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Euler-poincaré characteristics of complex projective hypersurfaces with isolated singularities.

By Mitsuyoshi Kato

College of Education, The University of Tokyo

In this report we would like to give a provisional explanation about results of our forthcoming paper [3].

1. Motivation. The following naive observation of a projective plane curve would tell us how to compute the Euler-Poincaré characteristic (simply, euler number) of a complex projective hypersurface with isolated singularities.

1.1. Let C be a complex plane curve defined by a homogeneous irreducible polynomial of degree d.

If P is a point of C, then an integer δ_P is defined by

$$\delta_P = \dim_{\mathbb{C}}(\overline{\mathcal{O}}_P/\mathcal{O}_P) ,$$

where $\mathcal{O}_P$ is the local ring of C at P and $\overline{\mathcal{O}}_P$ is its integral closure in the rational function field, see J.P. Serre [6]. Note that $\delta_P > 0$ if and only if P is a singular point of C.

The Plücker formula gives us the genus $g = g(M)$ of a non-singular model M of C in terms of the degree d of C and local algebro-geometric invariants δ_P at $P \varepsilon C$;

Plücker formula : $$g = \frac{(d-1)(d-2)}{2} - \sum_{P\varepsilon\Sigma C} \delta_P$$

where ΣC is the set of singular points of C.

1.2. A topological characterization of δ_P has been given by Milnor [4] as follows:

Taking an affine chart arround P, we may assume that for a sufficiently small number $\varepsilon > 0$, a ball $D^4(\varepsilon,P)$ with radius ε

centerd at P does not contain any singular point other than P and its boundary sphere $S^3(\varepsilon, P)$ intersects transversally C at γ_P circles; a link $L_P \subset S^3(\varepsilon, P)$. Milnor proves that $S^3 - L_P$ is a fiber boundle over a circle with a fiber a surface with γ_P holes.

The first betti number μ_P of the fiber will be called <u>Milnor multiplicity</u> of P. Then he obtains;

<u>Milnor formula</u> : $\quad 2\delta_P = \mu_P + \gamma_P - 1.$

1.3. We would like to know the genuine euler number $e(C)$ of the plane curve C. By definition of a non-singular model M of C, there is a resolution (or normalization)

$$\varphi : M \longrightarrow C$$

satisfying

(1) $\quad \varphi^{-1}(\Sigma C)$ is a finite set of points and

(2) $\quad \varphi | M - \varphi^{-1}(\Sigma C) : M - \varphi^{-1}(\Sigma C) \longrightarrow C - \Sigma C$

is a biholomorphic map, see Fig. 1.

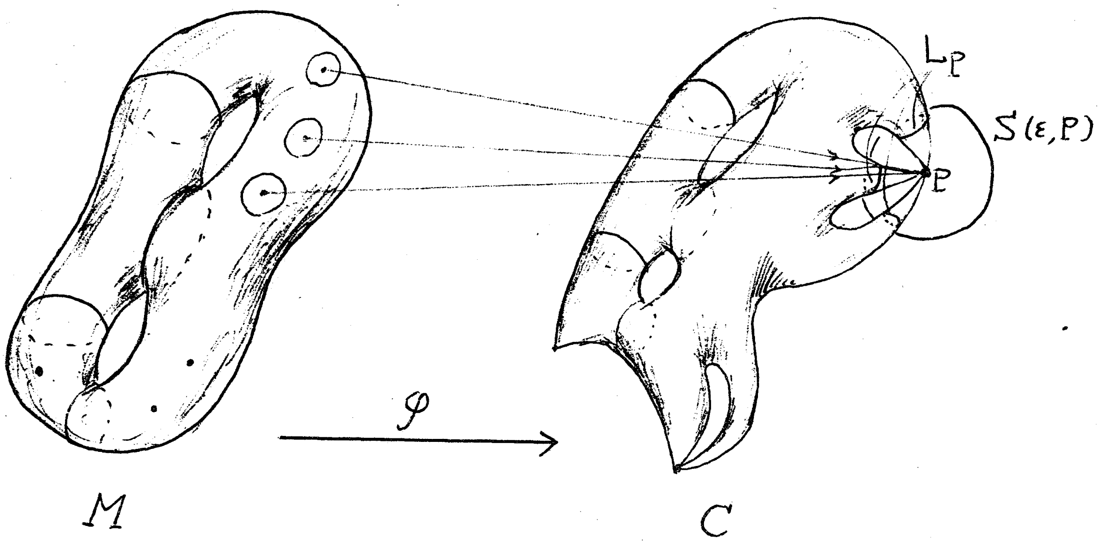

Fig 1

If $P \in C$, then the number $\# \varphi^{-1}(P)$ of points of $\varphi^{-1}(P)$ is equal to γ_P, since it is equal to the number of ends of $\varphi^{-1}(C \cap D^4(\varepsilon,P) - P)$ which is homeomorphic with $C \cap D^4(\varepsilon,P) - P$ having obviously γ_P ends for a sufficiently small number $\varepsilon > 0$. It follows that the genuine euler number $e(C)$ and the euler number $e(M)$ of the non-singular model M are related in a formula;

$$e(C) = e(M) - \sum_{P \in \Sigma C}(\gamma_P - 1)$$

Since $e(M) = 2 - 2g$, in this context, Plücker formula may be regarded as a formula giving the euler number $e(C)$ of C.

Combining Plücker and Milnor formulae we have that

$$\begin{aligned} e(M) &= 2 - 2g \\ &= 2 - \{(d-1)(d-2) - \sum_{P \in \Sigma C} 2\delta_P\} \\ &= 3d - d^2 + \sum_{P \in \Sigma C}(\mu_P + \gamma_P - 1). \end{aligned}$$

Hence we have;

Milnor-Plücker formula : $e(C) = 3d - d^2 + \sum_{P \in \Sigma C} \mu_P$.

2. Topological proof of Milnor-Plücker formula.

We ask ourselves the topological significance of Milnor-Plücker formula. First of all we remark the following two facts:

(I). The term $3d - d^2$ is the euler number of a non-singular projective plane curve of degree d.

(II). The Milnor multiplicity μ_P which appears in the formula as a correction term at each singular point P is equal to the middle betti number of

$$F_P = f^{-1}(a) \cap D^4(\varepsilon, P)$$

for a local equation f of C at P, for a sufficiently small number $\varepsilon > 0$ and a complex number a with sufficiently small absolute value $|a|$.

2.1. Difinition of an almost complex submanifold.

If we recall the notion of an almost complex submanifold, then the fact (I) follows from the adjunction formula, refer Hirzebruch [1].

A smooth 2n-submanifold M of an almost complex manifold W with a distinguished complex tangent bundle $\tau_{\mathbb{C}}W$ is almost complex, if the tangent bundle τM of M and the normal bundle νM of M in W are reduced to complex bundles $\tau_{\mathbb{C}}M$ and $\nu_{\mathbb{C}}M$ so that $\tau_{\mathbb{C}}M \oplus \nu_{\mathbb{C}}M$ is isomorphic with $\tau_{\mathbb{C}}W|M$ as complex bundles.

For an almost complex submanifold M of W we have the so-called adjunction formula;

$$c(\tau_{\mathbb{C}}M) \cdot c(\nu_{\mathbb{C}}M) = j^{*}c(\tau_{\mathbb{C}}W) ,$$

where $c(\ )$ denotes the total chern class and $j : M \hookrightarrow W$ is the inclusion map.

Let M be a smooth closed connected 2-dimensional submanifold of $\mathbb{C}P^2$ which represents a homology class $d\cdot[\mathbb{C}P^1]$, where $[\mathbb{C}P^1]$ denotes the canonical generator of $H_2(\mathbb{C}P^2;\mathbb{Z})$ represented by a projective line.

The following theorem implies the fact (I) and tells us the condition that M is "almost complex" in $\mathbb{C}P^2$ is very restrictive.

2.2. Theorem. *The surface M is almost comlpex in $\mathbb{C}P^2$ if and only if $e(M) = 3d - d^2$.*

Proof. Since the isomorphism classes of complex line bundles are completely determined by the 1-st chern classes which, in our case, should be equal to the euler classes of underlying real plane bundles, it follows from the adjunction formula that

M is almost complex in $\mathbb{C}P^2$

if and only if

$$X(\tau M) + X(\nu M) = j^{*}c_1(\mathbb{C}P^2) ,$$

where $X(\ )$ denotes the euler class.

If we evaluate these cohomology classes on the fundamental class $[M]$ of M, then we have that

$$X(\tau M) \cap [M] = e(M)$$

$$X(\nu M) \cap [M] = (\text{the self-intersection number of } M \text{ in } \mathbb{C}P^2) = d^2.$$

and

$$\begin{aligned} & j^* c_1(\mathbb{C}P^2) \cap [M] \\ &= c_1(\mathbb{C}P^2) \cap j_*[M] \\ &= 3\alpha \cap d\cdot[\mathbb{C}P^1] \\ &= 3d \ , \end{aligned}$$

where α is the dual of $[\mathbb{C}P^1]$.

Hence $X(\tau M) = j^*c_1(\mathbb{C}P^2) - X(\nu M)$ if and only if $e(M) = 3d - d^2$, completing the proof.

2.3. <u>A construction of an almost complex submanifold which "approximates" a plane curve</u> C.

Now we are in the crux of our observation.

Let C be a curve in a complex 2-dimensional manifold W. We would like to replace a singular part $C \cap D(\varepsilon,P)$ (in some local chart ; $\varepsilon > 0$ is sufficiently small) by a non-singular "smooth" part in $D(\varepsilon,P)$ for each $P \varepsilon \Sigma C$ to get an almost complex submanifold $\tilde{C}$ in W.

For this we take sufficiently small numbers $\varepsilon > \varepsilon' > 0$ so that a local equation

$$f : \mathbb{C}^2 \longrightarrow \mathbb{C} \text{ of } C \text{ arround } P$$

satisfies that

$f|D(\varepsilon,P) - P$ has no singular point and for some number $\delta > 0$, it has any value a with $|a| \leq \delta$ as a regular value, and for any $0 \leq \delta' \leq \delta$,

$$f^{-1}(0) \cap A[\varepsilon';\varepsilon;P] \text{ and } f^{-1}(\delta') \cap A[\varepsilon';\varepsilon;P]$$

are "almost parallel", where

$$A[\varepsilon';\varepsilon;P] = D(\varepsilon,P) - \mathrm{Int}\, D(\varepsilon';P).$$

Then we connect $f^{-1}(0) \cap S(\varepsilon,P)$ with $f^{-1}(\delta) \cap S(\varepsilon';P)$ by a bridge N in $f^{-1}[0,\delta] \cap A[\varepsilon';\varepsilon;P]$ such that N is diffeomorphic with $(f^{-1}(0) \cap S(\varepsilon,P)) \times [0, 1]$ and $f(N) = [0, \delta]$.

By a suitable choice of ε' and δ, we may take the bridge N so that it has a very mild slope in the sense that at each point $Q \in N$, tangent spaces of $f^{-1}(f(Q))$ and N at Q are sufficiently close as points in the grassmanien and that if we put

$$\tilde{C}(P) = (C - D(\varepsilon,P) - \Sigma C) \cup N \cup (f^{-1}(\delta) \cap D(\varepsilon';P)),$$

then $\tilde{C}(P)$ is a smooth submanifold.

We claim that $\tilde{C}(P)$ <u>is an almost complex submanifold in</u> W.

Note that $C - D(\varepsilon,P) - \Sigma C$ and $f^{-1}(\delta) \cap \mathrm{Int}\, D(\varepsilon';P)$ are already (almost) complex submanifold of W.

But N may not be almost complex. The classifying map (Gauss map in the local chart) of the tangent space of N restricted on ∂N is that of complex submanifold parts and hence it maps ∂N into the complex grassmannien. On the interior of N, the Gauss map of N is sufficiently close to a complex line field given by the complex tangent space of $f^{-1}(f(Q))$ at each point $Q \in \mathrm{Int}\, N$, since $f^{-1}(0) \cap A[\varepsilon';\varepsilon;P]$ and $f^{-1}(\delta') \cap A[\varepsilon';\varepsilon;P]$ are almost parallel for $\delta' \in f(N) = [0, \delta]$.

It follows that the Gauss map of N is homotopic to a map into a complex grassmannien relative to ∂N. This implies readily that $\tilde{C}(P)$ is almost complex in W.

Repeating this modification at each singular point we obtain an almost complex submanifold $\tilde{C} = \tilde{C}(\Sigma C)$. This almost complex submanifold $\tilde{C}$ will be called an <u>almost complex resolution of</u> C <u>in</u> W. If we take ε sufficiently small, then we could say that $\tilde{C}$ "approximates" C. In particular, it is clear that $\tilde{C}$ and C represent homologous cycles.

2.4. The topological proof of Milnor-Plücker formula.

By the sum formula of the euler numbers we have that

$$e(C) = e(\tilde{C}) - \sum_{P\varepsilon\Sigma C} e(F_P)$$
$$+ \sum_{P\varepsilon\Sigma C} e(C \cap D(\varepsilon,P)) ,$$

since $e(N) = e(f^{-1}(0) \cap S^3(\varepsilon,P)) = 0$.

Since $C \cap D(\varepsilon,P)$ is a cone, it follows from the fact (II) that

$$e(F_P) - e(C \cap D(\varepsilon,P)) = (1-\mu_P) - 1 = -\mu_P .$$

Note that if C is of degree d, then C represents a homology class $d\cdot[\mathbb{C}P^1]$ and hence so does $\tilde{C}$. By making use of Theorem 2.2 we have, therefore, the required formula :

$$e(C) = e(\tilde{C}) + \sum_{P\varepsilon\Sigma C} \mu_P$$
$$= 3d - d^2 + \sum_{P\varepsilon\Sigma C} \mu_P ,$$

completing the proof.

3. Conclusion.

The arguments in 2.3 which guarantees us the existence of an almost complex resolution of a curve in a complex surface work equally well in the case of complex hypersurfaces with isolated singularities to give an extension of Milnor-Plücker formula.

For this Milnor multiplicity μ_P is defined to be the degree of a map germ

$$\left(\frac{\partial f}{\partial z_0}, \ldots , \frac{\partial f}{\partial z_n}\right): (C^{n+1}, P) \Longrightarrow (C^{n+1}, 0)$$

where f is a local equation of the hypersurface arround P.

Indeed, μ_P is equal to the middle betti number of $F_P = f^{-1}(a) \cap D^{2n+2}(\varepsilon, P)$ for a sufficiently small non-zero complex number a and for a sufficiently small real number $\varepsilon > 0$.

It is known that every complex analytic set admits an integral fundamental class. A complex hypersurface V in $\mathbb{C}P^{n+1}$ is <u>of degree</u> d, if V represents a homology class $d \cdot [\mathbb{C}P^n]$.

<u>Theorem</u> A. <u>Let</u> V <u>be a complex hypersurface in</u> $\mathbb{C}P^{n+1}$ <u>of degree</u> d <u>with isolated singularities</u>. <u>Then the euler number</u> $e(V)$ of V <u>is given by</u>

$$e(V) = \frac{1}{d}\{(1-d)^{n+2} + (n+2)d - 1\} + (-1)^{n+1}\sum_{P \varepsilon \Sigma V} \mu_P .$$

Furthermore, Milnor-Plücker formula could be extended to the case of complete intersections of hypersurfaces.

A complex algebraic subset V in $\mathbb{C}P^{n+m}$ is <u>almost regular</u>, if V is defined by m homogeneous polynomials $f_1, \ldots, f_m$ and each point P of V is at most an isolated singular point of each of hypersurfaces defined by $f_1, \ldots, f_m$. If $f_1, \ldots, f_m$ define hypersurfaces of degrees $d_1, \ldots, d_m$, respectively, we shall say that the almost regular subset V in $\mathbb{C}P^{n+m}$ is of type $(d_1, \ldots, d_m)$. Milnor multiplicity μ_P at a point P is defined to be the middle betti number of

$$F_P = \mathfrak{f}^{-1}(\mathfrak{a}) \cap D^{2(n+m)}(\varepsilon, P)$$

for sufficiently small $\mathfrak{a} \varepsilon \mathbb{C}^{n+m}$ and $\varepsilon > 0$, where $\mathfrak{f} : \mathbb{C}^{n+m} \longrightarrow \mathbb{C}^m$ is the local equation of V arround P which is $(f_1, \ldots, f_m)$ restricted on a chart of $\mathbb{C}P^{n+m}$.

It could be shown that μ_P is invariant under analytic change of coordinates (the proof is not so easy) and $e(F_P) = 1 + (-1)^n \mu_P$.

For integers $n, d_1, \ldots, d_m$, let $c(n; d_1, \ldots, d_m)$ be a polynomial in the generator $\alpha \varepsilon H^2(\mathbb{C}P^{n+m}, \mathbb{Z})$ which is a part of degree $\leq n$ of a formal power series in α defined by $(1+\alpha)^{n+m+1}(1+d_1\alpha)^{-1} \cdot \ldots \cdot (1+d_m\alpha)^{-1}$.

The coefficient of α^n in $c(n;d_1, \ldots, d_m)$ is denoted by $c(n;d_1, \ldots, d_m)^{(n)}$.

Note that if $m = 1$, then we have that

$$c(n;d)^{(n)} = \frac{1}{d}\{(1-d)^{n+2} + (n+2)d - 1\} .$$

Theorem B. Let V be an almost regular subset of type $(d_1, \ldots, d_m)$ in $\mathbb{C}P^{n+m}$. Then the euler characteristic $e(V)$ of V is given by

$$e(V) = d_1 \cdot \ldots \cdot d_m \cdot c(n;d_1, \ldots, d_m)^{(n)} + (-1)^{n+1} \sum_{P \in \Sigma V} \mu_P$$

This enables us to determine the total chern homology class $\mathcal{C}(V)$ of V as follows :

Theorem C. Let V be an almost regular subset of type $(d_1, \ldots, d_m)$ in $\mathbb{C}P^{n+m}$.

Then we have that

$$j_* \mathcal{C}(V) = d_1 \cdot \ldots \cdot d_m \cdot c(n;d_1, \ldots, d_k) \frown [\mathbb{C}P^n] + (-1)^{n+1} \sum_{P \in \Sigma V} \mu_P$$

4. Problems and remarks.

We would like to know how much topological methods could contribute to the study of "topology of analytic sets (with singularities)".

We have provided the notion of an almost complex resolution of an analytic subset in a complex manifold, and seen that existence of it reduces the study of the subset to the study of local properties of singularities. Unfortunately, we have the existence theorem only for almost regular subsets.

Problem. 1. Under what condition does almost complex resolution exist ?

If we weaken the almost complex category to the smooth category, then we will see that any hypersurface admits a smooth resolution in the ambient complex manifold, [3]. Indeed, Theorem C is proved by making use of this.

The general notion of topological resolution will be found in [3]. Above all we should study the following :

Problem. 2. (Milnor-Plücker problem.)

Determine the euler number of a complex projective hypersurface (with non-isolated singularity) in terms of its degree and some invariants arround the singular set.

As for the mod 2 euler number, we have a solution in [3]. For a hypersurface in $\mathbb{C}P^3$ with "non-isolated nice singularity" Kodaira gives a solution by making use of classical methods.

Here is a technical problem.

Problem. 3. Give a topological definition of Milnor multiplicity μ_P at a singular point P of an almost regular subset in a projective space which readily implies that μ_P is a local analytic invariant.

At this conference, Hirzebruch and Morita called our attention to the index of an almost regular subset V in $\mathbb{C}P^{2n+m}$, refer Hirzebruch [2] and Morita [5].

For this, we define the index of V at P in $\mathbb{C}P^{2n+m}$ to be the index of $F_P = f^{-1}(a) \cap D^{2(2n+m)}(\varepsilon, P)$ and denote it by τ_P. Then the index $\tau(V)$ of V is given by $\tau(V)$ = (the index of a complete intersection of non-singular hypersurfaces of degrees $d_1, \ldots, d_m$) $- \sum_{P \in \Sigma V} \tau_P$.

For the first term in the right hand side of the formula above see p. 160, [1].

Reference

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