On a distance function between differentiable structures

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1. Let $M$, $N$ be smooth orientable manifolds with boundary and assume that the boundaries $\partial M$, $\partial N$ are diffeomorphic each other through a diffeomorphism $f$. Denote by $C(\partial M)$, $C(\partial N)$ the collar neighbourhoods of $\partial M$, $\partial N$ respectively and let

$$
\alpha : \partial M \times [0, 1) \to C(\partial M), \quad \beta : \partial N \times [0, 1) \to C(\partial N)
$$

be the diffeomorphisms. Then the map which sends $\alpha(x, t)$ ($x \in \partial M$, $t \in [0, 1)$) into $\beta(F(x), 1-t)$, defines a diffeomorphism $F = F(f)$ between $C(\partial M)$, $C(\partial N)$ and the identified space $M \cup^F N$ turns out to be a smooth manifold.

Lemma 1. Let $M_i$, $N_i$ ($i = 1, 2$) be smooth manifolds with boundary and let $f_i$ be a diffeomorphism between $\partial M_i$ and $\partial N_i$. If homeomorphisms $g_1 : M_1 \to M_2$ and $g_2 : N_1 \to N_2$ are diffeomorphic on some neighbourhoods of the closures of collar neighbourhoods $C(\partial N_1)$, $C(\partial N_2)$, then there are collar neighbourhoods $C(\partial M_2)$, $C(\partial N_2)$ and a diffeomorphism $F_2$ of $C(\partial M_2)$ onto $C(\partial N_2)$ so that $M_2 \cup_{F_2} N_2$ is homeomorphic to $M_1 \cup_{F_1} N_1$ by a homeomorphism $g_1 \cup g_2$ defined by

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\[ g_1 \cup g_2 (x) = \begin{cases} g_1 (x), & \text{if } x \in M_1 \\ g_2 (x), & \text{if } x \in N_1 \end{cases} \]

**Proposition 1**  
Let \( M_i, N_i, g_i \) (\( i = 1, 2 \)), \( f \), be as in Lemma 1. Suppose moreover that with respect to Riemannian metrics \( \mathcal{P}_i, \sigma_i \) (\( i = 1, 2 \)) on \( M_i, N_i \) respectively, the homeomorphism \( g_i \) (\( i = 1, 2 \)) satisfy that

\[ \mathcal{P}_i(x, y)/k_i \leq \sigma_i(g_i(x), g_i(y)) \leq k_i \mathcal{P}_i(x, y) \]

for \( x, y \in M_i \),

then there exist Riemannian metrics \( \tau_i \) on \( M_i \cup N_i \) (\( i = 1, 2 \)) such that

\[ \tau_1(x, y)/\max(k_1, k_2) \leq \tau_2(g_1 \cup g_2(x), g_1 \cup g_2(y)) \leq \max(k_1, k_2) \tau_1(x, y). \]

**Proof**  
Take a real valued smooth function \( \varphi \) such that

\[ 0 \leq \varphi(t) \leq 1, \quad \varphi(t) = 0 \text{ for } t \leq 0, \quad \varphi(t) = 1 \text{ for } t \geq 1, \]

\[ 0 \leq \varphi'(t) \quad \varphi'(t) = 0 \text{ for } t \leq 0 \text{ or } t \geq 1, \]

\[ \varphi(1-t) = 1 - \varphi(t) \]

and let

\[ \alpha_1: M_1 \times [0, 1) \to C(\partial M_1), \quad \beta_1: N_1 \times [0, 1) \to C(\partial N_1) \]

be diffeomorphisms onto the collar neighbourhoods. Then

\[ \alpha_2 = g_1 \circ \alpha_1 \left( (g_1^{-1}|_{\partial M_2}, \text{id}) \right), \quad \beta_2 = g_2 \circ \beta_1 \left( (g_2^{-1}|_{\partial N_2}, \text{id}) \right) \]

also are diffeomorphism of \( \partial M_2 \times [0, 1), \partial N_2 \times [0, 1) \)

onto collar neighbourhoods \( C(\partial M_2), C(\partial N_2) \), respectively, moreover, and the identification map \( F_2 \) obtained from \( \alpha_2, \beta_2, \) and

\[ (g_2|_{\partial N_1})^{-1} \circ f_1 \circ (g_1^{-1}|_{\partial M_2}) \]

satisfies that

\[ g_2 \circ F_1 = F_2 \circ g_1 \text{ on } C(\partial M_1). \]
Define quadratic forms $\tilde{\tau}_i$ on $M_i \cup_{F_i} N_i$ ($i = 1, 2$) by

$$
(\tilde{\tau}_i)_x = \begin{cases} 
(P_i)_x, & x \in M_i - C(\partial M_i), \\
\varphi(t(x))(P_i)_x + (1 - \varphi(t(x)))(F_i \ast \tilde{\sigma}_i)_x, & x \in C(\partial M_i), \\
(\tilde{\sigma}_i)_x, & x \in N_i - C(\partial N_i).
\end{cases}
$$

where $t(x)$ denotes the $t$-coordinate of $x$ in the collar (and $\sim$ indicates the quadratic form of the metric) neighbourhood. Then it is easy to see that the well-defined quadratic forms $\tilde{\tau}_i$ ($i = 1, 2$) give Riemannian metrics $\tau_i$ on $M_i \cup_{F_i} N_i$. Since

$$
\begin{align*}
\mathcal{P}_1(x, y)/k_1 & \lesssim \mathcal{P}_2(\mathcal{g}_1(x), \mathcal{g}_1(y)) \lesssim k_1 \mathcal{P}_1(x, y) \\
\mathcal{C}_1(F_1(x), F_1(y))/k_2 & \lesssim \mathcal{C}_2(\mathcal{g}_2(F_1(x), \mathcal{g}_2(F_1(y))) \\
& \lesssim k_2 \mathcal{C}_1(F_1(x), F_1(y)),
\end{align*}
$$

it holds that

$$
\begin{align*}
\mathcal{P}_1/k_1 & \lesssim \mathcal{g}_1 \ast \mathcal{P}_2 \lesssim k_1 \mathcal{P}_1, \\
F_1 \ast \mathcal{C}_1/k_2 & \lesssim \mathcal{g}_2 \ast (F_1 \ast \mathcal{C}_2) = (g_2 \ast F_1) \ast \mathcal{C}_2 \lesssim k_2 \mathcal{P}_1 \ast \mathcal{C}_1.
\end{align*}
$$

Therefore the metrics $\tau_i$ satisfy that

$$
\tilde{\tau}_i/\max(k_1, k_2) \lesssim \mathcal{g}_1 \ast \tilde{\tau}_2 \lesssim \max(k_1, k_2) \tilde{\tau}_i
$$

on $C(\partial M_i)$, thus from the construction of $g_1 \cup g_2$ we may conclude that

$$
\begin{align*}
\tau_1(x, y)/\max(k_1, k_2) & \lesssim \tau_2((g_1 \cup g_2)(x), (g_1 \cup g_2)(y)) \\
& \lesssim \max(k_1, k_2) \tau_1(x, y).
\end{align*}
$$

Let $M_i$ ($i = 1, 2$) be smooth manifolds with metrics $\mathcal{P}_i$ ($i = 1, 2$) and $f$ be a map of $M_1$ into $M_2$, then we define
$\ell(f; \mathcal{P}_1, \mathcal{P}_2)$ by

$$
\ell(f; \mathcal{P}_1, \mathcal{P}_2) = \inf \left\{ k \geq 1 / \mathcal{P}_1(x, y) / k \leq \mathcal{P}_2(f(x), f(y)) \leq k \mathcal{P}_1(x, y), \text{ for any } x, y \in M \right\}
$$

**Definition** Let $\Sigma_i (i = 1, 2)$ be differential structures on a combinatorial manifold $X$ represented by smooth manifolds $M_i (i = 1, 2)$ with Riemannian metrics $\mathcal{P}_i (i = 1, 2)$. The distance $d(\Sigma_1, \Sigma_2)$ between the differential structures is defined to be

$$
d(\Sigma_1, \Sigma_2) = \log \left( \inf \ell(f; \mathcal{P}_1, \mathcal{P}_2) \right),
$$

where the infimum is taken over all the piecewise linear equivalences of $M_1$ onto $M_2$ and all the Riemannian metrics $\mathcal{P}_1, \mathcal{P}_2$. It is known ([S]) that $d$ is actually a distance function.

**Theorem 1** Let $\Sigma_i, j (i = 1, 2, j = 1, 2)$ be differential structures on combinatorial manifolds $X_i$, then it holds that

$$
d(\Sigma_{1,1} \# \Sigma_{1,2}, \Sigma_{2,1} \# \Sigma_{2,2}) \leq \max(d(\Sigma_{1,1}, \Sigma_{2,1}), d(\Sigma_{1,2}, \Sigma_{2,2}))
$$

where $\Sigma_{1,1} \# \Sigma_{1,2}$ denotes the differential structure obtained by the connected sum.

**Proof** Represent $\Sigma_i, j$ by smooth manifolds $M_i, j$, and for $\varepsilon > 0$ take piecewise diffeomorphisms $g_i$ of $M_{i,1}$ into $M_{i,2}$ and Riemannian metrics $\mathcal{P}_{i,j}$ on $M_i, j$ so that

$$
\log \ell(g_i; \mathcal{P}_{i,1}, \mathcal{P}_{i,2}) \leq d(\Sigma_{i,1}, \Sigma_{i,2}) + \varepsilon
$$

Assume that $g_i$ are diffeomorphic on neighbourhoods of points $p_i \in M_{i,1}$, then after cutting out small imbedded disks around $p_i$, $M_i, j$ and $g_i$ turns out to satisfy the assumption of Proposition 1 with $k_i = \ell(g_i; \mathcal{P}_{i,1}, \mathcal{P}_{i,2})$. 
Since identified manifolds $M_{i,j} \cup M_{j,2}$ represent the connected sum $\Sigma_{i,j} \# \Sigma_{2,j}$, we have that

\[ d(\Sigma_{1,1} \# \Sigma_{2,1}, \Sigma_{1,2} \# \Sigma_{2,2}) \leq \max(\log k_1, \log k_2) \]

finishing the proof.

**Corollary 1** Let $\Gamma_k$ be the group of $k$-dimensional homotopy spheres, then it holds that

\[ d(\Sigma_1 + \Sigma_3, \Sigma_2 + \Sigma_3) = d(\Sigma_1, \Sigma_2) \]

for any $\Sigma_i \in \Gamma_k$ ($i = 1, 2, 3$).

**Corollary 2** The subset $\Gamma_k(a)$ of $\Gamma_k$ given by

\[ \Gamma_k(a) = \{ \Sigma \in \Gamma_k / d(s^k, \Sigma) \leq a \} \]

turns out to be a subgroup of $\Gamma_k$, where $s^k$ denotes the standard $k$-sphere.

**Corollary 3** Let $M_i$ ($i = 1, 2$) be $k$-dimensional manifolds such that $M_2 \sim M_1 \neq \Sigma$ (diffeomorphic) with $\Sigma \in \Gamma_k(a)$, then

\[ d(M_1, M_2) \leq a. \]

**Corollary 4** Let $\text{Diff } S^{k-1}$ denote the set of orientation preserving diffeomorphisms onto itself and let $\pi$ denote the projection of $\text{Diff } S^{k-1}$ onto $\Gamma_k$. Take the usual metric $||$ on $S^{k-1}$ induced from that of $R^k \cup S^{k-1}$, then it holds that

\[ d(s^k, \pi(f)) \leq \log \ell(f; ||, ||). \]

**Proof** Extend $f$ radially to a homeomorphism $g$ of disk $D^k$ onto itself which bounds the sphere $S^{k-1}$ and apply Lemma 1 to disks $D^k$, $g$, id and $f$:

\[
\begin{array}{c}
D^k \cup \partial D^k \xrightarrow{f} \partial D^k \subset D^k \\
\downarrow g \quad \quad \quad \quad \quad \quad \downarrow \text{id}
\end{array}
\]

to obtain a homeomorphism $g \cup \text{id}$ and a diffeomorphism $\mathbb{P}_2$ of $\partial D^k$ onto itself which can be chosen to identity. Since it is obvious that

$$\ell(f; \|, \|, \|) = \ell(g; \|, \|, \|),$$

Proposition 1 yields that

$$\partial(s_{k-1} \cup \mathbb{P}_2 \cup s_{k-1}, \varepsilon(f)) \leq \log \ell(f; \|, \|, \|).$$

2. The partial converse to Corollary 3 holds as in the following:

**Proposition 2** Let $f$ be a homeomorphism between $k$-dimensional manifolds $M_i$, $(i = 1, 2)$ with Riemannian metrics $\rho_i$ $(i = 1, 2)$ and assume that $f$ is diffeomorphic except finite number of points $P_1, \ldots, P_m \in M_1$ then

$$M_2 \cong M_1 \ast \Gamma \ast \bigcup_k \ell(s_1, \mathcal{J}_1, \mathcal{J}_2).$$

Proof Imbed small $k$-disks $D_i$ around $P_i$, then the images

$$f(D_i)$$

turn out to be summanifolds in $N$. Apply Lemma 1 to

manifolds $D_i, f(D_i)$, diffeomorphism $f \mid \partial D_i$ and homeomorphism $f^{-1}$

$$\begin{array}{ccc}
D_i \cup \partial D_i & \overset{f \mid \partial D_i}{\longrightarrow} & \partial(f(D_i)) \\
\downarrow \text{id} & & \downarrow f^{-1} \\
D_i \cup \partial D_i & \overset{id}{\longrightarrow} & \partial D_i \subset D_i
\end{array}$$

to obtain homotopy sphere; $\Sigma_i = D_i \cup f(D_i)$ and a homeomorphism $\text{id} \cup f^{-1}$ between the homotopy sphere and the sphere $S_i$. Because of Proposition 1 there are Riemannian metrics $\sigma_i$, $\sigma_2$ on $\Sigma_i$, $S_i$, respectively, so that

$$\ell(\text{id} \cup f; \sigma_1, \sigma_2) \leq \ell(f; \mathcal{J}_1, \mathcal{J}_2).$$

Therefore we have that

$$\ell \in \Gamma \ast (f; \mathcal{J}_1, \mathcal{J}_2).$$

On the other, since it is easy to see that

$$M_2 \cong M_1 \ast \Sigma_1 \ast \Sigma_2 \ast \cdots \ast \Sigma_m,$$

This finishes the proof.
In general concerning the first obstruction of Munkres ([M]) to smoothing f, we obtain the following:

**Proposition 3** Let $M_i$ ($i = 1, 2$) be smoothly triangulated manifolds with Remannian metrics $\rho_i$ ($i = 1, 2$) and let $L$ be a m-dimensional subcomplex of $M_1$. If a homeomorphism $f$ of $M_1$ onto $M_2$ is diffeomorphic mod. $L$, and if $\ell(f; \mathcal{F}_1, \mathcal{F}_2) < \ell_0 = 1.32$ for the positive root $\ell_0$ of $x^3 - x - 1 = 0$, then the first obstruction chain $\lambda(f)$ of Munkres to smoothing $f$ lies in

$$\Gamma_{k-m}(\ell(f)(1-(\ell^2(f)-\ell(f))^2)^{-1/4})$$

**Proof** Munkres obstruction is obtained as follows:

Take an m-simplex $\omega \in L$ and take trivializations of normal bundles as coordinate systems around $\omega$ and $f(\omega)$ so that the tubular neighbourhoods of $\omega$, $f(\omega)$ are diffeomorphic to $\omega \times R^{k-m}$, $f(\omega) \times R^{k-m}$, respectively, then if $\ell > 0$ is sufficiently small, $\pi \cdot f \cdot i_p$ is a homeomorphism of the $\ell$-disk $D_\ell$ around 0 into $R^{k-m}$ for the inclusion $i_p: R^{k-m} \to p \times R^{k-m}$ and for the projection $\pi: f(\omega) \times R^{k-m}$ $R^{k-m}$ thus the obstruction $\lambda(f)(\omega)$ is defined to be the homotopy sphere obtained by gluing the boundaries of $D_\ell$ and $\pi \cdot f \cdot i_p(D_\ell)$ through $\pi \cdot f \cdot i_p$.

Hence it is sufficient for the proof of Proposition 3 to compute $\ell(\pi \cdot f \cdot i_p; \mathcal{F}_1, \mathcal{F}_2)$ (see Proposition 1) and because of the regularity of $f$ at $L$ ([M] p.526 (4)) the computation is reduced to the following Assertion;
Assertion  Let $g$ be a map between manifolds $N_i$ ($i = 1, 2$) with Riemannian metrics $\mathcal{G}_i$ ($i = 1, 2$) satisfying that

$$\ell (g; \mathcal{G}_1, \mathcal{G}_2) < \kappa < \ell_0$$

then if $g$ is differentiable along any vector of an $m$-dimensional vector space $V \subset T_p(N_1)$, the angle $\Theta$ between the vector $\exp_2^{-1} g \cdot \exp_1 (y), 0$ and the plane $dg(V)$ is not too small, in fact $\Theta$ satisfies that

$$\cos \Theta < \kappa^3 - \kappa < 1,$$

for any $y$ in orthogonal linear subspace $W$ to $V$, provided $|y|$ is sufficiently small.

Proof of Assertion  Taking an $\ell$-disk $D_\ell$ of 0 in $T_p(N_1)$, we may assume that $\tilde{g} = \exp_2^{-1} g \cdot \exp_1$ also satisfies that

$$\ell (\tilde{g}; \ell, \ell) < \kappa < \ell_0$$

Let $x \in V$ be such that $|x| = |y|$, then it holds that

$$2 \left< f(x), f(y) \right> = |f(x)|^2 + |f(y)|^2 - |f(x) - f(y)|^2$$

$$< \kappa (|x|^2 + |y|^2) - |x - y|^2 / \kappa$$

$$= 2|x|^2 (\kappa - 1/\kappa)$$

also it holds that

$$2 \left< f(x), f(y) \right> > 2|x|^2 (1/\kappa - \kappa),$$

therefore we have that

$$|\cos (f(x), f(y))| < \kappa^3 - \kappa,$$

finishing the proof of Assertion.

Thus taking the regularity of $f$ into consideration, we may conclude that by an application of Assertion to $g = f \cdot i_p$,

$$\kappa^{-1} (1-(\kappa^3-\kappa^2)^{1/2}) \pounds_2 (\pi f_i_p(x), \pi f_i_p(y))/f_1 (x, y) \leq \kappa,$$
On a small disk around 0, completing the proof of Proposition 3.

3. The method in 1, 2 applies to obtain a weak estimation of the pinching of a exotic sphere. Let $M_1, M_2$ be combinatorially equivalent compact manifolds, then according to the construction of Hirsch-Munkres (111), we may have a sequence of compact manifolds $L_i (i=1\ldots k)$ such that

i) $L_i$ are combinatorially equivalent to $M_1, M_2$.

ii) $L_1 = N_1$, $L_k = M_2$ (diffeomorphic).

iii) $L_{i+1}$ is obtained by attaching of $S^j \times I^{n-j}$ to $L_i$ through a certain attaching map. ($L^j \in \Gamma^j$).

Now suppose $M_1, M_2$ have different (integral) Pontrjagin class, then for some $i, L_i, L_{i+1}$ have also different Pontrjagin classes. Since we know that manifolds having different Pontrjagin classes are of distance $\frac{1}{2} \log \frac{3}{2} (S_2)$, we have that

\[
\begin{align*}
(1) \quad 1/2 \log 3/2 \leq & \ d(L_1, L_{i+1}) \\
\leq & \ \max(d(L_1, L_i), d(S^j \times I^{n-j}, S^j \times I^{n-j})) \\
\leq & \ d(S^j, L^j).
\end{align*}
\]

Here the last inequality follows from an easily proved Lemma below:

Lemma 2. If $M_i, N_i$ denote a pair of combinatorially equivalent compact manifolds ($i=1, 2$) then

\[
\begin{align*}
d(M_1 \times M_2, N_1 \times N_2) \leq & \ \max (d(M_1, N_1), d(M_2, N_2))
\end{align*}
\]
On the other as is improved by Karcher (unpublished, see also \((S_2)\)) \(\delta\) -pinched Riemannian manifold \(M_\delta\) \((\delta \geq 9/16)\) has distance \(4(1-\sqrt{\delta})\) from the standard sphere \(S\), therefore if the exotic sphere \(\Sigma^j\) in (1) is expressed as a \(\delta\) -pinched manifold \(M_\delta\), \(\delta\) must satisfy that

\[
\frac{1}{2} \log \frac{3}{2} \leq 4(1 - \sqrt{\delta}).
\]

hence

\[
\delta \leq 0.64,
\]

thus we may conclude that a certain exotic sphere of dimension \(\leq 16\) which appears in the obstruction chain to smoothing a combinatorial equivalence can not be pinched by 0.64, because we know that there are compact \(16\) manifolds having different Pontrjagin classes.

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