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Lattice Green's Function for the Simple Cubic Lattice in Terms of a Mellin-Barnes Type Integral. II

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Abstract

The series representation of the lattice Green's function for the simple cubic lattice

$$I(\alpha) = \frac{1}{\pi^3} \iiint_0^{\pi} \frac{dx\ dy\ dz}{\alpha - i\varepsilon - \cos x - \cos y - \cos z}$$

around the singularity $\alpha = 1$ is obtained in fractional powers of α^2-1 (convergent for $|\alpha^2-1| < 1$), by the method of the analytic continuation using Mellin-Barnes type integral and also by use of the analytic continuation of ${}_3F_2(\ ,\ ,\ ;\ ,\ ;1)$ as a function of the parameter. It gives leading and full expansions near the singularity $\alpha = 1$.

1. INTRODUCTION

In the previous paper[1] lattice Green's function of the simple cubic lattice at the origin

$$I(a) = \frac{1}{\pi^3} \iiint_0^{\pi} \frac{dx\,dy\,dz}{a - i\varepsilon - \cos x - \cos y - \cos z} \qquad (1.\ 1)$$

which has the singularities at $a = 1$ and $a = 3$, was evaluated in series representation for $a \geqq 3$ in powers of $1/a^2$, for $0 \leqq a \leqq 1$ in powers of a^2, and for $1 \leqq a \leqq 3$ in powers of $(a^2 - 5)/4$ by the method of analytic continuation using Mellin-Barnes type integral. The exact values of I(0), I(1), I($\sqrt{5}$) were also given in terms of the product of the complete elliptic integrals. The method was successfully applied for the bcc lattice[2], the rectangular and the square lattices[3] and the tetragonal lattice[4]. In this paper the expansion of the lattice Green's function of the simple cubic lattice around the singularity $a = 1$, which were not given in the previous paper, is presented.

First $I(a)$ is expressed as a Mellin-Barnes type integral with the argument $a^2 - 1$. The integrand is a sum of two series expressed in the generalized hypergeometric function ${}_3F_2(\ ,\ ,\ ;\ ,\ ;1)$ which include the integration variable as a parameter. In order to obtain the expansion in powers of $a^2 - 1$, it is necessary to know the behavior of the integrand in the left-half plane of the integration variable.

The difficulty is that those series in the integrand are divergent in the left-half parameter plane while they are convergent in the right-half parameter-plane. We have succeeded in finding the behavior of the integrand in the left-half parameter-plane by constructing the analytic continuation of ${}_3F_2$ in the parameter plane. Then the series representation of $I(\alpha)$ around $\alpha^2 = 1$, which is convergent for $|\alpha^2-1| < 1$, is obtained by residue calculations in fractional powers of α^2-1.

2. SERIES REPRESENTATION AROUND $a^2 = 1$

For large absolute values of a ($a > 3$), the following integral expression using a hypergeometric function has been derived in the previous paper.[1]

$$I(a) = \frac{1}{\pi a}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} ds \frac{\Gamma(-s)[\Gamma(s+\frac{1}{2})]^2}{\Gamma(s+1)} \left(\frac{-4}{a^2}\right)^s$$

$$\times\ {}_2F_1(s+\tfrac{1}{2},\ s+1;\ 1;\ 1/a^2)\ , \qquad (2.\ 1)$$

$$|\arg(-4/a^2)| < \pi, \qquad (2.\ 1')$$

where δ is a small positive number and the path of the integration is taken as a straight line parallel to the imaginary axis. The restriction (2. 1') ensures the convergence of the integration, and -4 is to be taken $e^{-i\pi}4$ since we consider a in the lower half plane.[5]

Applying a formula

$${}_2F_1(\alpha,\ \beta;\ \gamma;\ z) = (1-z)^{-\alpha}\,{}_2F_1(\alpha,\ \gamma-\beta;\ \gamma;\ z/z-1)$$

to the hypergeometric function in the r.h.s. of the Eq. (2. 1) with $\alpha = s+\frac{1}{2}$, we obtain

$$I(a) = \frac{1}{\pi}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} ds \frac{\Gamma(-s)[\Gamma(s+\frac{1}{2})]^2(e^{-i\pi}4)^s}{\Gamma(s+1)}\left(\frac{1}{a^2-1}\right)^{s+1/2}$$

$$\times\ {}_2F_1(s+\tfrac{1}{2},\ -s;\ 1;\ 1/(1-a^2))\ . \qquad (2.\ 2)$$

Here we take the branch where $(a^2)^{1/2} = a$.

Using the representation of the hypergeometric function by Mellin-Barnes type integral, we have

$$I(a)=\frac{1}{\pi}\left(\frac{1}{2\pi i}\right)^2\int_{-\delta-i\infty}^{-\delta+i\infty}ds\int_{-\delta'-i\infty}^{-\delta'+i\infty}dt\,\frac{\Gamma(s+\frac{1}{2})\Gamma(-t)\Gamma(s+t+\frac{1}{2})\Gamma(-s+t)(e^{-i\pi/4})^s}{\Gamma(s+1)\Gamma(t+1)}$$

$$\times\left(\frac{1}{a^2-1}\right)^{s+t+1/2}, \qquad (2.\ 3)$$

where δ' is a small positive number chosen so as to make $Re(-s+t)>0$, i.e., $\delta'<\delta$.

Introducing a new variable $u=s+t$ and changing the order of the integration, we have

$$I(a)=\frac{1}{\pi}\left(\frac{1}{2\pi i}\right)^2\int_{-\delta''-i\infty}^{-\delta''+i\infty}du\ \Gamma(u+\tfrac{1}{2})(a^2-1)^{-u-1/2}$$

$$\times\int_{-\delta-i\infty}^{-\delta+i\infty}ds\ \frac{\Gamma(s+\frac{1}{2})\Gamma(s-u)\Gamma(u-2s)(e^{-i\pi/4})^s}{\Gamma(s+1)\Gamma(1+u-s)}, \qquad (2.\ 4)$$

where $\delta''=\delta+\delta'$. Note that $Re\ s-u=-\delta'<0$, and $Re\ u-2s=\delta'-\delta<0$.

Now s-integration is carried out by collecting the residues of the poles at $s=\frac{u}{2}+q$ and $\frac{u}{2}+\frac{1}{2}+q$ $(q=0,1,2,\ldots)$ in the right-half s-plane. Then we have

$$\frac{1}{2\pi i}\int ds\cdots=\frac{1}{2}\sum_{q=0}^{\infty}\frac{\Gamma(\frac{u}{2}+q+\frac{1}{2})\Gamma(q-\frac{u}{2})}{(2q)!\,\Gamma(\frac{u}{2}+q+1)\Gamma(1+\frac{u}{2}-q)}(e^{-i\pi/4})^{u/2+q}$$

$$-\frac{1}{2}\sum_{q=0}^{\infty}\frac{\Gamma(\frac{u}{2}+q+1)\Gamma(q-\frac{u}{2}+\frac{1}{2})}{(2q+1)!\,\Gamma(\frac{u}{2}+q+\frac{3}{2})\Gamma(\frac{1}{2}+\frac{u}{2}-q)}(e^{-i\pi/4})^{u/2+q+1/2}.$$

(2. 5)

It is shown that the summations with respect to q are divergent for $Re\ u \le -1$, while the integration path in the u-plane is to be closed to the left-half plane where $Re\ u < 0$ to obtain the series valid for $|\alpha^2| < 1$. Therefore it is necessary to transform the summations of Eq. (2. 5) and to get other expressions which are valid for $Re\ u \le 1$.

The r.h.s. of the Eq. (2. 5) is expressed in terms of a generalized hypergeometric function $_3F_2$ with the argument equal to unity, and leads

$$\frac{1}{2\pi i}\int ds\cdots = -\frac{1}{2\sqrt{\pi}}(4e^{-i\pi})^{u/2}\sin\frac{u}{2}\pi\,\frac{\Gamma(\frac{1}{2}+\frac{u}{2})[\Gamma(-\frac{u}{2})]^2}{\Gamma(\frac{1}{2})\Gamma(1+\frac{u}{2})}$$

$$\times\ {}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{u}{2},\ -\frac{u}{2},\ -\frac{u}{2};\ 1\\ \\ \frac{1}{2},\ 1+\frac{u}{2}\end{matrix}\right]$$

$$+\frac{i}{2\sqrt{\pi}}(4e^{-i\pi})^{u/2}\cos\frac{u}{2}\pi\,\frac{\Gamma(1+\frac{u}{2})[\Gamma(\frac{1}{2}-\frac{u}{2})]^2}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}+\frac{u}{2})}$$

$$\times\ {}_3F_2\left[\begin{matrix}1+\frac{u}{2},\ \frac{1}{2}-\frac{u}{2},\ \frac{1}{2}-\frac{u}{2};\ 1\\ \\ \frac{3}{2},\ \frac{3}{2}+\frac{u}{2}\end{matrix}\right] \qquad (2.\ 6)$$

The expansion of a generalized hypergeometric function in terms of hypergeometric functions of lower order[6] (Eq. (5. 1) in ref. 3) and the value of $_2F_1$ with the argument equal to unity lead to a formula

$$\frac{\Gamma(\alpha_3)}{\Gamma(\beta_1)\Gamma(\beta_2)}\ {}_3F_2\left[\begin{matrix}\alpha_1,\ \alpha_2,\ \alpha_3;\ 1\\ \beta_1,\ \beta_2\end{matrix}\right]$$

$$= \frac{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_2-\alpha_3)}$$

$$\times\ {}_3F_2\left[\begin{matrix}\beta_1-\alpha_3,\ \beta_2-\alpha_3,\ \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3;\ 1\\ \beta_1+\beta_2-\alpha_1-\alpha_3,\ \beta_1+\beta_2-\alpha_2-\alpha_3\end{matrix}\right]. \tag{2. 7}$$

Applying the formula[7] to the two ${}_3F_2$'s in (6) with $\alpha_3 = -\frac{u}{2}$ and $\alpha_3 = \frac{1}{2} - \frac{u}{2}$, respectively, we have

$$\frac{\Gamma(-\frac{u}{2})}{\Gamma(\frac{1}{2})\Gamma(1+\frac{u}{2})}\ {}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{u}{2},\ -\frac{u}{2},\ -\frac{u}{2};\ 1\\ \frac{1}{2},\ 1+\frac{u}{2}\end{matrix}\right]$$

$$= \frac{\Gamma(1+u)}{\Gamma(1+\frac{u}{2})\Gamma(\frac{3}{2}+\frac{3}{2}u)}\ {}_3F_2\left[\begin{matrix}\frac{1}{2}+\frac{u}{2},\ 1+u,\ 1+u;\ 1\\ 1+\frac{u}{2},\ \frac{3}{2}+\frac{3}{2}u\end{matrix}\right], \tag{2. 8}$$

and

$$\frac{\Gamma(\frac{1}{2}-\frac{u}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}+\frac{u}{2})}\ {}_3F_2\left[\begin{matrix}1+\frac{u}{2},\ \frac{1}{2}-\frac{u}{2},\ \frac{1}{2}-\frac{u}{2};\ 1\\ \frac{3}{2},\ \frac{3}{2}+\frac{u}{2}\end{matrix}\right]$$

$$= \frac{\Gamma(1+u)}{\Gamma(\frac{3}{2}+\frac{u}{2})\Gamma(2+\frac{3}{2}u)}\ {}_3F_2\left[\begin{matrix}1+\frac{u}{2},\ 1+u,\ 1+u;\ 1\\ \frac{3}{2}+\frac{u}{2},\ 2+\frac{3}{2}u\end{matrix}\right]. \tag{2. 9}$$

The two hypergeometric series in the ℓ.h.s. of Eq. (8) and (9) are convergent for $Re\ u > 1$ while those in the r.h.s. of Eqs. (8) and (9) are convergent for $Re\ u < 0$ and $Re\ u < 1$, respectively. The r.h.s. gives the analytic continuation of the ℓ.h.s. as a function of u.

Using the above transformations (8) and (9) and the series representation of ${}_3F_2$ (in the r.h.s.) and changing the order of the summation and the integration, we obtain

$$I(a) = -\frac{1}{2}(\pi)^{-3/2}(a^2-1)^{-1/2}\sum_{p=0}^{\infty}\frac{1}{p!}$$

$$\times\frac{1}{2\pi i}\int du\,\frac{\Gamma(\frac{1}{2}+u)\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}u+p)[\Gamma(1+u+p)]^2\sin\frac{1}{2}u\pi}{\Gamma(1+u)\Gamma(1+p+\frac{u}{2})\Gamma(\frac{3}{2}+p+\frac{3}{2}u)}\left(\frac{2e^{-i\pi/2}}{a^2-1}\right)^u$$

$$+\frac{i}{2}(\pi)^{-3/2}(a^2-1)^{-1/2}\sum_{p=0}^{\infty}\frac{1}{p!}$$

$$\times\frac{1}{2\pi i}\int du\,\frac{\Gamma(\frac{1}{2}+u)\Gamma(\frac{1}{2}-\frac{u}{2})\Gamma(1+p+\frac{1}{2}u)[\Gamma(1+p+u)]^2\cos\frac{1}{2}u\pi}{\Gamma(1+u)\Gamma(\frac{3}{2}+p+\frac{u}{2})\Gamma(2+p+\frac{3}{2}u)}\left(\frac{2e^{-i\pi/2}}{a^2-1}\right)^u . \tag{2. 10}$$

Closing the integration path to the left-half u-plane, the evaluation of the integrals is carried out by summing residues of poles in the left-half u-plane.

The poles of the integrand are located at $u = -\frac{1}{2} - q$ ($q = 0, 1, 2, \ldots$), and at $u = -1-p-q$ where $p+q+1$ is odd integer for the first integrand of Eq. (10) and is even integer for the second integrand with $q \geq 0$. The calculation is tedious but straightforward and we finally obtain

$$I(a) = I_{reg}(a) + I_{irreg}(a) \ , \qquad (2.\ 11a)$$

$$I_{reg}(a) = \frac{1}{4\sqrt{2\pi}} \sum_{q=0}^{\infty} \frac{i^q}{q!}$$

$$\times \left\{ (1+i) \frac{\Gamma(\frac{1}{4}+\frac{q}{2})\Gamma(\frac{1}{4}+\frac{3}{2}q)}{[\Gamma(\frac{3}{4}+\frac{q}{2})]^2} \ {}_3F_2 \begin{bmatrix} \frac{1}{4}-\frac{q}{2},\ \frac{1}{2}-q,\ \frac{1}{2}-q;\ 1 \\ \frac{3}{4}-\frac{q}{2},\ \frac{3}{4}-\frac{3}{2}q \end{bmatrix} \right.$$

$$\left. - (1-i) \frac{\Gamma(-\frac{1}{4}+\frac{q}{2})\Gamma(-\frac{1}{4}+\frac{3}{2}q)}{[\Gamma(\frac{1}{4}+\frac{q}{2})]^2} \ {}_3F_2 \begin{bmatrix} \frac{3}{4}-\frac{q}{2},\ \frac{1}{2}-q,\ \frac{1}{2}-q;\ 1 \\ \frac{5}{4}-\frac{q}{2},\ \frac{5}{4}-\frac{3}{2}q \end{bmatrix} \right\} \left(\frac{a^2-1}{4}\right)^q ,$$

(2. 11b)

$$I_{irreg}(a) = -i\frac{3}{2\pi} \sum_{r=0}^{\infty} \frac{[(\frac{1}{2})_r]^3}{r!(\frac{3}{4})_r(\frac{5}{4})_r} \ {}_3F_2 \begin{bmatrix} 1+2r,\ \frac{1}{2},\ -r;\ 1 \\ 1+r,\ 1+r \end{bmatrix} (a^2-1)^{2r+1/2}$$

$$+ i\frac{1}{4\pi} \sum_{r=0}^{\infty} \frac{(1)_r[(\frac{3}{2})_r]^3}{[(2)_r]^2(\frac{5}{4})_r(\frac{7}{4})_r} \ {}_3F_2 \begin{bmatrix} 2+2r,\ \frac{1}{2},\ -r;\ 1 \\ 2+r,\ 2+r \end{bmatrix} (a^2-1)^{2r+3/2} ,$$

(2. 11c)

where $I_{reg}(a)$ and $I_{irreg}(a)$ represent regular and irregular parts of $I(a)$ at $a = 1$, and the leading singularity is

$(a^2 - 1)^{1/2}$. For $a^2 < 1$, the irregular part does not contribute to the imaginary part but to the real part of $I(a)$. ${}_3F_2$'s in the irregular part are finite series and give rational numbers.

The generalized hypergeometric function ${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; 1)$ converges when $\sigma \equiv \Sigma\beta_i - \Sigma\alpha_i > 0$, and the convergence becomes faster as σ increases, i.e., the degree of the convergence is of the order of $\sum_{n=1}^{\infty} (1/n^{\sigma+1})$. From a point of view of the convergence of ${}_3F_2(1)$, it is more convenient to transform $I_{reg}(a)$ into another form, though Eq. (2. 11) is a desired expression as far as it goes. Using a transformation[8] of ${}_3F_2$, we have

$$I_{reg}(a) = \frac{1}{\sqrt{2}} \frac{1}{\pi^{3/2}} \sum_{q=0}^{\infty} \frac{\Gamma(q-\frac{1}{2})}{q!} \, {}_3F_2 \left[\begin{matrix} \frac{1}{2} - q, \ \frac{1}{2} - q, \ \frac{1}{2}; \ 1 \\ \frac{3}{4} - \frac{q}{2}, \ \frac{5}{4} - \frac{q}{2} \end{matrix} \right] \left(\frac{a^2-1}{8}\right)^q$$

$$+ \frac{1}{(2\pi)^{3/2}} \sum_{q=0}^{\infty} [(-1)^q + i] \frac{\Gamma(\frac{1}{2}+q)[\Gamma(\frac{1}{4}+\frac{q}{2})]^2}{q![\Gamma(\frac{3}{4}+\frac{q}{2})]^2}$$

$$\times \, {}_3F_2 \left[\begin{matrix} \frac{1}{4} - \frac{q}{2}, \ \frac{1}{4} - \frac{q}{2}, \ \frac{1}{4} + \frac{q}{2}; \ 1 \\ \frac{1}{2}, \ \frac{3}{4} + \frac{q}{2} \end{matrix} \right] \left(\frac{a^2-1}{2}\right)^q$$

$$+ \frac{1}{\sqrt{2}\pi^{3/2}} \sum_{q=0}^{\infty} [(-1)^q - i] \frac{\Gamma(\frac{1}{2}+q)[\Gamma(\frac{3}{4}+\frac{q}{2})]^2}{q!\Gamma(\frac{5}{4}+\frac{q}{2})\Gamma(\frac{1}{4}+\frac{q}{2})}$$

$$\times \, {}_3F_2 \left[\begin{matrix} \frac{3}{4} - \frac{q}{2}, \ \frac{3}{4} - \frac{q}{2}, \ \frac{3}{4} + \frac{q}{2}; \ 1 \\ \frac{3}{2}, \ \frac{5}{4} + \frac{q}{2} \end{matrix} \right] \left(\frac{a^2-1}{2}\right)^q . \qquad (2. 11d)$$

The convergence indices σ in ${}_3F_2$'s in (2. 11d) are all $\frac{1}{2}$+q, while those in (2. 11b) are $\frac{1}{4}+\frac{q}{2}$ and $\frac{3}{4}+\frac{q}{2}$, showing better convergence than the original ${}_3F_2$ in (2. 11b).

Now we investigate the radius of the convergence of Eq. (2. 11). Consider the double series $\Sigma\Sigma A_{qp}x^q y^p$ generalized from the first term of (2. 11d), where

$$A_{qp} = \frac{[\Gamma(\frac{1}{2} -q+p)]^2\Gamma(\frac{1}{2}+p)\Gamma(\frac{3}{4} - \frac{q}{2})\Gamma(\frac{5}{4} - \frac{q}{2})}{q!p!8^q[\Gamma(\frac{1}{2} -q)]\Gamma(\frac{3}{4} - \frac{q}{2}+p)\Gamma(\frac{5}{4} - \frac{q}{2}+p)} .$$

Put p = λq, then from

$$\frac{1}{r} \equiv \lim_{q\to\infty} \left|\frac{A_{q+1,\ p}}{A_{q,\ p}}\right| = \left|\frac{1-2\lambda}{8(1-\lambda)^2}\right|$$

$$\frac{1}{s} \equiv \lim_{q\to\infty} \left|\frac{A_{q,\ p+1}}{A_{q,\ p}}\right| = \left|\frac{4(1-\lambda)^2}{(1-2\lambda)^2}\right|$$

we have

$$\frac{1}{r} = \left|\frac{\pm\sqrt{s} - s}{2}\right| \qquad (2.\ 12)$$

by elliminating λ. The double series $\Sigma A_{qp}x^q y^p$ converges absolutely in the region $|x| < r$, and $|y| < s$, where r and s are determined by Eq. (2. 12). For s = 1, we have r = 1. That is, the first term in Eq. (2. 11d) converges for $|a^2-1| < 1$, i.e., $0 < a < \sqrt{2}$ for real a. The radii of convergence of other terms in Eqs. (2. 11d) and (2. 11c) are also shown to be $|a^2-1| < 1$.

The expression (2. 11) includes only a^2, while the original form (1. 1) depends on a such that $I(a-i\varepsilon) = -I(-a+i\varepsilon)$. This suggests that the expression (2. 1) has a branch point at $a^2 = 0$. That is the reason why Eq. (2. 11) is convergent for $|a^2-1| < 1$.

For $a^2 = 1$, only the terms of q = 0 in Eq. (2. 11b) do not vanish, and ${}_3F_2(1)$ for q = 0 can be expressed in gamma functions[9] and the exact value of I(1) announced in the previous paper (Eq. (33) in ref. 1) is derived. The leading term is given by

$$I(a) = \frac{\pi}{2}(1+\sqrt{2}i)[\Gamma(\tfrac{5}{8})\Gamma(\tfrac{7}{8})]^{-2} - \frac{3i}{2\pi}(a^2-1)^{1/2} + 0(a^2-1). \tag{2. 13}$$

The third term gives a real part for $a^2 < 1$.

Equations (2. 11a), (2. 11c) and (2. 11d) are the series representation of $I(a)$ around $a = 1$, convergent for $|a^2-1| < 1$.

3. CONCLUSION

The lattice Green's function of the simple cubic lattice is expanded at the singularity $\alpha = 1$ by the method of the analytic continuation in terms of Mellin-Barnes type integral. In the process of calculation it is shown that the analytic continuation of a generalized hypergeometric function ${}_3F_2(\ ,\ ,\ ;\ ,\ ;1)$ in a complex-parameter plane allows us to obtain the series representation of $I(\alpha)$ in fractional powers of α^2-1. The result is given in Eq. (2. 11) and the series is convergent for $|\alpha^2-1| < 1$. It gives insights of the nature of the singularity and simple and rapid subroutines for numerical calculations near the singularity.

The numerical calculation of Eq. (2. 11) reproduces the values in the table by Morita and Horiguchi[10]. The values of first several terms of ${}_3F_2$ used are listed in Appendix.

APPENDIX VALUES OF $_3F_2$

The values of $_3F_2(\; , \; , \; ; \; , \; ; 1)$ in Eqs. (2. 11c) and (2. 11d) are calculated by a subroutine based on the definition of $_3F_2$. Those in $I_{irreg}(a)$ are finite series and give rational numbers. Those in Eq. (2. 11d) are infinite series with $\sigma = \frac{1}{2} + q$ and the convergence becomes faster as q increases. Here we list the values of $_3F_2$'s in Eq. (2. 11d) for the first several terms of q. The values of them for large q can be calculated rapidly.

$$F_a(q) \equiv {}_3F_2 \begin{bmatrix} \frac{1}{2} - q, \frac{1}{2} - q, \frac{1}{2}; 1 \\ \frac{3}{4} - \frac{q}{2}, \frac{5}{4} - \frac{q}{2} \end{bmatrix}$$

$$F_b(q) \equiv {}_3F_2 \begin{bmatrix} \frac{1}{4} - \frac{q}{2}, \frac{1}{4} - \frac{q}{2}, \frac{1}{4} + \frac{q}{2}; 1 \\ \frac{1}{2}, \frac{3}{4} + \frac{q}{2} \end{bmatrix}$$

$$F_c(q) \equiv {}_3F_2 \begin{bmatrix} \frac{3}{4} - \frac{q}{2}, \frac{3}{4} - \frac{q}{2}, \frac{3}{4} + \frac{q}{2}; 1 \\ \frac{3}{2}, \frac{5}{4} + \frac{q}{2} \end{bmatrix}.$$

q	$F_a(q)$	$F_b(q)$	$F_c(q)$
0	$2^{-3/2}\pi^2[\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})]^{-2}$	$\pi[\Gamma(\frac{3}{4})]^2[\Gamma(\frac{5}{8})]^{-4}$	$\pi[\Gamma(\frac{5}{4})]^2[\Gamma(\frac{7}{8})]^{-4}$

q	$F_a(q)$	$F_b(q)$	$F_c(q)$
0	$2^{-3/2}\pi^2[\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})]^{-2}$	$\pi[\Gamma(\frac{3}{4})]^2[\Gamma(\frac{5}{8})]^{-4}$	$\pi[\Gamma(\frac{5}{4})]^2[\Gamma(\frac{7}{8})]^{-4}$
	.1428125286 E+01	.1114018565 E+01	.1830796988 E+01
1	.1764390572 E+01	.1095404897 E+01	.1046372292 E+01
2	-.2086411047 E+02	.1821375868 E+01	.1036961125 E+01
3	.1825835244 E+03	.3477117659 E+01	.1310778150 E+01
4	-.1518523862 E+04	.6832596364 E+01	.1891707243 E+01
5	.1241300083 E+05	.1353545539 E+02	.2948617543 E+01
6	-.1006733423 E+06	.2690471748 E+02	.4824681185 E+01
7	.8130444869 E+06	.5357641342 E+02	.8159238967 E+01
8	-.6549578345 E+07	.1068076513 E+03	.1413060823 E+02
9	.5267460718 E+08	.2130836520 E+03	.2491485647 E+02
10	-.4231589025 E+09	.4253236238 E+03	.4454815485 E+02
11	.3396715097 E+10	.8492762411 E+03	.8055058548 E+02
12	-.2724952366 E+11	.1696281019 E+04	.1469922940 E+03
13	.2185065481 E+12	.3388737728 E+04	.2702993008 E+03

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