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On a distance function between differentiable structures*

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1. Let M, N be smooth orientable manifolds with boundary and assume that the boundaries ∂M, ∂N are diffeomorphic each other through a diffeomorphism f. Denote by $C(\partial M)$, $C(\partial N)$ the collar neighbourhoods of ∂M, ∂N respectively and let

$$\alpha : \partial M \times [0,1) \to C(\partial M), \quad \beta : \partial N \times [0,1) \to C(\partial N)$$

be the diffeomorphisms. Then the map which sends $\alpha(x, t)$ $(x \in \partial M, t \in [0, 1))$ into $\beta(F(x), 1-t)$, defines a diffeomorphism $F = F(f)$ between $C(\partial M)$, $C(\partial N)$ and the identified space $M \underset{F}{\cup} N$ turns out to be a smooth manifold.

Lemma 1. Let M_i, N_i $(i = 1,2)$ be smooth manifolds with boundary and let f_1 be a diffeomorphism between ∂M_1 and ∂N_1. If homeomorphisms $g_1 : M_1 \to M_2$ and $g_2 : N_1 \to N_2$ are diffeomorphic on some neighbourhoods of the closures of collar neighbourhoods $C(\partial M_1)$, $C(\partial N_1)$, then there are collar neighbourhoods $C(\partial M_2)$, $C(\partial N_2)$ and a diffeomorphism F_2 of $C(\partial M_2)$ onto $C(\partial N_2)$ so that $M_2 \underset{F_2}{\cup} N_2$ is homeomorphic to $M_1 \underset{F_1}{\cup} N_1$ by a homeomorphism $g_1 \cup g_2$ defined by

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$$g_1 \cup g_2\,(x) = \begin{cases} g_1\,(x)\,, & \text{if } x \in M_1 \\ g_2\,(x)\,, & \text{if } x \in N_1 \end{cases}$$

<u>Proposition 1</u> Let M_i, N_i, g_i $(i = 1,2)$, f, be as in Lemma 1. Suppose moreover that with respect to Riemannian metrics ρ_i . σ_i $(i = 1,2)$ on M_i, N_i respectively, the homeomorphism g_i $(i = 1,2)$ satisfy that

$$\rho_i(x, y)/k_i \leq \sigma_i(g_i(x), g_i(y)) \leq k_i\,\rho_i(x, y) \quad \text{for } x, y \in M_i \quad ,$$

then there exist Riemannian metrics τ_i on $M_i \underset{F_i}{\cup} N_i$ $(i = 1,2)$ such that

$$\tau_1(x, y)/\max(k_1, k_2) \leq \tau_2(g_1 \cup g_2(x), g_1 \cup g_2(y)) \leq \max(k_1, k_2)\,\tau_1(x, y).$$

Proof Take a real valued smooth function φ such that
$0 \leq \varphi(t) \leq 1$, $\varphi(t) = 0$ for $t \leq 0$, $\varphi(t) = 1$ for $t \geq 1$,
$0 \leq \varphi'(t)$ $\varphi'(t) = 0$ for $t \leq 0$ or $t \geq 1$,
$\varphi(1 - t) = 1 - \varphi(t)$
and let

$$\alpha_1 : M_1 \times [0, 1) \to C(\partial M_1), \quad \beta_1 : N_1 \times [0, 1) \to C(\partial N_1)$$

be diffeomorphisms onto the collar neighbourhoods. Then
$\alpha_2 = g_1 \circ \alpha_1\,((g_1^{-1}|_{\partial M_2}), \mathrm{id})$, $\beta_2 = g_2 \circ \beta_1\,((g_2^{-1}|_{\partial N_2}), \mathrm{id})$
also are diffeomorphism of $\partial M_2 \times [0, 1)$, $\partial N_2 \times [0, 1)$ onto collar neighbourhoods $C(\partial M_2)$, $C(\partial N_2)$, respectively, moreover ~~and~~ the identification map F_2 obtained from α_2, β_2, and $(g_2|_{\partial N_1}) \cdot f_1 \cdot (g_1^{-1}|_{\partial M_2})$ satisfies that

$$g_2 \circ F_1 = F_2 \cdot g_1 \quad \text{on} \quad C(\partial M_1).$$

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Define quadratic forms $\tilde{\tau}_i$ on $M_i \underset{F_i}{\cup} N_i$ $(i = 1,2)$ by

$$(\tilde{\tau}_i)_x = \begin{cases} (\tilde{\rho}_i)_x & , x \in M_i - C(\partial M_i), \\ \varphi(t(x))(\tilde{\rho}_i)_x + (1-\varphi(t(x)))(F_i{}^*\tilde{\sigma}_i)_x, & x \in C(\partial M_i), \\ (\tilde{\sigma}_i)_x & , x \in N_i - C(\partial N_i). \end{cases}$$

where $t(x)$ denotes the t-coordinate of x in the collar neighbourhood (and ~ indicates the quadratic form of the metric). Then it is easy to see that the well defined quadratic forms $\tilde{\tau}_i$ $(i = 1,2)$ give Riemannian metrics τ_i on $M_i \underset{F_i}{\cup} N_i$. Since

$$\rho_1(x, y)/k_1 \leq \rho_2(g_1(x), g_1(y)) \leq k_1 \ \rho_1(x, y)$$

$$\sigma_1(F_1(x), F_1(y))/k_2 \leq \sigma_2(\cancel{g_2F_1(x)}, g_2F_1(x), g_2F_1(y)) \leq k_2 \sigma_1(F_1(x), F_1(y)) \quad ,$$

it holds that

$$\tilde{\rho}_1/k_1 \prec g_1{}^*\tilde{\rho}_2 \prec k_1 \tilde{\rho}_1 \quad ,$$

$$F_1{}^* \tilde{\sigma}_1/k_2 \prec g_1{}^*(F_2{}^*\tilde{\sigma}_2) = (g_2 F_1)^*\tilde{\sigma}_2 \prec k_2F_1{}^*\tilde{\sigma}_1 .$$

Therefore the metrics τ_i satisfy that

$$\tilde{\tau}_1/\max(k_1, k_2) \prec g_1{}^*\tilde{\tau}_2 \prec \max(k_1, k_2)\tilde{\tau}_1$$

on $C(\partial M_i)$, thus from the construction of $g_1 \cup g_2$ we may conclude that

$$\tau_1(x, y)/\max(k_1, k_2) \leq \tau_2((g_1 \cup g_2(x), (g_1 \cup g_2)(y)) \leq \max(k_1, k_2)\, \tau_1(x, y).$$

Let M_i $(i = 1,2)$ be smooth manifolds with metrics $\rho_i (i = 1,2)$ and f be a map of M_1 into M_2, then we define

$\ell(f: \rho_1, \rho_2)$ by

$$\ell(f: \rho_1, \rho_2) = \inf\{k \geq 1 / \rho_1(x,y)/k \leq \rho_2(f(x), f(y)) \leq k\rho_1(x, y), \quad \text{for any } x, y \in M\}$$

<u>Definition</u> Let $\Sigma_i (i = 1,2)$ be differential structures on a combinatorial manifold X represented by smooth manifolds $M_i (i = 1,2)$ with Riemannian metrics $\rho_i (i = 1,2)$. The distance $d(\Sigma_1, \Sigma_2)$ between the differential structures is defined to be

$$d(\Sigma_1, \Sigma_2) = \log \left(\inf \ell (f: \rho_1, \rho_2) \right) ,$$

where the infimum is taken over all the piecewise linear equivalences (f) of of M_1 onto M_2 and all the Riemannian metrics ρ_1, ρ_2. It is known ([S]) that d is actually a distance function.

<u>Theorem 1</u> Let $\Sigma_{i,j}$ $(i = 1,2, j = 1,2)$ be differential structures on cominatorial manifolds X_i ((i = 1, 2) respectively), then it holds that

$$d(\Sigma_{1,1} \# \Sigma_{1,2}, \Sigma_{2,1} \# \Sigma_{2,2}) \leq \max(d(\Sigma_{1,1}, \Sigma_{2,1}), d(\Sigma_{1,2}, \Sigma_{2,2}))$$

where $\Sigma_{i,1} \# \Sigma_{i,2}$ denotes the differential structure obtained by the connected sum.

Proof Represent $\Sigma_{i,j}$ by smooth manifolds $M_{i,j}$, and for $\varepsilon > 0$ take piecewise diffeomorphisms g_i of $M_{i,1}$ into $M_{i,2}$ and Riemannian metrics $\rho_{i,j}$ on $M_{i,j}$ so that

$$\log \ell(g_i; \rho_{i,1}, \rho_{i,2}) \leq d(\Sigma_{i,1}, \Sigma_{i,2}) + \varepsilon$$

Assume that g_i are diffeomorphic on neighbourhoods of points $p_i \in M_{i,1}$, then after cutting out small imbedded disks around p_i, $M'_{i,j}$ and g_i turns out to satisfy the assumption of Proposition 1 with $k_i = \ell(g_i; \rho_{i,1}, \rho_{i,2})$.

Since identified manifolds $M'_{1,j} \cup M'_{2,j}$ represent the connected sum $\Sigma_{1,j} \# \Sigma_{2,j}$, we have that

$$d(\Sigma_{1,1} \# \Sigma_{2,1},\ \Sigma_{1,2} \# \Sigma_{2,2}) \leq \max(\log k_1, \log k_2)$$

finishing the proof.

<u>Corollary 1</u> Let Γ_k be the group of k-dimensional homotopy spheres, then it holds that

$$d(\Sigma_1 + \Sigma_3,\ \Sigma_2 + \Sigma_3) = d(\Sigma_1, \Sigma_2)$$

for any $\Sigma_i \in \Gamma_k$ $(i = 1,2,3)$.

<u>Corollary 2</u> The subset $\Gamma_k(a)$ of Γ_k given by

$$\Gamma_k(a) = \{ \Sigma \in \Gamma_k / d(S^k, \Sigma) \leq a \}$$

turns out to be a subgroup of Γ_k, where S^k denotes the standard k-sphere.

<u>Corollary 3</u> Let M_i $(i = 1,2)$ be k-dimensional manifolds such that $M_2 \approx M_1 \# \Sigma$ (diffeomorphic) with $\Sigma \in \Gamma_k(a)$, then

$$d(M_1, M_2) \leq a.$$

<u>Corollary 4</u> Let Diff S^{k-1} denote the set of orientation preserving diffeomorphisms onto itself and let π denote the projection of Diff S^{k-1} onto Γ_k. Take the usual metric $|\ |$ on S^{k-1} induced from that of $R^k \supset S^{k-1}$, then it holds that

$$d(S^k, \pi(f)) \leq \log \ell(f; |\,|, |\,|).$$

Proof Extend f radically to a homeomorphism g of disk D^k onto itself which bounds the sphere S^{k-1} and apply Lemma 1 to disks D^k, g, id and f:

$$\begin{array}{ccccccc} D^k & \supset & \partial D^k & \xrightarrow{f} & \partial D^k & \subset & D^k \\ \downarrow{\scriptstyle g} & & & & & & \downarrow{\scriptstyle id} \\ D^k & & & & & & D^k \end{array}$$

to obtain a homeomorphism $g \cup id$ and a diffeomorphism F_2 of ∂D^k onto itself which can be chosen to (be) identity. Since it is obvious that

$$\ell(f; |\,|, |\,|) = \ell(g; |\,|, |\,|),$$

Proposition 1 yields that

$$d(S^{k-1} \underset{F_2}{\cup} S^{k-1}, \pi(f)) \leq \log \ell(f; |\,|, |\,|).$$

2. The partial converse to Corollary 3 holds as in the following:

Proposition 2 Let f be a homeomorphism between k-dimensional manifolds M_i, $(i = 1,2)$ with Riemannian metrics ρ_i($i = 1,2$) and assume that f is diffeomorphic except finite number of points $P_1, \ldots P_m \in M_1$ then $M_2 \approx M_1 \# \Sigma$(diffeomorphic) with $\Sigma \in \Gamma_k(\log \ell(f; \rho_1, \rho_2))$.

Proof Imbed small k-disks D_i around P_i, then the images $f(D_i)$ turn out to be summanifolds in N. Apply Lemma 1 to manifolds D_i, $f(D_i)$, diffeomorphism $f|_{\partial D_i}$ and homeomorphism id, f^{-1}

$$\begin{array}{ccccccc} D_i & \supset & \partial D_i & \xrightarrow{f|\partial D_i} & \partial(f(D_i)) & \subset & f(D_i) \\ \downarrow & & id & & & & \downarrow f^{-1} \\ D_i & \supset & \partial D_i & \xrightarrow{id} & \partial D_i & \subset & D_i \end{array}$$

to obtain homotopy spheres $\Sigma_i = D_i \underset{F_1}{\cup} f(D_i)$ and a homeomorphism $id \cup f^{-1}$ between the homotopy sphere and the sphere S_i. Because of Proposition 1 there are Riemannian metrics $\sigma_1{}^i$, $\sigma_2{}^i$ on Σ_i, S_i, respectively, so that

$$\ell(id \cup f; \sigma_1{}^i, \sigma_2{}^i) \leq \ell(f: \rho_1, \rho_2) .$$

Therefore we have that

$$\Sigma_i \in \Gamma_k(\log \ell(f; \rho_1, \rho_2)).$$

On the other, since it is easy to see that

$$M_2 \approx M_1 \# \Sigma_1 \# \Sigma_2 \cdots \# \Sigma_m ,$$

This finishes the proof.

In general concerning the first obstruction of Munkres ([M]) to smoothing f, we obtain the following:

<u>Proposition 3</u> Let M_i $(i = 1,2)$ be smoothly triangulated manifolds with Remannian metrics ρ_i $(i = 1,2)$ and let L be a m-dimensional subcomplex of M_1. If a homeomorphism f of M_1 onto M_2 is diffeomorphic mod. L, and if $\ell(f: \rho_1, \rho_2) < \ell_0 \doteqdot 1{,}32$ for the positive root ℓ_0 of $x^3-x-1 = 0$, then the first obstruction chain $\lambda(f)$ of Munkres to smoothing f lies in

$$\Gamma_{k-m}\overset{(\log(}{(}\ell(f)\ (1-(\ell^3(f) - \ell(f))^2)^{-1/4})$$

Proof Munkres obstruction is obtained as follows: Take an m-simplex $\sigma \in L$ and take trivializations of normal bundles as coordinate systems around σ and $f(\sigma)$ so that the tubular neighbourhoods of σ, $f(\sigma)$ are diffeomorphic to $\sigma \times R^{k-m}$, $f(\sigma) \times R^{k-m}$, respectively, then if $\varepsilon > 0$ is sufficiently small, $\pi \cdot f \circ i_p$ is a homeomorphism of the ε- disk D_ε around 0 into R^{k-m} for the inclusion i_p: $R^{k-m} \to p \times R^{k-m}$ and for the projection $\pi : f(\sigma) \times R^{k-m}$ R^{k-m} thus the obstruction $\lambda(f)(\sigma)$ is defined to be the homotopy sphere obtained by glueing the boundaries of D_ε and $\pi \cdot f \circ i_p(D_\varepsilon)$ through $\pi \cdot f \cdot i_p$.

Hence it is sufficient for the proof of Proposition 3 to compute $\ell(\pi \cdot f \circ i_p; \rho_1, \rho_2)$ (see Proposition 1) and because of the regularity of f at L ([M] p.526 (4)) the compulation is reduced to the following Assertion;

Assertion Let g be a map between manifolds N_i $(i = 1,2)$ with Riemannian metrics σ_i $(i = 1,2)$ satisfying that

$$\ell(g: \sigma_1, \sigma_2) < \kappa < \ell_0$$

then if g is differentiable along any vector of an m dimensional vector space $V \subset T_p(N_1)$, the angle θ between the vector $\overrightarrow{\exp_2^{-1}\cdot g\cdot \exp_1(y), 0}$ and the plane $dg(V)$ is not too small, in fact θ satisfies that

$$\cos\theta < \kappa^3 - \kappa < 1,$$

for any y in orthogonal linear subspace W to V, provided $|y|$ is sufficiently small.

Proof of Assertion Taking an ε-disk D_ε of 0 in $T_p(N_1)$, we may assume that $\tilde{g} = \exp_2^{-1}\cdot g\cdot \exp_1$ also satisfies that

$$\ell(\tilde{g}: |\ |, |\ |) < \kappa < \ell_0$$

(on D_ε.)

Let $x \in V$ be such that $|x| = |y|$, then it holds that

$$\begin{aligned} 2\langle f(x), f(y)\rangle &= |f(x)|^2 + |f(y)|^2 - |f(x) - f(y)|^2 \\ &< \kappa(|x|^2 + |y|^2) - |x-y|^2/\kappa \\ &= 2|x|^2(\kappa - 1/\kappa) \end{aligned}$$

also it holds that

$$2\langle f(x), f(y)\rangle > 2|x|^2 \quad (1/\kappa - \kappa),$$

therefore we have that

$$|\cos(\overrightarrow{f(x)\,0}, \overrightarrow{f(y)\,0})| < \kappa^3 - \kappa,$$

finishing the proof of Assertion.

Thus taking the regularity of f into consideration, (we) may conclude that by an application of Assertion to $g = f\cdot i_p$,

$$\kappa^{-1}(1-(\kappa^3-\kappa)^2)^{1/2} \le \rho_2(\pi\cdot f\cdot i_p(x), \pi\cdot f\cdot i_p(y))/\rho_1(x,y) \le \kappa$$

On a small disk around 0, completing the proof of Proposition 3.

3. The method in 1, 2 applies to obtain a weak estimation of the pinching of a exotic sphere. Let M_1, M_2 be combinatorially equivalent compact manifolds, then according to the construction of Hirch-Munkres (M), we may have a sequence of compact manifolds L_i ($i=1\ldots k$) such that

i) L_i are combinatorially equivalent to M_1, M_2.

ii) $L_1 = M_1$, $L_k = M_2$ (diffeomorphic).

iii) L_{i+1} is obtained by attaching of $\Sigma^j \times I^{n-j}$ to L_i through a certain attaching map. ($\Sigma^j \in \Gamma^j$).

Now suppose M_1, M_2 have different (integral) Pontrjagin class, then for some i, L_i, L_{i+1} have also different Pontrjagin classes. Since we know that manifolds having different Pontrjagin classes are of distance $\geq 1/2 \log 3/2$ (S_2), we have that

$$(1) \quad 1/2 \log 3/2 \leq d(L_i, L_{i+1}) \leq \max(d(L_i, L_i), d(S^j \times I^{n-j}, \Sigma^j \times I^{n-j})) \leq d(S^j, \Sigma^j).$$

Here the last inequality follows from an easily proved Lemma below:

Lemma 2 If M_i, N_i denote a pair of combinatorially equivalent compact manifolds ($i=1, 2$) then

$$d(M_1 \times M_2, N_1 \times N_2) \leq \max(d(M_1, N_1), d(M_2, N_2))$$

On the other as is improved by Karcher (unpublished, see also (S_3)) δ-pinched Riemannian manifold M_δ ($\delta \geqq 9/16$) has distance $4(1-\sqrt{\delta})$ from the standard sphere S, therefore if the exotic sphere Σ^j in (1) is expressed as a δ-pinched manifold M_δ, δ must satisfy that

$$1/2 \ \log \ 3/2 \leqq 4(1-\sqrt{\delta}).$$

hence

$$\delta \leqq 0.64 \ ,$$

thus we may conclude that a certain exotic sphere of dimension ≤ 16 which appears in the obstruction chain to smoothing a combinatorial equivalence can not be pinched by 0.64, because we know that there are compact 16 manifolds having different Pontrjagin classes.

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