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Smooth S^1-action and bordism

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This talk is based on a work which is done in part jointly with H. Taniguchi. Details will appear elsewhere.

Let G be a compact Lie group and $\mathcal{F}$ and $\mathcal{F}'$ be families of subgroups of G such that $\mathcal{F}' \subset \mathcal{F}$. Following Conner-Floyd I call an effective action of G on a manifold M $(\mathcal{F}, \mathcal{F}')$-free if $G_x \in \mathcal{F}$ for all $x \in M$ and $G_x \in \mathcal{F}'$ for all $x \in \partial M$. When $F' = \phi$ then ∂M must be empty. The bordism group $\Omega_n(G; \mathcal{F}, \mathcal{F}')$ of all orientation preserving $(\mathcal{F}, \mathcal{F}')$-free smooth G-actions on compact smooth manifolds is defined as follows. (M, ψ) and (M', ψ') are bordant iff there is (W, ψ) such that

$$\partial W \supset M \cup -M, \quad \Psi | M = \psi, \quad \Psi | M' = \psi',$$

Ψ is $\mathcal{F}$-free and $\Psi | \partial W - (M \cup M')$ is $\mathcal{F}'$-free. There is an exact sequence

$$\cdots \longrightarrow \Omega_n(G; \mathcal{F}') \xrightarrow{i_*} \Omega_n(G; \mathcal{F}) \xrightarrow{j_*} \Omega_n(G; \mathcal{F}, \mathcal{F}') \xrightarrow{\partial_*} \cdots .$$

Similarly the U-bordism group $\Omega_n^U(G; \mathcal{F}, \mathcal{F}')$ is defined where we consider U-manifolds and U-structure preserving actions.

Now consider the case $G = S^1$. We set

$$\mathcal{F}_\ell^+ = \{ Z_k \mid k \leq \ell \} \quad \text{and} \quad \mathcal{F}_\ell^+ = \mathcal{F}_\ell \cup \{ S^1 \}.$$

<u>Theorem</u>. <u>The sequences</u>

$$0 \longrightarrow \Omega_n^U(S^1;\ \mathfrak{F}^+_{\ell-1}) \xrightarrow{i_*} \Omega_n^U(S^1;\ \mathfrak{F}^+_{\ell}) \xrightarrow{j_*} \Omega_n^U(S^1;\ \mathfrak{F}^+,\ \mathfrak{F}^+_{\ell-1}) \longrightarrow 0$$

$$0 \longrightarrow \Omega_n(S^1;\ \mathfrak{F}^+_{\ell-1}) \longrightarrow \Omega_n(S^1;\ \mathfrak{F}^+_{\ell}) \longrightarrow \Omega_n(S^1;\ \mathfrak{F}^+_{\ell},\ \mathfrak{F}^+_{\ell-1}) \longrightarrow 0$$

are split exact $(1 < \ell)$.

Geometrical contents of the theorem are as follows. For the sake of simplicity hereafter I restrict myself only to U-cases. Consider a triple (X, V, ψ) where

X is a compact U-manifolds,

V is a complex vector bundle over X,

ψ is an effective S^1-action on V by isomorphisms.

Let

$$H = \{g \mid g \in S^1,\quad \psi(g)x = x \quad \forall x \in X\}.$$

If $H \neq S^1$ then $H = \mathbb{Z}_\ell$ for some ℓ. We say that the action ψ is of order ℓ. In that case there is a unique S^1-action φ on X such that

$$\psi(g)x = \varphi(g)^\ell x.$$

Definition. ψ is strictly $\mathfrak{F}^+_\ell$-free $(\ell > 1)$, iff

1) ψ is of order ℓ,

2) the action φ (as above) is $\mathfrak{F}^+_1$-free,

3) ψ restricted on $V - X$ is $\mathfrak{F}_{\ell-1}$-free.

If (X, V, ψ) is strictly $\mathfrak{F}^+_\ell$-free then $(D(V), \psi)$ is $(\mathfrak{F}^+_\ell, \mathfrak{F}_{\ell-1})$-free hence $(\mathfrak{F}^+_\ell, \mathfrak{F}^+_{\ell-1})$-free where $D(V)$ is the disk bundle of V. The bordism group $B_n^U(S^1;\ \mathfrak{F}^+_\ell) = \{[X, V, \psi]\}$ is defined in an obvious way where ψ is strictly $\mathfrak{F}^+_\ell$-free and

$\dim X + 2 \dim_{\mathbb{C}} V = n$.

Proposition.

$$B_n^U(S^1;\ \mathcal{F}_\ell^+) \cong \Omega_n^U(S^1;\ \mathcal{F}_\ell^+,\ \mathcal{F}_{\ell-1}^+)$$

in a natural way.

Moreover the homomorphism j_* is transformed into the "fixed point homomorphism for $\mathbb{Z}_\ell$"

$$\Omega_n^U(S^1;\ \mathcal{F}_\ell^+) \longrightarrow B_n^U(S^1;\ \mathcal{F}_\ell^+)$$

in the following sense.

Let (M, ψ) be an $\mathcal{F}^+$-free action. Then there are 2 kinds among the components X of the fixed point set of $\psi(\mathbb{Z}_\ell)$.

1st kind: $G_x = \mathbb{Z}_\ell$ for some $x \in X$.

2nd kind: $G_x = S^1$ for all $x \in X$.

Proposition. j_* is transformed into the homomorphism given by

$$[M, \psi] \longmapsto \sum [X_i, V_i, \psi]$$

where X_i runs over the components of the 1st kind of the fixed point set of $\psi(\mathbb{Z}_\ell)$ and V_i is the normal bundle of X_i in M.

In the rest of this talk I shall give a splitting

$$B_n^U(S^1;\ \mathcal{F}_\ell^+) \longrightarrow \Omega_n^U(S^1;\ \mathcal{F}_\ell^+),\quad 2 \le \ell\ ,$$

which looks very simple.

First consider the case $\ell = 2$. Since ψ is free on $V - X$, $S(V)/\psi = \mathbb{P}_\psi(V)$ is a smooth manifold. Let W_ψ be the disk bundle of $S(V) \longrightarrow \mathbb{P}_\psi(V)$ and

$$\mathbb{P}_\psi(V \times \mathbb{C}) = D(V) \cup W_\psi.$$

Clearly the action ψ extends on $\mathbb{P}_\psi(V \times \mathbb{C})$. The fixed point set of $\psi(\mathbb{Z}_\ell)$ equals

$$X \cup \mathbb{P}_\psi(V)$$

where X is of the 1st kind and $\mathbb{P}_\psi(V)$ is of the 2nd kind. Hence

$$B_*^U(S^1;\, \mathcal{F}_2^+) \longrightarrow \Omega_*^U(S^1;\, \mathcal{F}_2^+)$$

$$[X,\, V,\, \psi] \longmapsto [\mathbb{P}_\psi(V \times \mathbb{C}),\, \psi]$$

is a splitting for j_*.

For general ℓ we construct a strictly $\mathcal{F}_2^+$-free S^1-action ψ on V which covers φ^2 and commutes with ψ as follows. The group $\mathbb{Z}_\ell$ acts on V by automorphism (via ψ). Hence it gives a decomposition

$$V = \sum V(\ell_i)$$

where

$$\psi(g)v = g^{\ell_i}v\,, \qquad g \in \mathbb{Z}_\ell,\quad v \in V(\ell_i).$$

The integer ℓ_i is determined modulo ℓ so that we may assume

$$0 < \ell_i < \ell\,,$$

since ψ is strictly $\mathcal{F}_\ell^+$-free. I shall write this as

$$\psi(g) = \psi'(g)^{\ell_i}\,, \quad g \in \mathbb{Z}_\ell, \quad \text{on } V(\ell_i),$$

where $\psi'(g)$ is scalar multiplication. Consider the S^1-action on $V(\ell_i)$ given by

$$g \longmapsto \psi(g)\,\psi'(g)^{-\ell_i}.$$

There is a unique ψ'' on $V(\ell_i)$ such that

$$\psi''(g)^\ell = \psi(g)\,\psi'(g)^{-\ell_i}.$$

ψ'' covers φ and hence can be summed up:

$$\psi''(g)\sum v_i = \sum \psi''(g)v_i\ , \quad v_i \in V(\ell_i).$$

Define

$$\psi_1(g) = \psi''(g)^2\,\psi'(g)$$

ψ_1 commutes with ψ . Set

$$\begin{cases} \mathbb{P}_\psi(V) = S(V)/\psi_1 \\ \mathbb{P}_\psi(V \times \mathbb{C}) = D(V) \cup W_\psi \end{cases}$$

as before. ψ is extended over $\mathbb{P}_\psi(V \times \mathbb{C})$.

The following lemma can be checked by calculations.

Lemma. The action ψ on $\mathbb{P}_\psi(V \times \mathbb{C})$ is $\mathcal{F}^+_{\ell-1}$-free outside of X.

It follows that

$$\begin{aligned} {}^t\mathbb{P} : B^U_*(S^1;\ \mathcal{F}^+_\ell) &\longrightarrow \Omega^U_*(S^1;\ \mathcal{F}^+_\ell) \\ [X,\ V,\ \psi] &\longmapsto [\mathbb{P}_\psi(V \times \mathbb{C}),\ \psi] \end{aligned}$$

is a splitting for j_*.

Corollary.

$$\Omega^U_*(S^1) = \Omega^U_*(S^1;\ \mathcal{F}^+_1) \oplus \sum_{2\leq \ell} {}^t\mathbb{P}(B^U_*(S^1;\ \mathcal{F}^+_\ell)).$$

I want call $\mathbb{P}_\psi(V)$ and $\mathbb{P}_\psi(V \times \mathbb{C})$ twisted complex projective space bundle although they are not bundles in the usual sense. Let me remark the following:

Proposition. Let $\dim_{\mathbb{C}} V = k$. $F = \mathrm{Fix}\,\varphi \subset X$.

There is a map

$$\pi : \mathbb{P}_\psi(V) \longrightarrow X/\varphi$$

which is

$$\text{a } \mathbb{C}P^{k-1}\text{-bundle on } F = F/\varphi \subset X/\varphi\ ,$$

an $\mathbb{R}P^{2k-1}$*-bundle on* $X/\varphi - F$.

This construction $\mathbb{P}_{\psi}(V \times \mathbb{C})$ can be used to give an elementary proof of Kooniowski's and Atiyah-Singer's formula.

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