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<td>DOI, YUTAKA</td>
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A MONOTONE MAP ON 2-MANIFOLDS, WHOSE IMAGE IS
HOMEOMORPHIC TO ITS DOMAIN SPACE, A 2-MANIFOLD.

BY

YUTAKA DOHI

§0. The main results in this note are Theorem A (see §2) and
Theorem B (see §3) which are extensions of Whyburn's theorem [1];
Any monotone mapping of a plane onto a plane is compact. In §1,
extensions of a well-known Moore's theorem concerning a decomposi-
tion of a 2-sphere, are stated without proof, Theorem 1, 2, which
are used to show Theorem A and B. Terminologies are explained also
in §1, whose meanings will be found among Lemmata 1~10. Finally
the author regrets that he has not enough informations concerning
these problems: Anyone gets the same results?

§1. A map \( f(X) = Y \) is compact (connected) iff the inverse image
\( f^{-1}(B) \) of any compact (connected) set \( B \) of \( Y \) is compact (connected).

Lemma 1. Suppose \( M \) is a connected compact \( n \)-manifold, \( n \geq 1 \),
and \( W \) is its connected open subspace with a totally disconnected
complement \( K = X - W \). Any compact and connected map \( f(W) = W \), has a
unique extension \( g(N) = M \), which is also compact and connected, whose
restriction on \( K \) is a homeomorphism on \( K \).

A map \( f(X) = Y \) is monotone iff the inverse image \( f^{-1}(y) \) of any point \( y \) of \( Y \) is always compact and connected in \( X \). A fundamental domain of a 2-manifold is a homeomorphic image of a connected open set in a plane.

Lemma 2. A subset \( C \) in a 2-manifold \( M \) is cellular iff \( C \) is compact, connected and is contained in such a fundamental domain \( W \) of \( M \), that the complement \( W - K \) is connected.

Lemma 3. Let \( X \) be a locally compact Hausdorff space and \( \{ K_i \}_{i=1}^{\infty} \) be its countable compact covering, then for any open set \( U \) of \( X \), there is an integer \( n \) such that \( U \cap \text{Int } K_n \neq \emptyset \).

Lemma 4. A separable metric \( n \)-manifold \( M \) has a countable connected open cover \( M = \bigcup_{i=1}^{\infty} M_i \) such that the closure \( \overline{M_i} \) is compact and \( \overline{V_i} \subset M_i \) for each \( i \).

Lemma 5. Given an injective map \( f: M_1 \rightarrow M_2 \) from a \( m \)-manifold without boundary \( M_1 \) into a \( m \)-manifold \( M_2 \), then the map \( f \) is an imbedding and its image \( f(M_1) \) is an open set in the interior of \( M_2 \).

A map \( f(X) = Y \) is quasi-compact iff the set \( B \) of \( Y \), whose inverse image \( f^{-1}(B) \) is closed in \( X \), is also closed in \( Y \). A map \( f(X) = Y \) is upper semi-continuous (u.s.c.) iff for any open set \( U \) of \( X \),
the set $\tilde{U}$ in $X$, defined by $\tilde{U} = \bigcup \{ f^{-1}(y) \mid y \in Y, f^{-1}(y) \subseteq U \}$, is open in $X$.

Lemma 6. A map $f(X)=Y$ is closed iff the map $f$ is quasi-compact and u.s.c.

Lemma 7. Let $f(X)=Y$ be a closed map. If $X$ is a separable metric space, so is $Y$.

Lemma 8. Let the map $f(X)=Y$ be quasi-compact such that the inverse image of any point of $Y$ is connected. Then if $X$ is locally connected, so is $Y$.

Lemma 9. If the map $f(X)=Y$ is closed and monotone, then it is compact and connected.

Lemma 10. A monotone map $f(X)=Y$ from a locally compact space $X$ onto a Hausdorff space $Y$ is u.s.c.

Lemma 11. Let $f(X)=Y$ be a monotone map from a locally compact metric space $X$ onto a Hausdorff space $Y$. For any compact set $K$ in $Y$, the inverse image $f^{-1}(K)$ is also compact in $X$.

A disjoint closed cover $G$ of a space $X$ is called a decomposition of a space $X$, and its quotient space $X'/=X/G$ is called a decomposition space of $X$ by $G$, where the projection $\phi:X\rightarrow X'$ is clearly quasi-compact.

A decomposition $G$ of a space $X$ is u.s.c. iff the projection $\phi:X\rightarrow X/G$ is u.s.c.; and it is compact (connected) iff each element of which is
compact (connected); it is non-separating iff for any element $K \in G$, the complement $X - K$ is connected. A map $f(X) = Y$ induces naturally a decomposition of $X$, $G(f) = \{ f^{-1}(y) \mid y \in Y \}$, and we denote its decomposition space by $\mathcal{G}: X \to X/G(f)$, where the combined map $n = f^{-1}: X/G(f) \to Y$ is well defined, and bijective. It is clear that the map $h$ is homeomorphism iff the map $f$ is quasi-compact.

Moore's Theorem. Given a non-trivial decomposition $G(X)$ of a space $X$, where the space $X$ is a 2-sphere or 2-plane, which is monotone, u.s.c. and non-separating, then the decomposition space $X/G$ is homeomorphic to the space $X$.

Lemma 12. Given a 2-sphere $X$ and its connected open subset $W$ with a decomposition $G(W)$ which is u.s.c. and monotone. Let $K$ be the complement of $W$ in $X$, and $G(K)$ be its decomposition whose element is a component of $K$. Define the decomposition of a 2-sphere $X$, by $G(X) - G(W) U G(K)$, then the decomposition space $X/G(X)$ is a Hausdorff space.

Theorem 1. ([12]) Given a connected open subspace $W$ of a 2-sphere $S^2$ and its non-trivial decomposition $G(W)$, which is monotone, u.s.c. and non-separating. Then the decomposition space $W/G(W)$ is homeomorphic to $W$. 
Theorem 2. Let $M$ be a separable metric 2-manifold without boundary, $G$ be its u.s.c. cellular decomposition. Then the decomposition space $M/G$ is homeomorphic to $M$.

Suppose $A$ is a subset of a space $X$. A point $a$ of $A$ is a cut point of $A$, iff the complement $A - a$ is disconnected. A subspace $A$ is a true cyclic element of $X$, iff it is maximal in a sense that it has no cut points. A cactoid is a locally connected continuum every true cyclic element of which is a 2-sphere.

Theorem 3. (Moore) Every monotone image of a 2-sphere, which is a locally connected continuum, is a cactoid and every cactoid is the image under some monotone mapping.

§2. Theorem A. Let $W$ be a connected open subspace of a 2-sphere $S^2$, and $M$ be a topological 2-manifold in the most large sense. Given a monotone map $f: W \to M$, whose image $f(W)$ has a non-vacuous interior in $M$, then the image $f(W)$ is in the interior of $M$, and homeomorphic to $W$, moreover, the map $f: W \to f(W)$ is closed, compact, and connected, and the natural decomposition space by $f$, $g(W) = W'$ is homeomorphic to $W$.

Proof. The map $f: W \to f(W)$, is monotone, u.s.c. by Lemma 10, and non-separating (see the later argument), so the natural decom-
position space by \( f, g(W) = W' \) is homeomorphic to \( W \) by Theorem 1.

Now the injective map \( fg^{-1}: W' \to M \), is an imbedding and \( fg^{-1}(W') = f(W) \) is open subspace of \( M \), that is, \( f(W) \) is homeomorphic to \( W \). (see Lemma 5.) Since the map \( fg^{-1}: W' \to f(W) \) is a homeomorphism, the map \( f: W \to f(W) \) is quasi-compact, so \( f \) is closed by Lemma 6. A closed monotone map \( f(X) = Y \) is compact and connected by Lemma 9. Now we show that \( f \) is non-separating. Take a point \( y \in f(W) \), then the inverse image \( f^{-1}(y) = N \) is compact and connected in \( W \), since the map \( f \) is monotone. The 2-manifold \( W \) has a connected open cover \( W = \bigcup_{i=1}^{b} W_i \) such that \( \overline{W_i} \subseteq W \) for each \( i \). (see Lemma 4) There is an integer \( n_0 \) such that \( W_n \supset N, n > n_0 \), because \( N \) is compact. Take an open set \( U \subseteq \text{Int} f(W) \), such that \( \overline{U} \) is compact in \( \text{Int} f(W) \). Since \( \overline{U} = \bigcup_{i=1}^{b} f(\overline{W_i}) \) is a compact covering of a compact Hausdorff space, (see Lemma 3), there is an integer \( n_1 \), such that for any \( n > n_1 \), \( f(\overline{W_n}) \) contains an open 2-cell \( C^2 \) of \( M \). Choose an integer \( n \), such that \( n > n_0 + n_1 \). Since \( H = f^{-1}(f(\overline{W_n})) \) is compact by Lemma 11, the restriction \( f:H \to f(\overline{W_n}) \) is a closed map, so \( H \) is connected by Lemma 9. Thus \( H \) is a continuum in \( W \). There are a finite number of 2-disks in the 2-sphere \( S^2 \), say \( D_1, D_2, \ldots, D_d \), whose union is denoted by \( D = \bigcup_{i=1}^{d} D_i \), which satisfy that \( D \cap H = \emptyset \), \( (D_i) \supset (S^2 - W) \). We may assume that the boundary
\( \mathcal{D} \) consists of a finite number of disjoint 1-spheres, say \( S_1, S_2, \ldots, S_d \), that is \( \mathcal{D} = \bigcup_{j=1}^{d} S_j \). For each \( j \), the inverse set \( f^{-1}(S_j) \subset W \), is a continuum by the same argument for \( H \), so the union \( f^{-1}\mathcal{D} = \bigcup_{j=1}^{d} f^{-1}S_j \) is compact in \( W \) and has less than \( (s+1) \) components. Let \( Q \) be the component of the complement \( W-(f^{-1}\mathcal{D}) \), which contains a connected set \( H \), then it is clear that \( \overline{Q} \) is compact in \( W \) and \( \partial Q \) has a finite number of components, which implies that the complement \( (S^2-Q) \) has a finite number of components, say \( E_1, E_2, \ldots, E_m \) (\( m \leq s \)), that is \( S^2-Q = \bigcup_{i=1}^{m} E_i \). Since \( f \) has a connected decomposition, we know that \( f^{-1}fQ = Q \), so the restriction \( f:Q \to f(Q) \), is a quasi-compact map, because \( f:Q \to f(Q) \) is a closed map. (See Lemma 9.) Consider the decomposition space of the 2-sphere \( S^2 \), \( \phi:S \to K \), by its monotone decomposition defined by \( G(S^2) = \{E_1, \ldots, E_m\} \cup \{f^{-1}f(x) | x \in Q\} \), which is a Hausdorff space by Lemma 12. So the map \( \phi(S^2) \to K \) is closed by Lemma 10 and 9, which implies \( K \) is a locally connected separable metric space which is also compact and connected. (See Lemmata 7 and 8.) After all the monotone image \( K \) of a 2-sphere is a cactoid, by Theorem 3. The map \( h=f^{-1}\phi(Q) \to f(Q) \) is a homeomorphism because \( f:Q \to f(Q) \) is quasi-compact, so the inverse image \( h^{-1}(C^2) \) of an open 2-cell \( C^2 \subset f(Q) \), is a non-degenerate connected subset of \( K \), which has no cut point of
\text{h}'(\sigma^2), \text{whence we may say that the cactoid } K \text{ has at least one } E_{\sigma} \text{-set, namely one } 2\text{-sphere } \Omega \subset K. \text{ There is a point } p \in \Omega, \text{ such that } K-\{p\} \text{ is connected, because generally any simple link or } E_{\sigma} \text{-set in a connected set } X \text{ contains at most a countable number of cut points of } X. \text{ Now we define a connected subset } K_\phi \text{ in } K, \text{ by } K_\phi = \phi(\Omega) - \{p\}, \text{ in which the subset } 
abla = \Omega - \{\phi(E_1), \ldots, \phi(E_m), p\} \text{ is closed and open, so we know that } K_\phi = \nabla, \text{ that is, } K \text{ is a } 2\text{-sphere } \Omega. \text{ The reason why the set } \nabla \text{ is closed and open in } K_\phi: \text{ It is clear that the closure of } \nabla \text{ in } K \text{ is contained in } \Omega, \text{ because the compact set } \Omega \text{ is closed in a Hausdorff space } K \text{ and } \nabla \subset \Omega, \text{ which means that } \overline{\nabla} \cap K_\phi = \nabla, \text{ that is } \nabla \text{ is closed in } K_\phi. \text{ Next, the map from a } 2\text{-manifold } \nabla \text{ without a boundary into a } 2\text{-manifold } M, h: \nabla \to M, \text{ is injective, the image } h(\nabla) \text{ is open in } M, \text{ by Lemma 5, so } h(\nabla) \text{ is also open in } h(K_\phi). \text{ So } \nabla = h^{-1}(h\nabla) \text{ is open in } K_\phi, \text{ because } h|_{K_\phi} \text{ is a homeomorphism. Finally the closed monotone map } \phi: S^2 \to K \text{ is connected by Lemma 9, so the inverse image of a connected set } K-\phi f^{-1}(y), \text{ where } K \text{ is a } 2\text{-sphere and } \phi f^{-1}(y) \text{ is a point of } \phi(\Omega), \text{ is connected, that is, the complement } S^2-f^{-1}(y) \text{ is connected, so } w-f^{-1}(y) \text{ is connected.}

\S 3. \text{Theorem B. Let } f: M_1 \to M_2 \text{ be a monotone map, } \text{Int} f(M_1) \cap \phi, \text{ from a separable metric } 2\text{-manifold } M_1 \text{ without a boundary into a } 2\text{-manifold}
\( M_2 \). If any element \( K \) of the monotone decomposition \( G \) of \( X \), defined by \( G = \{ f^{-1}(y) | y \in f(M_1) \} \), is contained in a fundamental domain \( W \) of \( M_1 \), then the map \( f \) is closed, \( M \approx f(M_1) \) (homeomorphic) and \( f(M_1) \subset M_2 \).

Proof. There is such a compact set \( A \) in \( M_1 \) that \( \text{Int} f(A) \neq \emptyset \) in \( M_2 \) and \( f^{-1}(A) = A \). Take a countable open cover \( N = \{ N_i \} \) of \( M_1 \) such that the closure \( \overline{N}_i \) is compact and \( \overline{N}_i \subset N_{i+1} \) for each \( i = 1, 2, \ldots \), and a 2-disk \( D \) in the interior of \( f(M_1) \). Since \( \{ D \cap f(\overline{N}_i) \}_{i=1}^{\infty} \) is a compact cover of a 2-disk \( D \), there is an integer \( m \) such that the interior of \( (D \cap f(N_m)) \) in \( D \) is non-vacuous, that is \( \text{Int} f(N_m) \neq \emptyset \) in \( M_2 \), whence we define \( A = f^{-1}f(N_m) \), which is a compact set in \( M_1 \). There is a fundamental domain \( W_0 \) such that the image \( fW_0 \) is open in \( f_\# W_0 \), \( f^{-1}fW_0 = W_0 \), and for any \( K \in G_{W_0} = \{ K \in G | K \subset W_0 \} \), the complement \( (W_0 - K) \) is connected.

For any element \( K_\mu \in G \), we may choose such a fundamental domain \( W_\mu \) containing \( K_\mu \), as \( f^{-1}fW_\mu = W_\mu \). Take a fundamental domain \( W \) of \( K_\mu \), then \( \tilde{W} = \{ K_\mu \in G | K_\mu \subset W \} \) is an open set of \( M_1 \), because \( G \) is u.s.c. decomposition of \( M_1 \). Let \( W_\mu \) be the component of \( \tilde{W} \), which contains \( K_\mu \). It is clear that \( W_\mu \) be a desired one. There is an open set \( V_\mu \) of \( M_1 \), such that \( K_\mu \subset V_\mu \subset \overline{V_\mu} \subset W_\mu \), because a metric space \( M_1 \) is normal. Define \( \overline{V_\mu} \) by \( \overline{V_\mu} = \{ K \in G | K \subset V_\mu \} \), then we have \( A = \bigcup_{\mu} (A \cap \overline{V_\mu}) = \bigcup_{\mu} (A \cap \overline{V_\mu}) = \bigcup_{\mu} (A \cap \overline{V_\mu}) <^\# W_\mu \), because \( A \) is compact. There is an integer \( m \), such that
Int f(A) \cap \text{Int } f(A \cap \overline{w_{n+1}}) \not\subseteq \Phi, \text{ in other words, } \text{Int } (f \circ \alpha) \not\subseteq \Phi \text{ in } M_2. \text{ Now } W_0 - W_n \text{ is a desired one, by the Theorem A. For any } K \in G, \text{ the complement } (W_n - K) \text{ is connected. Take a path } \sigma : [0, 1] \rightarrow M_1, \text{ such that } \sigma(0) \in W_0 \text{ and } \sigma(1) \in K. \text{ Since the set } f^{-1} \sigma([0, 1]) \text{ is compact and connected, there is a sequence of fundamental domain } W_0, W_1, \ldots, W_n = W_n \text{ such that } f^{-1} \sigma([0, 1]) \subseteq \bigcup_{i=0}^{n} W_i \text{ and } (\bigcup_{i=0}^{n} W_i) \cap W_{i+1} \not\subseteq \Phi (j=0, 1, \ldots, n-1). \text{ It is clear that } W_0 \cap W_1 = f^{-1} f(W_0 \cap W_1), \text{ so } G_{1-n} = \{ K \in G | K \subseteq W_0 \cap W_1 \} \text{ is a decomposition of the intersection } W_0 \cap W_1, \text{ which is monotone, u.s.c. and non-separating. Using Theorem 1, it is known that } (W_0 \cap W_1) \cong f(W_0 \cap W_1) \text{ which is in } M_2 \text{ and open in } M_2. \text{ It implies that } \text{Int } fW_1 \not\subseteq \Phi. \text{ By Theorem A we know that } f(W_1) \text{ is open in } M_2 \text{ and for any } K \in G_1 = \{ K \in G | K \subseteq W_1 \} \text{ (} W_1 - K \text{) is connected. After the same type of arguments, we know that } (W_n - K) \text{ is connected. Finally by Theorem 2, the decomposition space } \Phi : M_1 \rightarrow M_1 / G \text{ is homeomorphic to } M_1, \text{ and the map } h = f^{-1} : M_1 \rightarrow M_2 \text{ is a homeomorphism, which implies that } M_1 \cong f(M_i), f(M_i) \subseteq M_2 \text{ and the map } f : M_1 \rightarrow M_2 \text{ is quasi-compact. Since the map } f : M_1 \rightarrow M_2 \text{ is also u.s.c., the monotone map is closed, that is compact and connected.}
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Tokyo Metropolitan Junior College of Commerce
(April, 1972)