirichlet spaces are regular. An almost polar set was called "essentially polar" in the former paper and "polar" in the latter. In both papers, almost polar sets were identified with the sets of $\lambda$-capacity zero, the $\lambda$-capacity being defined suitably according to the respective situations.

In §2, we will study almost polar sets together with q.e. finely continuous functions and present some fundamental properties that they possess. Assertions (i)$\sim$(x) of §2 are the generalizations of those established in [2 ; §3, 4], while (xi)$\sim$(xiv) are our versions of those in Blumenthal-Getoor [1 ; VI]. The second assertion states that each almost polar set is m-negligible. But the converse is not necessarily true. Proposition (viii) asserts that the resolvent of M is absolutely continuous with respect to m if and only if each almost polar set is semipolar. The final assertion states that the next two conditions (C) and (C') are equivalent.

(C) Each semipolar set is almost polar,

(C') A function is q.e. finely continuous if and only if it is q.e. cofinely continuous.

In particular, condition (C) is met when M is m-symmetric $(M = \hat{M})$:

## §2. Potential theory for $M$ and $\hat{M}$

Our assertions will be listed up.

(i) Let $E$ be universally measurable.

The next three conditions are equivalent.

($\alpha$) $$m(E) = 0$$

($\beta$) $p(t, x, E) = 0$ m-a.e. $x$ $X$, for each $t > 0$.

($\gamma$) $G_\alpha(x, E) = 0$ m-a.e. $x$ $X$, for every $\alpha > 0$.

(equivalently for some $\alpha > 0$).

Here $p(t, x, E)$ (resp. $G_\alpha(x, E)$) is the transition probability (resp. the resolvent kernel) of the process $M$.

<u>Proof.</u> We only show the implication $(\beta) \Rightarrow (\alpha)$ :

$$0 = \lim_{t \downarrow 0} \int_X T_t I_E(x) m(dx) = \lim_{t \downarrow 0} \int_X \hat{T}_t 1(x) I_E(x) m(dx) \geqq \int_X I_E(x) m(dx)$$

where $I_E$ denotes the indicator of $E$.

(ii) If $N$ is almost polar, then $m(N) = 0$.

<u>Proof.</u> There is a Borel almost polar set $E \supset N$. Then $E$ satisties $(\beta)$. Hence $m(E) = 0$.

(iii) Let $A$ be a Borel set and put

$$T_t^0 f(x) = E_x(f(X_t) \; ; \; t < \sigma_A)$$

$$H_A^\alpha f(x) = E_x(e^{-\alpha\sigma_A} f(X\sigma_A)) ,$$

Then we have

(2. 1) $$(f, T_t^0 g) = (\hat{T}_t^0 f, g) , \quad t > 0 ,$$

(2. 2) $$(f, H_A^\alpha G_\alpha g) = (\hat{H}_A^\alpha \hat{G}_\alpha f, g) , \quad \alpha > 0 ,$$

for any non-negative Borel functions $f$ and $g$. Here $G_\alpha$ is the resolvent of $T_t$ .

Proof. (2. 2) is equivalent to (2. 1) . (2. 1) was proved by Dynkin [ 4 ; Lemma 14. 1] for the Brownian motion by making use of a method of time reversion. The same method has been extended to m-symmetric Markov processes in [ 2 ; Theorem 3. 5] . The argument there is independent of the symmetry of $T_t$ and only the relation of duality (1. 1) is enough to get (2. 1) for open A . Next (2. 1) for any Borel A can be obtained just as in [ 1 ; pp 262] by noticing that any semi-polar set (resp. cosemi-polar set) is of potential zero (resp. copotential zero) and hence m-negligible according to (i) .

For a nearly Borel set $E \subset X$ , we will write

$$(2.\ 3)\qquad \begin{cases} e_E^\alpha(x) = H_E^\alpha 1(x) \\ e_E(x) = H_E^{0+}1(x) = P_x(\sigma_E < +\infty)\ . \end{cases}$$

Here are two consequences of the relation (2. 2) .

(iv) Almost polarity and almost copolarity are equivalent.

Proof. Let E be Borel and almost polar. Then, for any $g \in C_0(X)$ (the space of continuous functions with compact supports) , $0 = (1, H_E^\alpha G_\alpha g) = (\hat{H}_E^\alpha \hat{G}_\alpha 1, g)$ . Hence $0 = \hat{H}_E^\alpha \hat{G}_\alpha 1 \geqq \hat{H}_E^\alpha \hat{G}_\beta 1$ m-a.e. for all $\beta \geqq \alpha$ . But then $\hat{e}_E^\alpha(x) = \lim_{\beta\to+\infty} \beta \hat{H}_E^\alpha \hat{G}_\beta 1(x) = 0$ for m-a.e. $x \in X$ , proving that E is almost copolar.

(v) Let A be nearly Borel and finely open. Suppose that a Borel subset $E \subset A$ has the property that

$$\hat{e}_E(x) = 0 \quad \text{m-a.e. on } A .$$

Then $E$ is almost polar.

Proof. Take any compactum $K \subset E$. Since $\hat{e}_K = 0$ m-a.e. on $A$, we have $0 = (\hat{H}_K \hat{G}_\alpha f, I_A) = (f, H_K^\alpha G_\alpha I_A)$ for any $f \in C_0(X)$. Therefore $H_K^\alpha G_\beta I_A = 0$ m-a.e. for all $\beta \geq \alpha$. $H_K^\alpha$ is supported by $K$ but $\lim_{\beta \to +\infty} \beta G_\beta I_A(y) = 1$ for $y \in K (\subset A)$ because $A$ is finely open. We get $e_K^\alpha(x) = 0$ m-a.e. Now it suffices to find, for strictly positive $f \in L^1(X ; m)$, an increasing sequence of compact sets $K_n \subset E$ such that

$$(f, e_E^\alpha) = \lim_{n \to +\infty} (f, e_{K_n}^\alpha) .$$

Definition. A function $f$ defined q.e. on $X$ is called q.e. finely continuous if the following conditions are satisfied : there exists a nearly Borel almost polar set $B$ such that $X - B$ is finely open and $f$ is nearly Borel measurable and finely continuous on $X - B$.

(vi) If $f$ is q.e. finely continuous and if $f \geq 0$ m-a.e. on $X$, then $f \geq 0$ q.e. on $X$.

Proof. Let $B$ be the set appeared in the above definition of q.e. fine continuity of $f$. Then the set $A = (X - B) \cap \{x ; f(x) < 0\}$ is nearly Borel and finely open. By the assumption, $m(A) = 0$. Since $A$ is nearly Borel, there are, for a strictly positive function $h \in L^1(X ; m)$, some Borel sets $A'$ and $A''$ such that $A' \subset A \subset A''$ and $P_{h \cdot m}(X_t \in A'' - A' \text{ for some } t \geq 0) = 0$, which means that $A'' - A'$ is almost polar. Since $m(A) = 0$, $\hat{e}_{A'} = 0$ m-a.e. on $A$

trivially and $A'$ is almost polar by (v). Hence $A'' = A' + (A'' - A')$ is almost polar and so is $A$.

The following characterization of almost polar sets already appeared in [ 2 ; Theorem 3. 12]. We say that a set $E$ is M-invariant if $P_x(X_t \in E$ for every $t \geqq 0) = 1$ for every $x \in E$.

(vii) A set $N$ is almost polar if and only if there exists a Borel set $B \supset N$ such that $m(B) = 0$ and $X - B$ is M-invariant.

Proof. Let $N$ be almost polar then there is a Borel set $B_0 \supset N$ such that $e_{B_0}(x) = 0$ m-a.e. Since $e_{B_0}$ is excessive, it is nearly Borel and finely continuous. Hence, by the previous assertion, $e_{B_0}(x) = 0$ q.e. , that is, except on some Borel almost polar set $B_1$. Apply the same argument to the function $e_{B_0 \cup B_1}$. In this way, we get a sequence $B_0, B_1, \cdots, B_k, \cdots$ of Borel almost polar sets. It suffices to put $B = \bigcup_{k=0}^{\infty} B_k$.

Now we will give some criteria for the absolute continuity of the resolvent in terms of the relationship among almost polarity, polarity and semipolarity.

(viii) The following four conditions are mutually equivalent.

(α) A set is almost polar if and only if it is polar.

(β) Any almost polar set is semipolar.

(γ) $m$ is a reference measure for $M$ : a set is of potential zero if and only if it is m-negligible.

(δ) $G_\alpha(x, \cdot)$ is absolutely continuous with respect to $m$ for each

$\alpha > 0$ and $x \in X$ .

Proof. $(\gamma)$ and $(\delta)$ are equivalent in view of the first assertion (i) . $(\alpha)$ implies $(\beta)$ . Suppose that the condition $(\beta)$ is satisfied. Let $E$ be an m-negligible Borel set. Then $G_\alpha(x, E) = 0$ m-a.e. $x \in X$ , by virtue of (i) . But $G_\alpha(\ , E)$ is $\alpha$-excessive and finely continuous. Hence $G_\alpha(x, E) = 0$ q.e. by (vi) and moreover except on a semipolar set by the present assumption. Since any semipolar set is of potential zero, we have $G_\alpha(x, E) = \lim_{\beta\to\infty} \beta G_{\beta+\alpha} G\alpha I_E(x) = 0$ , $x \in X$ , arriving at $(\delta)$ . Evidently $(\delta)$ implies $(\alpha)$ . The proof is finished.

Remark 1°. Assertion (viii) is a generalization of [ 2 ; Theorem 3. 13] . Combining (vii) and (viii) , we get the following criterion : $G_\alpha(x, \cdot\,)$ is not absolutely continuous with respect to $m$ for some $\alpha > 0$ and $x \in X$ if and only if there exists an m-negligible Borel set $E$ such that $X - E$ is M-invariant but $E$ is not thin.

2°. In the case that $M = \hat{M}$ , the above conditions in (viii) are also equivalent to the following one $(\varepsilon)$ [5] .

$(\varepsilon)$ The transition probability $p(t, x, \cdot\,)$ is absolutely continuous with respect to $m$ for each $t > 0$ and $x \in X$ .

The next proposition says that we can reduce the nearly Borel measurability of q.e. finely continuous functions to the Borel measurability.

(ix) A function $f$ is q.e. finely continuous if and only if there exists a Borel almost polar set $B$ such that $X - B$ is M-invariant and

$f$ is Borel measurable and finely continuous on $X - B$ .

<u>Proof.</u> Let $f$ be q.e. finely continuous. Then by the definition and (vii) , there is a Borel almost polar set $B_0$ such that $X - B_0$ is M-invariant and $f$ is nearly Borel and finely continuous on $X - B_0$ . For a fixed natural number $M$ we define the trancated function $f^M$ of $f$ by $f^M = (f \wedge M) \vee (-M)$ on $X - B_0$ . We extend $f^M$ by setting its value to be zero on $B_0$ . by the fine continuity of $f^M$ on $X - B_0$ , we have

$$\lim_{n\to\infty} nG_n f^M(x) = f^M(x) , \quad x \in X - B_0 .$$

On the other hand, there are Borel functions $f_1$ and $f_2$ such that $f_1 \leqq f^M \leqq f_2$ on $X$ and $\int_X (f_2(x) - f_1(x))m(dx) = 0$ . But, for any

$$h \in C_0(X) , \quad (h, G_n(f_2 - f_1)) = (\hat{G}_n h, f_2 - f_1) = 0$$

yielding that $G_n f_1 = G_n f_2$ m-a.e. and hence q.e. owing to (vi) . Therefore there is a Borel almost polar set $B_n$ such that $G_n f^M(x) = G_n f_1(x)$ for every $x \in X - B_n$ . Put $B_M = B_0 \cup \bigcup_{n=1}^{\infty} B_n$ , then

$$f^M(x) = \lim_{n\to\infty} nG_n f^M(x) = \lim_{n\to\infty} nG_n f_1(x) , \quad x \in X - B_M .$$

Consequently $f^M$ is Borel measurable on $X - B_M$ . According to (vii) , there is a Borel almost polar set $B \supset \bigcup_M B_M$ such that $X - B$ is M-invarinat. Then $f(x) = \lim_{M\to\infty} f^M(x)$ is Borel measurable on $X - B$ , completing the proof.

(x) Let $\{f_n\}$ be a decreasing sequence of α-excessive functions with limit $f$ and suppose that $f = 0$ m-a.e. Then $f = 0$ q.e.

This proposition corresponds to Blumenthal-Getoor [ 1 ; VI(3. 2)] .

The proof is quite the same. We do not know whether in our case every semipolar set is cosemipolar. But by making use of (x) and following the same line as in Blumenthal-Getoor [ 1 ; VI (1. 19)] , we get

(xi) Each semipolar set is the sum of a cosemipolar set and an almost polar set.

(xii) For any Borel sets $A$ and $B$ , we have

$$(g, H_A^\alpha H_B^\alpha G_\alpha h) = (\hat{H}_B^\alpha \hat{H}_A^\alpha \hat{G}_\alpha g, h)$$

for any non-negative Borel functions $g$ and $h$ .

Proof. This is a consequence of (2. 2) . Take non-negative $g$ and $h$ in $C_0(X)$ . Since $H_B^\alpha G_\alpha h$ (resp. $\hat{H}_A^\alpha \hat{G}_\alpha g$) is an $\alpha$-excessive (resp. $\alpha$-coexcessive) function, we have

$$\begin{aligned}(g, H_A^\alpha H_B^\alpha G_\alpha h) &= \lim_{\beta\to+\infty} \beta(g, H_A^\alpha G_\beta H_B^\alpha G_\alpha h)\\ &= \lim_{\beta\to+\infty} \beta(g, H_A^\alpha G_\alpha (I - (\beta - \alpha)G_\beta) H_B^\alpha G_\alpha h)\\ &= \lim_{\beta\to+\infty} \beta(\hat{H}_B^\alpha \hat{G}_\alpha (I - (\beta - \alpha)\hat{G}_\beta) \hat{H}_A^\alpha \hat{G}_\alpha g, h)\\ &= \lim_{\beta\to+\infty} \beta(\hat{H}_B^\alpha \hat{G}_\beta \hat{H}_A^\alpha \hat{G}_\alpha g, h) = (\hat{H}_B^\alpha \hat{H}_A^\alpha \hat{G}_\alpha g, h) .\end{aligned}$$

(xiii) Let $A$ be a Borel set. Denote by $A^r$ (resp. ${}^rA$) the totality of regular (resp. coregular) points of $A$ . Then ${}^rA - A^r$ is written as

$${}^rA - A^r = N_1 + N_2$$

with a Borel semi-polar set $N_1$ and a nearly Borel almost polar set $N_2$ . The same conclusion holds for $A^r - {}^rA$ .

Proof. Since ${}^rA$ is co-nearly Borel, there are Borel sets $\hat{A}'$ and $\hat{A}''$ such that $\hat{A}' \subset {}^rA \subset \hat{A}''$ and $\hat{A}'' - \hat{A}'$ is almost copolar. There are also Borel sets $A'$ and $A''$ such that $A' \subset A^r \subset A''$ and $A'' - A'$ is almost polar. Put $F = \hat{A}' - A''$, then $F$ is a Borel set, $F \subset {}^rA - A^r$ and the set $({}^rA - A^r) - F$ is almost polar in view of (iv). By the preceding identity, we have

$$(g,\ H^\alpha_A H^\alpha_F G_\alpha h) = (\hat{H}^\alpha_F \hat{H}^\alpha_A \hat{G}_\alpha g,\ h)\ .$$

Since $F \cup {}^rF \subset {}^rA$, we see that $\hat{H}^\alpha_F \hat{H}^\alpha_A \hat{G}_\alpha g = \hat{H}^\alpha_F \hat{G}_\alpha g$. Hence, by (2. 2), $(g,\ H^\alpha_A H^\alpha_F G_\alpha h) = (g,\ H^\alpha_F G_\alpha h)$. Now choose $h_n$ such that $G_\alpha h_n \uparrow 1$. We have $(g,\ H^\alpha_A e^\alpha_F) = (g,\ e^\alpha_F)$ for every $g \in C_0(X)$. Using (vi) we get $e^\alpha_F = H^\alpha_A e^\alpha_F$ q.e. If $x \in F$, then $x \notin A^r$ and $H^\alpha_A e^\alpha_F(x) \leqq H^\alpha_A 1(x) < 1$. Thus, there is an almost polar Borel set $N'$ such that $e^\alpha_F(x) < 1$ for $x \in N_1 = F - N'$. $N_1$ is then a Borel semipolar set because $e^\alpha_{N_1}(x) \leqq e^\alpha_F(x) < 1$ for $x \in N_1$. Now ${}^rA - A^r = N_1 + N_2$ with $N_2 = [({}^rA - A^r) - F] + F \cap N'$ is the desired expression.

(xiv) The following two conditions are equivalent.

(C) Each semipolar set is almost polar.

(C') A function is q.e. finely continuous if and only if it is q.e. cofinely continuous.

Proof. Assume the condition (C). Consider a q.e. finely continuous function $f$. By (ix), there is an almost polar Borel set $B$ such that $X_0 = X - B$ is finely open and $f$ is Borel measurable and finely continuous on $X_0$. For a real number $a$, put $E_a = \{x \in X_0 ; f(x) < a\}$. Since $X - E_a$ is finely closed,

$$N_a = {}^r(X - E_a) - (X - E_a) \subset {}^r(X - E_a) - (X - E_a)^r$$

which is almost polar on account of (xiii) and (C). Notice that $E_a - N_a = E_a - {}^r(X - E_a)$ is cofinely open. Choose an almost polar Borel set $N'_a \supset N_a$ and set $\hat{B}_0 = B \cup (\bigcup_{a:\ \text{rational}} N'_a)$. By virtue of (vii), there exists an almost polar Borel set $\hat{B} \supset \hat{B}_0$ such that $X - \hat{B}$ is M-invariant. Now, for any rational $a$, the set $\{x \in X - \hat{B} ; f(x) < a\} = E_a - \hat{B}$ is cofinely open because $E_a - \hat{B} = (E_a - N_a) \cap (X - \hat{B})$ and both $E_a - N_a$ and $X - \hat{B}$ are cofinely open. This shows that $f$ is q.e. cofinely continuous.

Coming to the converse, assume (C'). Consider a compact thin set $K$ and put $B_n = \{x \in X ; e_K^\alpha(x) \geq 1 - \frac{1}{n}\}$. Then $\bigcap B_n$ is empty. Since $e_K^\alpha$ is $\alpha$-excessive, it is q.e. cofinely continuous in view of (C'). Hence there is a Borel polar set $N$ such that $B_n - N$ is Borel and cofinely closed for every $n$ (owing to (ix)). On the other hand, we have (see Blumenthal-Getoor [1 ; VI (4. 10)])

$$H_{B_n}^\alpha e_K^\alpha = e_K^\alpha \quad \text{on } X .$$

Let $g$ be a strictly positive continuous and m-integrable function. Then, just as in the proof of (xii), $(\hat{H}_{B_n}^\alpha g, e_K^\alpha) = (g, e_K^\alpha)$. Since $N$ is almost polar, the left hand side is equal to $(\hat{H}_{B_n - N}^\alpha g, e_K^\alpha)$, which decreases to zero as $n \to +\infty$, because the measures $\hat{H}_{B_n - N}^\alpha(x, \cdot)$, $x \in X$, are supported by a cofinely closed set $B_n - N$ and $\bigcap(B_n - N)$ is empty. Therefore $e_K^\alpha = 0$ m-a.e., yielding that $K$ is almost polar.

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