

On mod 3 characteristic classes for F_4 -bundles.

by Hirosi Toda (Kyoto Univ.)

Let F_4 be simply connected exceptional Lie group of rank 4.

The mod p characteristic classes for F_4 -bundles are in one-to-one

correspondence with the elements of the mod p cohomology ring

$H^*(BF_4; Z_p)$ of the classifying space BF_4 of F_4 .

For $p \neq 3$, the structure of $H^*(BF_4; Z_p)$ is determined from

that of $H^*(F_4; Z_p)$ by use of Borel's theorems:

$$H^*(BF_4; Z_2) = Z_2[x_4, x_6, x_7, x_{16}, x_{24}],$$

$$H^*(BF_4; Z_p) = Z_p[x_4, x_{12}, x_{16}, x_{24}] \quad \text{for } p \neq 2, 3,$$

where $x_i \in H^i$, $x_6 = \mathfrak{S}_q^2 x_4$, $x_7 = \mathfrak{S}_q^3 x_4$, $x_{24} = \mathfrak{S}_q^8 x_{16}$ for $p = 2$,

and $\mathfrak{P}^1 x_i = x_j$ if $j = i + 2p - 2$ and $p \geq 5$.

For $p = 3$, however, it seems very difficult to determine

$H^*(BF_4; Z_3)$ directly from the result

$$(1) \quad H^*(F_4; Z_3) = Z_3[x_8] / (x_8^3) \otimes \wedge(x_3, x_7, x_{11}, x_{15}),$$

where $x_7 = \mathfrak{P}^1 x_3$, $x_8 = \Delta x_7$ and $x_{15} = \mathfrak{P}^1 x_{11}$.

In fact, S. Araki [1] showed that the elements x_{11} and x_{15}

are not transgressive, hence we cannot apply Borel's method to

the case $\bar{p} = 3$.

The purpose of the present speech is to determine the structure of $H^*(BF_4; Z_3)$ by use of the fibering

$$(2) \quad \mathbb{P} \rightarrow BSpin(9) \xrightarrow{\rho} BF_4,$$

where $\mathbb{P} = F_4/Spin(9)$ is Cayley plane. Since the natural map

$BSpin(9) \rightarrow BSO(9)$ has mod 3 trivial fibre BZ_2 , the map induces

an isomorphism of $H^*(; Z_3)$. Thus we identify $H^*(BSpin(9); Z_3)$

with $H^*(BSO(9); Z_3)$, which is the polynomial algebra of Pontrjagin

classes $P_i \in H^{4i}$;

$$H^*(BSpin(9); Z_3) = Z_3 [p_1, p_2, p_3, p_4].$$

First we prepare the following

Lemma 1. The image of

$$\rho^* : H^*(BF_4; Z_3) \rightarrow H^*(BSpin(9); Z_3)$$

is contained in the subalgebra A generated by

$$p_1, \bar{p}_2 = p_2 - p_1^2, \quad \bar{p}_5 = p_4 p_1 + p_3 \bar{p}_2,$$

$$\bar{p}_9 = (p_3^2 - p_4 p_1^2)(p_3 + \bar{p}_2 p_1) \quad \text{and} \quad \bar{p}_{12} = p_4^3 + p_4^2 \bar{p}_2 + p_4 \bar{p}_2^2,$$

thus A is isomorphic to $Z_3[p_1, \bar{p}_2, \bar{p}_5, \bar{p}_9, \bar{p}_{12}] / \langle r_{15} \rangle$

$$\text{for } r_{15} = \bar{p}_5^3 + \bar{p}_5^2 \bar{p}_2 \bar{p}_1 - \bar{p}_{12} \bar{p}_1^3 - \bar{p}_9 \bar{p}_2^3.$$

This lemma is proved as follows. Let BT be the classifying space of a maximal torus T of $\text{Spin}(9)$. The Weyl group $\bar{\Phi}' = \bar{\Phi}(\text{Spin}(9))$ of $\text{Spin}(9)$ operates on $H^*(BT; Z_3) = Z_3[t_1, t_2, t_3, t_4]$ as permutations and the change of signs. As is well known, under the homomorphism ρ^{L*} induced by the natural map $\rho^L: BT \rightarrow B\text{Spin}(9)$, $H^*(B\text{Spin}(9); Z_3)$ is mapped isomorphically onto the subalgebra $H^*(BT; Z_3)^{\bar{\Phi}'}$ invariant under $\bar{\Phi}'$ which is a polynomial algebra of the variables $\rho^{L*}(p_i)$ (= the i -th elementary symmetric function on $t_1^2, t_2^2, t_3^2, t_4^2$). For the Weyl group $\bar{\Phi} = \bar{\Phi}(F_4)$ of F_4 , the image of $(\rho\rho')^* = \rho'^*\rho^*$ is contained in the invariant subalgebra $H^*(BT; Z_3)^{\bar{\Phi}} = \rho'^*A$. $\bar{\Phi}$ is generated by $\bar{\Phi}'$ and an element which operates on $Z_3[t_1, t_2, t_3, t_4]$ as the reflection with respect to the plane $t_1 + t_2 + t_3 + t_4 = 0$ (cf. [2]). Then the lemma is proved by purely algebraic computations.

Consider the mod 3 cohomology spectral sequence $(E_r^*, *)$ associated with the fibering (2), where

$$E_2^{*,*} = H^*(BF_4; Z_3) \otimes \{1, w, w^2\}, w \in H^8(\mathbb{P}; Z_3), \text{ and}$$

$E_\infty^{*,*}$ is a graded algebra associated to $Z_3[p_1, p_2, p_3, p_4]$.

Thus the differential d_r is trivial unless $r = 9, 17$.

By the aid of Lemma 1, we have the following

Theorem 1. $H^*(BF_4; Z_3) = Z_3[x_{36}, x_{48}] \otimes$

$$\left[Z_3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} + \wedge(x_9) \otimes Z_3[x_{26}] \otimes \{1, x_{20}, x_{21}, x_{25}\} \right].$$

Schetch of the proof. For instance, we compute $H^*(BF_4; Z_3)$

step by step as follows.

(i) There exists x_4 such that $\rho^*(x_4) = p_1$. (easy)

(ii) There exists x_8 such that $\rho^*(x_8) = p_2$, since

$$p_2 + p_1^2 = -\beta^1(p_1) = \rho^*(-\beta^1 x_4)$$

(iii) The transgression image x_9 of w is non-zero, since

$$d_9(1 \otimes w) = 0 \text{ implies rank } (E_\infty^{0,8} + E_\infty^{8,0}) = 3 > \text{rank } (H^8(B \text{ Spin}(9)))$$

= 2.

(iv) The natural map

$$(3) \quad Z_3[x_4, x_8] + \{x_9\} \rightarrow H^*(BF_4)$$

is monic and it is epic for $\text{deg} < 12$. The first half follows from

(i), (ii), (iii) and $Z_3[p_1, p_2] \subset H^*(B \text{ Spin}(9))$.

(v) The relation $x_4 x_9 = 0$ holds and (3) is epic for $\text{deg} < 16$. Assume that $x_4 x_9 = d_9(x_4 \otimes W) \neq 0$, then $E_\infty^{4,8} = 0$ and thus $\rho^* : H^{12}(BF_4) \rightarrow H^{12}(B \text{ Spin}(9))$ is epic. But this contradicts to the result $p_3 \notin \text{Im } \rho^*$ of Lemma 1. Thus we have the assestion of (v). Remark that $x_4 \otimes W \in E_\infty^{4,8}$ represents $p_3 \pmod{\text{Im } \rho^*}$. Similarly we have

(vi) The relation $x_8 x_9 = 0$ holds and (3) is epic for $\text{deg} < 20$.

$x_8 \otimes W$ is a parmanent cycle and represents $p_4 \pmod{\text{Im } \rho^* + \{p_3 p_1\}}$.

(vii) There exists an element x_{20} such that $\rho^* x_{20} = \overline{p_5}$
 $= p_4 p_1 + p_3 p_2 - p_3 p_1^2$. The parmanent cycle $x_4 x_8 \otimes W$ represents both of $p_2 (\pm p_3 + \text{lower term})$ and $p_1 (\pm p_4 + \text{lower term})$. This shows that $p_4 p_1 \pm p_3 p_2 + (\text{lower term})$ is in $E_\infty^{0,20}$, i.e., in the mage of ρ^* . Such an element should be $\overline{p_5} \pmod{(p_1, p_2)}$ by Lemma 1.

Thus the existence of x_{20} is proved.

(viii) $x_4^2 \otimes w^2$ represents x_{12}^2 since $x_4 \otimes W$ does x_{12} .

By the same reason as (iii) we have

$$(ix) \quad x_{21} = d_{17}(x_4 \otimes W^2) \neq 0, \quad x_{25} = d_{17}(x_8 \otimes W^2) \neq 0$$

$$\text{and } x_{26} = d_{17}(x_9 \otimes W^2) \neq 0.$$

Continuing the computation a little more, we have the existence of the elements x_{36}, x_{48} such that

$$(x) \quad \rho^*(x_{36}) = \bar{p}_9 \quad \text{and} \quad \rho^*(x_{48}) = \bar{p}_{12}.$$

Finally we prove the theorem by the aid of comparison theorem.

Corollary $\text{Ker } \rho^*$ is an ideal generated by $x_9, x_{21}, x_{25}, x_{26}$, and ρ^* maps the subalgebra generated by $x_4, x_8, x_{20}, x_{36}, x_{48}$ isomorphically onto $\text{Im } \rho^* = A$.

Let \widetilde{BF}_4 be the 4-connective fibre space over BF_4 . we have a fibering

$$(4) \quad \widetilde{BF}_4 \rightarrow BF_4 \xrightarrow{p} K(Z, 4).$$

The loop-space of BF_4 is a 3-connective fibre space over F_4 ,

say \widetilde{F}_4 . From [3]

$$H^*(\widetilde{F}_4; Z_3) = \wedge (x_{11}, x_{15}, x_{19}, x_{23}) \otimes Z_3[x_{18}]$$

where $x_{15} = \beta^1 x_{11}$, $x_{19} = \Delta x_{18}$ and $x_{23} = \beta^1 x_{19}$. By spectral

sequence arguments we have

$$H^*(\widetilde{BF}_4; Z_3) = \Lambda(y_{19}) \otimes Z_3 [y_{12}, y_{16}, y_{20}, y_{24}]$$

for $\deg < 55$. On the other hand

$$H^*(K(Z, 4); Z_3) = Z_3 [x_4, x_8, x_{20}, x_{26}, \dots] \otimes \Lambda(x_9, x_{21}, x_{25}, \dots),$$

where $x_8 = \beta^1 x_4$, $x_{20} = \beta^3 x_8$, $x_9 = \Delta x_8$, $x_{21} = \Delta x_{20}$, $x_{25} = \beta^4 x_9$,

$x_{26} = \Delta x_{25}$. Consider the cohomology spectral sequence associated with

the fibering, then by dimensional reasons we have

Lemma 2. $\Delta \beta^4 \Delta \beta^1 x_4 = \pm p^*(x_{26}) \neq 0.$

This leads us to the following

Theorem 2. In Theorem 1, we can choose

$$x_8 = \beta^1 x_4, \quad x_9 = \Delta \beta^1 x_4,$$

$$x_{20} = \beta^3 \beta^1 x_4, \quad x_{21} = \Delta \beta^3 \beta^1 x_4,$$

$$x_{25} = \beta^4 \Delta \beta^1 x_4, \quad x_{26} = \Delta \beta^4 \Delta \beta^1 x_4$$

and $x_{48} = \beta^3 x_{36}$

Corollary, For the above choice of generators, the following

relations hold:

$$x_4 x_9 = x_8 x_9 = x_4 x_{21} = x_8 x_{25} = x_{20} x_{21} = x_{20} x_{25} = 0,$$

$$x_4 x_{25} = -x_8 x_{21} = x_9 x_{20}, \quad x_4 x_{26} = -x_9 x_{21},$$

$$x_8 x_{26} = -x_9 x_{25}, \quad x_{21} x_{25} = -x_{20} x_{26}.$$

These relations are obtained from the relation $x_4 x_9 = 0$

by applying cohomology operations.

References

- 1 S. Araki : On the non-commutativity of Pontrjagin rings mod 3 of some compact exceptional groups, Nagoya J. Math., 17(1960), 225-260.
- 2 A. Borel and F. Hirzebruch : Characteristic classes and homogeneous spaces I, Amer. J. of Math., 80(1958), 458-538.
- 3 M. Mimura : The homotopy groups of Lie groups of low rank, J. Math. Kyoto Univ., 6(1967), 131-176