数理解析研究所講究録
158巻 1972年 12-18

The notion of near-complex subvariety

by H. Hironaka

(Note by S. Nakano. Nakano is responsible for possible errors.)

§1. Introduction Let X be a compact complex analytic manifold and suppose X is diffeomorphic to the complex projective space $\mathbb{P}^n(\mathbb{C})$. Is, then, X analytically isomorphic to $\mathbb{P}^n(\mathbb{C})$?

F.Hirzebruch and K.Kodaira [1] solved this problem affirmatively under the condition that X carries a Kähler metric (and an additional condition when $n = \dim_{\mathbb{C}}(X)$ is even). In this lecture we discuss this problem without Kähler condition, but in the case when X is a member of a smooth family of complex analytic manifolds, other members being isomorphic to $\mathbb{P}^n(\mathbb{C})$. The conclusion is:

Main Theorem. Let $\pi:\mathfrak{X}\longrightarrow D$ be a smooth proper morphism from a complex analytic manifold $\mathfrak{X}$ onto the unit disc $D = \{x \in \mathbb{C} \mid |x| < 1\}$, so that $\pi^{-1}(x)$ is a compact complex analytic manifold of dimension n (independent of x) for every $x \in D$. Suppose $\pi^{-1}(x)$ is analytically isomorphic to $\mathbb{P}^n(\mathbb{C})$ except for $\pi^{-1}(0) = X_0$, then X_0 is also isomorphic to $\mathbb{P}^n(\mathbb{C})$.

In §2 we shall indicate how Kähler condition was made use of in the proof of Hirzebruch-Kodaira and give an example. In §3 we shall discuss the detour by which we arrive at our aim without Kähler condition, namely the notion of near-complex subvarieties.

§2. Let us consider our family

$$\pi : \mathfrak{X} \longrightarrow D.$$

We can set up a diffeomorphism f

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\quad f \quad} & X_0 \times D \\ {\scriptstyle\pi}\downarrow & \curvearrowright & \downarrow{\scriptstyle \text{proj}} \\ D & \xrightarrow{\quad \text{id} \quad} & D \end{array}$$

so that $H^{2q}(X_0,\mathbb{Z}) \cong H^{2q}(\mathbb{P}^n(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}\, g^q$, where g is the generator of $H^2(\mathbb{P}^n(\mathbb{C}),\mathbb{Z})$ dual to the hyperplane. (In reality, f can be taken to be a real analytic homeomorphism. This will be used in §3.) Hence we can speak of positive or negative cohomology classes.

Now theorem 6 in Herzebruch-Kodaira [1] can be modified to

Theorem 1 Let X be an n-dimensional compact complex analytic manifold and let a complex line bundle (invertible sheaf) L on X be given. Denote by g ($\in H^2(X,\mathbb{Z})$) the Chern class of L. Assume

(1) $\dim_{\mathbb{C}} H^\circ(X,L) \geq n+1$,

(2) $H^{2d}(X,\mathbb{Z}) = \mathbb{Z}\cdot g^d \cong \mathbb{Z}$ (for $0 \leq d \leq n$),

(3) X is a Moishezon space, (i.e. there exist n algebraically independent global meromorphic functions on X,)

(4) any complex analytic subvariety of X determines a non-negative cohomology class.

Then we can conclude that X is analytically isomorphic to $\mathbb{P}^n(\mathbb{C})$.

In case of Hirzebruch-Kodaira, Kähler condition was made use of in order to establish (4) as well as (1) and (3). In our case of a smooth family, it is not too hard to establish (1)----(3) without Kähler condition, while for (4) we need the analysis we develop in §3.

Here is an example of a manifold which satisfies (1) --- (3) but not (4):

Take a non-singular curve Γ in the product $\mathbb{P}^1 \times \mathbb{P}^1 = S$, with the property that $\Gamma\cdot(u \times \mathbb{P}^1)$ consists of three points and $\Gamma\cdot(\mathbb{P}^1 \times v)$ of 5 points for generic u and v. Embed S into $\mathbb{P}^3$ as a quadratic surface, and blow up $\mathbb{P}^3$ with Γ as the center. We obtain a projective threefold X' and X' contains the subvariety S', the proper transform of S. S' is isomorphic to S: $S' \cong S \cong \mathbb{P}^1 \times \mathbb{P}^1$. It can be seen that S' can be blown down, i.e. there exist a compact complex manifold X and a morphism $p: X' \longrightarrow X$, so that $p|_{X'-S'}$ is an isomorphism and on $S' \cong \mathbb{P}^1 \times \mathbb{P}^1$, p is nothing but the projection to the first factor.

It can be shown that X satisfies (1)---(3) but not (4). ($p(S')$ gives a negative class.) Thus we see the condition (4) is essential in the theorem.

§3 Given a complex analytic manifold Y_0, we denote by $\mathcal{T}_{Y_0,y}$ and $T_{Y_0,y}$ the complex tangent space and real tangent space respectively, to the manifold Y_0 at a point y on it. We denote by $\mathcal{T}_{Y_0}$ and T_{Y_0} the bundles $\bigcup_{y\in Y_0} \mathcal{T}_{Y_0,y}$ and $\bigcup_{y\in Y_0} T_{Y_0,y}$ respectively. We have a canonical isomorphism

$T_{Y_0} \cong \mathrm{Re}(\mathcal{T}_{Y_0} \oplus \overline{\mathcal{T}}_{Y_0})$.

Now let a compact complex analytic manifold X be given. A near-complex subvariety $\mathcal{Y}$ of X is a quadruple $(Y, Y_0, \mathcal{E}_0, \rho_0)$, where

- Y is a closed subset of X,
- Y_0 is a connected dense open subset of Y and has a structure of even dimensional oriented C^∞-manifold, (a submanifold of X with induced topology,)
- $\mathcal{E}_0$ is a complex subbundle of $\mathcal{T}_X|_{Y_0}$,
- ρ_0 is a real bundle isomorphism $\mathrm{Re}(\mathcal{E}_0 \oplus \overline{\mathcal{E}}_0) \cong T_{Y_0}$, and maps the natural orientation on $\mathrm{Re}(\mathcal{E}_0 \oplus \overline{\mathcal{E}}_0)$ to the given orientation of Y_0.

In reality, we add further conditions of technical character, e.g. Y can be triangulated so that $Y - Y_0$ be a subcomplex of dimension $\leqq \dim_{\mathbb{R}} Y_0 - 2$. In this way, Y determines a homology class in $H_*(X,\mathbb{Z})$ and, if we go over to rational coefficients, determines a cohomology class in $H^*(X,\mathbb{Q})$.

An example of a near-complex subvariety is given by an analytic subvariety Y of X. We take Y_0 to be the set of the simple points of Y and $\mathcal{E}_0 = \mathcal{T}_{Y_0}$, $\rho_0 : \mathrm{Re}(\mathcal{E}_0 \oplus \overline{\mathcal{E}}_0) \to T_{Y_0}$ is the canonical one.

Another example will appear in connection with a smooth family of compact complex analytic manifolds: Let $\pi : \mathfrak{X} \to D$ be a smooth family over the disc $D = \{x \in \mathbf{C} \mid |x| < 1\}$. We take a differentiable trivialization of $\mathfrak{X}$:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\ f\ } & X_0 \times D \\ \pi \downarrow & \curvearrowright & \downarrow \mathrm{proj} \\ D & \xrightarrow{\ id\ } & D \end{array}, \qquad X_0 = \pi^{-1}(0).$$

If Y is an analytic subvariety of X_0, then it determines a near-complex subvariety in X_0 as in the preceeding example. We put this in $X_0 \times x$ on the right hand side of the above diagram and pull back everything onto $X_x = \pi^{-1}(x)$ by f. Then a near-complex subvariety on X_x is obtained.

If we restrict ourselves to the family in the main theorem, all X_x are isomorphic to $\mathbb{P}^n(\mathbb{C})$ except $x = 0$, and we shall have a family of near-complex subvarieties

$$(*) \qquad \{\mathcal{Y}_x \mid x \in D - \{0\}\}$$

of $\mathbb{P}^n(\mathbb{C})$. These can be and <u>will be taken to be real analytic near-complex subvarieties</u>, by choosing a real analytic trivialization f.

For a near-complex subvariety $\mathcal{Y} = (Y, Y_0, \mathcal{E}_0, \rho_0)$ in $X = \mathbb{P}^n(\mathbb{C})$ with $\dim_{\mathbb{R}} Y_0 = 2(n-d)$, and for a point $y \in Y_0$, we consider the set

$$P(y) = \{L \in \mathrm{Grass}_{\mathbb{C}}(\mathcal{T}_{X,y}, d) \mid L \text{ and } T_{Y_0,y} \text{ intersect properly with multiplicity } +1\},$$

where $\mathrm{Grass}_{\mathbb{C}}(V,d)$ denotes the complex Grassmann variety of d-dimensional vector subspaces of the given vector space V. Since L and $T_{Y_0,y}$ are oriented, we can speak of positivity of the intersection. $P(y)$ is an open set of

$\mathrm{Grass}_{\mathbb{C}}(\mathcal{T}_{X,y},d)$ and $B = \bigcup_{y \in Y_0} P(y)$ is an open set of $A = \bigcup_{y \in Y_0} \mathrm{Grass}_{\mathbb{C}}(\mathcal{T}_{X,y},d)$.

Making use of a suitable metric, we estimate volumes of A and B, and define the <u>positivity rate</u> $r(\mathcal{Y})$ of $\mathcal{Y}$ by

$$r(\mathcal{Y}) = \mathrm{vol}(B)/\mathrm{vol}(A).$$

On the other hand, we define the absolute degree $\delta(\mathcal{Y})$ of $\mathcal{Y}$. This is the maximal number of intersection points of Y and the variable linear subvariety L^d of $\mathbb{P}^n(\mathbb{C})$, each intersection point being counted once irrespective of orientation, and maximum being taken for $L \in \mathrm{Grass}(\mathbb{P}^n(\mathbb{C}),d)$ -(a set of measure 0). This number is well defined because we have a real analytic near-complex subvariety.

Going back to our family $\pi: \mathcal{X} \longrightarrow D$, $\pi^{-1}(x) \cong \mathbb{P}^n(\mathbb{C})$ for $x \neq 0$, we can derive the non-negativity of the class of an analytic subvariety of $X_0 = \pi^{-1}(0)$ from the following facts:

<u>Theorem 2</u> If $\mathcal{Y}$ is a real analytic near-complex subvariety of $\mathbb{P}^n(\mathbb{C})$ with the property

$$\delta(\mathcal{Y})(1 - r(\mathcal{Y})) < 1,$$

then $\mathcal{Y}$ determines a non-negative homology class.

<u>Proposition</u> For the near-complex subvarieties $\mathcal{Y}_x$ of $\mathbb{P}^n(\mathbb{C})$ described in (*), $\delta(\mathcal{Y}_x)$ remains bounded and $r(\mathcal{Y}_x) \rightarrow 1$ for $x \rightarrow 0$. (Hermitean metrics on the fibers $\mathbb{P}^n(\mathbb{C})$ are induced by a fixed one on the total space $\mathcal{X}$.)

Reference

[1] F.Hirzebruch and K.Kodaira: On the complex projective spaces, Jour. Math. pures appl. (9) vol.36 (1957) pp.201–216