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A MONOTONE MAP ON 2-MANIFOLDS, WHOSE IMAGE IS HOMEOMORPHIC TO ITS DOMAIN SPACE, A 2-MANIFOLD.

BY

YUTAKA DOHI

§0. The main results in this note are Theorem A (see §2) and Theorem B (see §3) which are extensions of Whyburn's theorem [1]; Any monotone mapping of a plane onto a plane is compact. In §1, extensions of a well-known Moore's theorem concerning a decomposition of a 2-sphere, are stated without proof, Theorem 1, 2, which are used to show Theorem A and B. Terminologies are explained also in §1, whose meanings will be found among Lemmata 1~10. Finally the author regrets that he has not enough informations concerning these problems: Anyone gets the same results?

§1. A map f(X)=Y is compact (connected) iff the inverse image f^-1(B) of any compact (connected) set B of Y is compact (connected).

Lemma 1. Suppose M is a connected compact n-manifold, n>1, and W is its connected open subspace with a totally disconnected complement K=X-W. Any compact and connected map f(W)=W, has a unique extension g(W)=M, which is also compact and connected, whose
restriction on \( K \) is a homeomorphism on \( K \).

A map \( f(X) = Y \) is monotone iff the inverse image \( f^{-1}(y) \) of any point \( y \) of \( Y \) is always compact and connected in \( X \). A fundamental domain of a 2-manifold is a homeomorphic image of a connected open set in a plane.

Lemma 2. A subset \( C \) in a 2-manifold \( M \) is cellular iff \( C \) is compact, connected and is contained in such a fundamental domain \( W \) of \( M \), that the complement \( W-K \) is connected.

Lemma 3. Let \( X \) be a locally compact Hausdorff space and \( \{K_n\}_{n=1}^{\infty} \) be its countable compact covering, then for any open set \( U \) of \( X \), there is an integer \( n \) such that \( U \cap \text{Int } K_n \neq \emptyset \).

Lemma 4. A separable metric \( n \)-manifold \( M \) has a countable connected open cover \( M = \bigcup_{n=0}^{\infty} M_n \) such that the closure \( \overline{M}_n \) is compact and \( \overline{M}_n \subseteq M_{n+1} \) for each \( i \).

Lemma 5. Given an injective map \( f: M_1 \rightarrow M_2 \) from a \( m \)-manifold without boundary \( M_1 \) into a \( m \)-manifold \( M_2 \), then the map \( f \) is an imbedding and its image \( f(M_1) \) is an open set in the interior of \( M_2 \).

A map \( f(X) = Y \) is quasi-compact iff the set \( B \) of \( Y \), whose inverse image \( f^{-1}(B) \) is closed in \( X \), is also closed in \( Y \). A map \( f(X) = Y \) is upper semi-continuous (u.s.c.) iff for any open set \( U \) of \( X \),
the set \( \tilde{U} \) in \( X \), defined by \( \tilde{U} = \bigcup \{ \tilde{f}^{-1}(y) \mid y \in Y, f^{-1}(y) \subset U \} \), is open in \( X \).

**Lemma 6.** A map \( f(X) = Y \) is closed iff the map \( f \) is quasi-compact and u.s.c.

**Lemma 7.** Let \( f(X) = Y \) be a closed map. If \( X \) is a separable metric space, so is \( Y \).

**Lemma 8.** Let the map \( f(X) = Y \) be quasi-compact such that the inverse image of any point of \( Y \) is connected. Then if \( X \) is locally connected, so is \( Y \).

**Lemma 9.** If the map \( f(X) = Y \) is closed and monotone, then it is compact and connected.

**Lemma 10.** A monotone map \( f(X) = Y \) from a locally compact space \( X \) onto a Hausdorff space \( Y \) is u.s.c.

**Lemma 11.** Let \( f(X) = Y \) be a monotone map from a locally compact metric space \( X \) onto a Hausdorff space \( Y \). For any compact set \( K \) in \( X \), the inverse image \( f^{-1}(K) \) is also compact in \( X \).

A disjoint closed cover \( G \) of a space \( X \) is called a decomposition of a space \( X \), and its quotient space \( X' = X/G \) is called a decomposition space of \( X \) by \( G \), where the projection \( \phi : X \to X' \) is clearly quasi-compact. A decomposition \( G \) of a space \( X \) is u.s.c. iff the projection \( \phi : X \to X/G \) is u.s.c., and it is compact (connected) iff each element of which is...
compact (connected); it is non-separating iff for any element $K \in G$, the complement $X - K$ is connected. A map $f(X) = Y$ induces naturally a decomposition of $X$, $G(f) = \{ f^{-1}(y) \mid y \in Y \}$, and we denote its decomposition space by $\phi : X \to X/G(f)$, where the combined map $n = f \circ \phi : X/G(f) \to Y$ is well defined, and bijective. It is clear that the map $n$ is homeomorphism iff the map $f$ is quasi-compact.

Moore's Theorem. Given a non-trivial decomposition $G(X)$ of a space $X$, where the space $X$ is a 2-sphere or 2-plane, which is monotone, u.s.c. and non-separating, then the decomposition space $X/G$ is homeomorphic to the space $X$.

Lemma 12. Given a 2-sphere $X$ and its connected open subset $W$ with a decomposition $G(W)$ which is u.s.c. and monotone. Let $K$ be the complement of $W$ in $X$, and $G(K)$ be its decomposition whose element is a component of $K$. Define the decomposition of a 2-sphere $X$, by $G(X) - G(W)\cup G(K)$, then the decomposition space $X/G(X)$ is a Hausdorff space.

Theorem 1. ([2]) Given a connected open subspace $W$ of a 2-sphere $S^2$ and its non-trivial decomposition $G(W)$, which is monotone, u.s.c. and non-separating. Then the decomposition space $W/G(W)$ is homeomorphic to $W$. 

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Theorem 2. Let $M$ be a separable metric 2-manifold without boundary, $G$ be its u.s.c. cellular decomposition. Then the decomposition space $M/G$ is homeomorphic to $M$.

Suppose $A$ is a subset of a space $X$. A point $a$ of $A$ is a cut point of $A$, iff the complement $A - a$ is disconnected. A subspace $A$ is a true cyclic element of $X$, iff it is maximal in a sense that it has no cut points. A cactoid is a locally connected continuum every true cyclic element of which is a 2-sphere.

Theorem 3. (Moore) Every monotone image of a 2-sphere, which is a locally connected continuum, is a cactoid and every cactoid is the image under some monotone mapping.

§2. Theorem A. Let $W$ be a connected open subspace of a 2-sphere $S^2$, and $M$ be a topological 2-manifold in the most large sense. Given a monotone map $f: W \to M$, whose image $f(W)$ has a non-vacuous interior in $M$, then the image $f(W)$ is in the interior of $M$, and homeomorphic to $W$, moreover, the map $f: W \to f(W)$ is closed, compact, and connected, and the natural decomposition space by $f$, $g(W) = W'$ is homeomorphic to $W$.

Proof. The map $f: W \to f(W)$, is monotone, u.s.c. by Lemma 10, and non-separating (see the later argument), so the natural decom-
position space by \( f, g(W) = W' \) is homeomorphic to \( W \) by Theorem 1.

Now the injective map \( f g^{-1} : W' \to M \), is an imbedding and \( f g^{-1}(W') = f(W) \) is open subspace of \( M \), that is, \( f(W) \) is homeomorphic to \( W \). (see Lemma 5.) Since the map \( f g^{-1} : W' \to f(W) \) is a homeomorphism, the map \( f: W \to f(W) \) is quasi-compact, so \( f \) is closed by Lemma 6. A closed monotone map \( f(X) = Y \) is compact and connected by Lemma 9. Now we show that \( f \) is non-separating. Take a point \( y \in f(W) \), then the inverse image \( f^{-1}(y) = N \) is compact and connected in \( W \), since the map \( f \) is monotone. The 2-manifold \( W \) has a connected open cover \( W = \bigcup_{i=1}^{\infty} W_i \) such that \( W_i \) is compact and \( \overline{W_i} \subset W_{i+1} \) for each \( i \). (see Lemma 4) There is an integer \( n_0 \) such that \( W_n \supset N, n > n_0 \), because \( N \) is compact. Take an open set \( U \subset \text{Int} f(W) \), such that \( U \) is compact in \( \text{Int} f(W) \). Since \( \bigcup_{i=1}^{\infty} f(\overline{W_i}) \) is a compact covering of a compact Hausdorff space, (see Lemma 3), there is an integer \( n_1 \), such that for any \( n > n_1 \), \( f(\overline{W_n}) \) contains an open 2-cell \( C^2 \) of \( M \). Choose an integer \( n \), such that \( n > n_0 + n_1 \). Since \( H = f^{-1}f(\overline{W_n}) \) is compact by Lemma 11, the restriction \( f: H \to f(\overline{W_n}) \) is a closed map, so \( H \) is connected by Lemma 9. Thus \( H \) is a continuum in \( W \). There are a finite number of 2-disks in the 2-sphere \( S^2 \), say \( D_1, D_2, \ldots, D_d \), whose union is denoted by \( D = \bigcup_{i=1}^{d} D_i \), which satisfy that \( D \cap H = \emptyset \). We may assume that the boundary
$\partial D$ consists of a finite number of disjoint 1-spheres, say $S_1, S_2, \ldots, S_d$, that is $\partial D = \bigcup_{i=1}^{d} S_i$. For each $j$, the inverse set $f^{-1}(S_j) \subset W$, is a continuum by the same argument for $H$, so the union $f^{-1}(\partial D) = \bigcup_{j=1}^{d} f^{-1}(S_j)$ is compact in $W$ and has less than $(s+1)$ components. Let $Q$ be the component of the complement $W - (f^{-1}(\partial D))$, which contains a connected set $H$, then it is clear that $\partial Q$ is compact in $W$ and $\partial Q$ has a finite number of components, which implies that the complement $(\partial^2 - Q)$ has a finite number of components, say $E_1, E_2, \ldots, E_m$, $(m \leq s)$, that is $\partial^2 - Q = \bigcup_{i=1}^{m} E_i$. Since $f$ has a connected decomposition, we know that $f^{-1}(Q) = Q$, so the restriction $f:Q \to f(Q)$, is a quasi-compact map, because $f:Q \to f(Q)$ is a closed map. (See Lemma 9.) Consider the decomposition space of the 2-sphere $S^2$, $\phi : S \to K$, by its monotone decomposition defined by $G(S^2) = \{E_1, \ldots, E_m\} \cup \{f^{-1}(x) | x \in Q\}$, which is a Hausdorff space by Lemma 12. So the map $\phi(S^2) = K$ is closed by Lemma 10 and 9, which implies $K$ is a locally connected separable metric space which is also compact and connected. (See Lemmata 7 and 8.) After all the monotone image $K$ of a 2-sphere is a cactoid, by Theorem 3. The map $h=f \circ \phi : Q \to f(Q)$ is a homeomorphism because $f:Q \to f(Q)$ is quasi-compact, so the inverse image $h^{-1}(C^2)$ of an open 2-cell $C^2 \subset f(Q)$, is a non-degenerate connected subset of $K$, which has no cut point of
h^1(C^2), whence we may say that the cactoid K has at least one E_o-set, namely one 2-sphere O \subset K. There is a point p \in O, such that K \setminus \{p\} is connected, because generally any simple link or E_o-set in a connected set X contains at most a countable number of cut points of X. Now we define a connected subset K_o in K, by K_o = \phi(O) \setminus \{p\}.

K_o = \phi(O) \setminus \{p\} = K \setminus \{\phi(E_1), \ldots, \phi(E_m), p\}, in which the subset Z = O \setminus \{\phi(E_1), \ldots, \phi(E_m), p\} is closed and open, so we know that K_o = Z, that is, K is a 2-sphere O. The reason why the set Z is closed and open in K_o is clear that the closure of Z in K is contained in O, because the compact set O is closed in a Hausdorff space K and Z \subset O, which means that Z \cap K_o = Z, that is Z is closed in K_o. Next, the map from a 2-manifold Z without a boundary into a 2-manifold M, h: Z \to M, is injective, the image h(Z) is open in M, by Lemma 5, so h(Z) is also open in h(K_o). So Z = h^d(hZ) is open in K_o, because h|K_o is a homeomorphism. Finally the closed monotone map \phi : S^2 \to K is connected by Lemma 9, so the inverse image of a connected set K \setminus \phi^{-1}(y), where K is a 2-sphere and \phi^{-1}(y) is a point of \phi(O), is connected, that is, the complement S^2 \setminus \phi^{-1}(y) is connected, so w^{-1}(y) is connected.

§3. Theorem B. Let f: M_1 \to M_2 be a monotone map, Int f(M_1) \cup \phi, from a separable metric 2-manifold M_1 without a boundary into a 2-manifold.
M₂. If any element K of the monotone decomposition G of X, defined
by \( G = \{ f^{-1}(y) | y \notin f(M_1) \} \), is contained in a fundamental domain \( W \) of \( M_1 \),
Then the map \( f \) is closed, \( M \cong f(M_1) \) (homeomorphic) and \( f(M_1) \subset \text{int}M_2 \).

Proof. There is such a compact set \( A \) in \( M_1 \) that \( \text{int} f(A) \neq \emptyset \) in \( M_2 \) and \( f^{-1}f(A) = A \). Take a countable open cover \( N_i = \{ N_i \} \) of \( M_1 \) such that
the closure \( \overline{N_i} \) is compact and \( \overline{N_i} \subset N_{i+1} \) for each \( i = 1, 2, \ldots \), and a
2-disk \( D \) in the interior of \( f(M_1) \). Since \( \{ D \cap f(\overline{N_i}) | i = 1, 2, \ldots \} \) is a
compact cover of a 2-disk \( D \), there is an integer \( m \) such that the
interior of \( (D \cap f(\overline{N_m})) \) in \( D \) is non-vacuous, that is \( \text{int} f(\overline{N_m}) \neq \emptyset \) in \( M_2 \),
whence we define \( A = f^{-1}f(\overline{N_m}) \), which is a compact set in \( M_1 \). There is a
fundamental domain \( W_0 \) such that the image \( fW_0 \) is open in \( \text{int}M_2 \), \( f^{-1}fW_0 = W_0 \),
and for any \( K \in G_{W_0} = \{ K \in G | K \subset W_0 \} \), the complement \( (W_0 - K) \) is connected.
For any element \( K_{\nu} \in G \), we may choose such a fundamental domain \( W_{\nu} \)
containing \( K_{\nu} \) as \( f^{-1}fW_\nu = W_\nu \). Take a fundamental domain \( W \) of \( K_{\nu} \), then
\( \widehat{W} = \{ K_{\mu} \in G | K_{\mu} \subset W \} \) is an open set of \( M_1 \), because \( G \) is u.s.c. decomposi-
tion of \( M_1 \). Let \( W_{\nu} \) be the component of \( \widehat{W} \), which contains \( K_{\nu} \). It
is clear that \( W_{\nu} \) be a desired one. There is an open set \( V_{\nu} \) of \( M_1 \),
such that \( K_{\nu} \subset V_{\nu} \subset f^{-1}W_\nu \), because a metric space \( M_1 \) is normal. Define
\( \widehat{V}_{\nu} \) by \( \widehat{V}_{\nu} = \{ K \in G | K \subset V_{\nu} \} \), then we have \( A = f\cap (A \cap \overline{V}_{\nu}) = f\cap (A \cap \overline{V}_{\nu}) = f\cap (A \cap \overline{V}_{\nu}) \subset f\cap W_\nu \),
because \( A \) is compact. There is an integer \( m \), such that

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Int \ f(A) \cap \text{Int} \ (f(A \cap 1W_{w_0})) = \emptyset$, in other words, \text{Int} \ (f_{w_0}) = \emptyset \text{ in } M_2. \text{ Now } W_0 = W_{w_0} \text{ is a desired one, by the Theorem A. For any } K_0 \in G, \text{ the complement } (W_0 - K_0) \text{ is connected. Take a path } \gamma : [0, 1] \rightarrow M_1, \text{ such that } \gamma(0) \in W_0 \text{ and } \gamma(1) \in K_0. \text{ Since the set } f^{-1}f([0, 1]) \text{ is compact and connected, there is a sequence of fundamental domain } W_0, W_1, \ldots, W_n = W_\mu \text{ such that } f^{-1}f([0, 1] \subseteq \bigcup_{i=0}^{n} W_i \text{ and } (\bigcup_{i=0}^{j} W_i) \cap W_\mu = \emptyset \text{ for } j = 0, 1, \ldots, n-1). \text{ It is clear that } W_0 \cap W_1 = f^{-1}f(W_0 \cap W_1), \text{ so } G_{\text{bi}} = \{K \in G | (W_0 \cap W_1) \subseteq K\} \text{ is a decomposition of the intersection } W_0 \cap W_1, \text{ which is monotone, u.s.c. and non-separating. Using Theorem 1, it is known that } (W_0 \cap W_1) \approx f(W_0 \cap W_1) \text{ which is in } \overset{o}{M}_2 \text{ and open in } M_2. \text{ It implies that } \text{Int } f_{W_1} \neq \emptyset. \text{ By Theorem A we know that } f(W_1) \text{ is open in } \overset{o}{M}_2 \text{ and for any } K \in G \subseteq \{K \in G | K \subseteq W_1\} \text{ (} W_1 - K \text{) is connected. After the same type of n arguments, we know that } (W_\mu - K_\mu) \text{ is connected. Finally by Theorem 2, the decomposition space } \phi : M_1 \rightarrow M_2 = M_1 / G \text{ is homeomorphic to } M_1, \text{ and the map } h = f^{-1} : M_1 \rightarrow M_2 \text{ is a homeomorphism, which implies that } M_1 \approx f(M_1), \text{ f}(M_1) \subseteq \overset{o}{M}_2 \text{ and the map } f : M_1 \rightarrow M_2 \text{ is quasi-compact. Since the map } f : M_1 \rightarrow M_2 \text{ is also u.s.c., the monotone map is closed, that is compact and connected.}
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