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Pontrjagin classes of rational homology manifold

(Report on work by Don Zagier [3])

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1. L-classes in the equivariant case.

Let X be a compact oriented rational homology manifold and assume that a compact Lie group G acts on X by orientation preserving simplicial homeomorphisms. Then there is defined the equivariant signature $\mathrm{sign}(g, X) \in \mathbb{C}$ for any $g \in G$ as follows (see [1]).

(i) If $\dim X \equiv 1 \pmod 2$, we put $\mathrm{sign}(g, X) = 0$.

(ii) If $\dim X = 4k$, then the cup product defines a non-degenerate quadratic form B on $H^{2k}(X; \mathbb{Q})$. Let

$$H^{2k}(X; \mathbb{Q}) = V^{+} \oplus V^{-}$$

be an equivariant decomposition of the G-vector space $H^{2k}(X; \mathbb{Q})$ such that B is positive (negative) definite on V^{+} (V^{-}). Then we define

$$\mathrm{sign}\ (g, X) = \mathrm{Trace}\ (g|V^{+}) - \mathrm{Trace}\ (g|V^{-})$$

Observe that if g acts on X trivially, then $\mathrm{sign}\ (g,X) = \mathrm{sign}\ X$, where $\mathrm{sign}\ X$ is the ordinary signature of X.

(iii) If $\dim X = 4k + 2$, then we can give a complex vector space structure to $H^{2k+1}(X; \mathbb{R})$ such that the action of G preserves this structure. We define

$$\mathrm{sign}\ (g, X) = 2i\,\mathrm{Im}\ (\mathrm{Trace}\ g|H^{2k+1}(X; \mathbb{R})).$$

Now Thom defined the Pontrjagin classes (or equivalently the L-classes) for any rational homology manifold. Then Milnor simplified the Thom's definition by using a t-regularity argument and the (ordinary) signature.

Recently Zagier has generalized this procedure to the equivariant case.

Precisely, assume that a finite group G acts on a compact oriented rational homology manifold X. Then Zagier has defined the "equivariant L-class"

$$L(g, X) \in H^*(X; \mathbb{C})$$

for any $g \in X$. This class can be used to calculate the ordinary L-class of the rational homology manifold X/G, by virtue of the following theorem. This theorem is one of the main remults of Zagier.

Theorem 1. Let G be a finite group and X a compact oriented rational homology G-manifold. Let

$$\pi : X \longrightarrow X/G$$

be the natural projection. Then

$$\frac{1}{\deg \pi} \pi^* L(X/G) = \frac{1}{|G|} \sum_{g \in G} L(g, X).$$

Here $\deg \pi$ is the degree of the map π (we do not assume that the action of G is effective) and $L(X/G)$ is the Thom-Milnor L-class of the rational homology manifold X/G.

We will sketch the definition of the class $L(g, X)$ for the case when X is a differentiable G-manifold.

The proof of Theorem 1 in the differntiable case then follows from a calculation depending on Milnor's definition of the L-class $L(X/G)$ and the Atiyah-Singer G-signature theorem.

The general case (i.e. the case when X is only a rational homology G-manifold) follows from a parallel extension in the equivariant context of Milnor's argument.

Thus let X be a compact oriented differentiable G-manifold, where G is a finite group. Let $X^g = \{x \in X \mid gx = x\}$, the fixed point set of g. Then by Atiyah-Singer [1], Theorem (G-signature theorem)

$$\operatorname{sign}(g, X) = L'(g, X)\,[X^g]$$

for a certain class $L'(g, X) \in H^*(X^g; \mathbb{C})$, defined below.

Now the right hand side of the above equation depends only on the top dimensional components of the class $L'(g, X^g)$. However to define the equivariant L-class, lower terms of $L'(g, X)$ are also necessary. Since the "correct" class $L'(g, X)$ for our purpose differs from the original one defined by Atiyah-Singer [1] by powers of two, we define it explicitely.

Let N^g be the normal bundle of X^g in X. Then N^g can be decomposed equivariantly as follows,

$$N^g = N^g(-1) \oplus \sum_{0<\theta<\pi} N^g_\theta$$

where $N^g(-1)$ is a real bundle over X^g on which g acts as -1. N^g_θ is a complex bundle on which g acts as $e^{i\theta}$. We define

$$L_{-1}(N^g(-1)) = e(N^g(-1))L(N^g(-1))^{-1}$$

where $L(N^g(-1))$ is the L-class of the real bundle $N^g(-1)$ and $e(N^g(-1))$ is the Euler class. For the complex part N^g_θ, we define

$$L_\theta(N^g_\theta) = \left(\coth\frac{i\theta}{2}\right)^q \prod_j \frac{\coth(X_j + \frac{i\theta}{2})}{\coth\frac{i\theta}{2}}$$

where $q = \dim_{\mathbb{C}} N^g_\theta$ and X_j is the usual formal class such that the Chern classes are the elementary symmetric polynomials in x_j's. Now we define

$$L'(g, X) = L(X^g)L_{-1}(N^g(-1)) \prod_{0<\theta<\pi} L_\theta(N^g_\theta).$$

We are now prepared to define the equivariant L-class, $L(g, X)$. Let $j: X^g \hookrightarrow X$ be the inclusion map. Then we simply define

$$L(g, X) = j_! L'(g, X)$$

where $j!$ is the Gysin homomorphism.

We will give two applications on Theorem 1 in §§2, 3.

One is the case of linear actions of complex projective space $P_n\mathbb{C}$ (§2) and the other is the action of the symmetric group of degree n, S_n, on

$$S^n = \underbrace{X \times \cdots \times X}_{n \text{ times}} \qquad (\S 3)$$

2. Complex projective space

Let $P_n\mathbb{C} = \{[z_0, z_1, \cdots, z_n] \mid z_i \in \mathbb{C}\}$ be n-dimensional complex projective space. We define a finite group G_a by

$$G_a = G_{a_0} \times G_{a_1} \times \cdots \times G_{a_n}$$
$$G_{a_j} = \{\lambda \mid \lambda^{a_j} = 1\}.$$

Then G_a acts on $P_n\mathbb{C}$ by

$$(\lambda_0, \lambda_1, \cdots, \lambda_n)[z_0, z_1, \cdots, z_n] = [\lambda_0 z_0, \lambda_1 z_1, \cdots, \lambda_n z_n]$$
$$(\lambda_0, \lambda_1, \cdots, \lambda_n) \in G_a, \quad [z_0, z_1, \cdots, z_n] \in P_n\mathbb{C}.$$

Let $\pi : P_n\mathbb{C} \longrightarrow P_n\mathbb{C}/G_a$ be the natural projection. Then Bott has calculated

Theorem 2 (Bott)

$$\pi^* L(P_n\mathbb{C}/G_a) = \frac{1}{d} \sum_{0 \le \xi < \pi} \prod_{j=0}^{n} \frac{a_j x}{\tanh(a_j(x + i\xi))}$$

where d is the greatest common divisor of the natural numbers $a_0, a_1, \cdots, a_n$ and $x \in H^2(P_n\mathbb{C})$ is the standard generator.

The sum on the right hand side is taken over all real numbers $\xi \in [0, \pi)$. However the product $\prod_{j=0}^{n} \frac{a_j x}{\tanh(a_j(x + i\xi))}$ is equal to zero, unless there is at least one a_j such that $a_j \xi$ is a multiple of π (because $x^{n+1} = 0$). Therefore the sum is well-defined.

Now this theorem can be obtained from Theorem 1 as follows.

Proof of Theorem 2 using Theorem 1. By Theorem 1, we have

$$(1) \qquad \pi^* L(P_n\mathbb{C}/G_a) = \frac{\deg \pi}{|G_a|} \sum_{g \in G_a} L(g, P_n\mathbb{C}).$$

But it is easy to see that

$\frac{\deg \pi}{|G_a|} = \frac{1}{d}$. Hence we have only to show that

(2) $$\sum_{g\in G_a} L(g, P_n\mathbb{C}) = \sum_{0\le\xi<\pi} \prod_{j=0}^{n} \frac{a_j x}{\tanh (a_j(x+i\xi))} .$$

Let $g = (\zeta_0, \zeta_1, \dots, \zeta_n)$, $\zeta_j \in G_{a_j}$. Then

(3) $$P_n\mathbb{C}^g = \{ [z_0, z_1, \dots, z_n] \mid \zeta_j z_j = \zeta z_j \quad \text{for } j = 0, 1, \dots, n \quad \text{some } \zeta\in S^1\} = \bigcup_{\zeta\in S^1} X(\zeta)$$

where $X(\zeta) = \{ [z_0, z_1, \dots, z_n]\in P_n\mathbb{C}^g \mid \zeta_j z_j = \zeta z_j \text{ for all } j \}$.

Clearly if $\zeta\notin\{\zeta_0, \zeta_1, \dots, \zeta_n\}$. Then $X(\zeta) = \emptyset$, while $X(\zeta_j)$, is isomorphic to $P_s\mathbb{C}$, where $s+1$ is the number of indices i with $\zeta_i = \zeta_j$.

Now by the definition of the equivariant L-class, we have

$L(g, P_n\mathbb{C}) = j_! L'(g, P_n\mathbb{C})$ where $j: P_n\mathbb{C}^g \to P_n\mathbb{C}$ is the inclusion.

Thus we must calculate the class

$L'(g, P_n\mathbb{C}) \in H^*(P_n\mathbb{C}^g; \mathbb{C})$.

Let $L'(g, P_n\mathbb{C})_\zeta$ be the component of $L'(g, P_n\mathbb{C})$ corresponding to the connected component $X(\zeta)\subset P_n\mathbb{C}^g$.

As mentioned earlier, $X(\zeta)$ is isomorphic to $P_s\mathbb{C}$ and it is easy to check that, to calculate $L(g, P_n\mathbb{C})_\zeta$, we may assume that

$X(\zeta) = P_s\mathbb{C}\subset P_n\mathbb{C}$, where

$P_s\mathbb{C} = \{ [z_0, z_1, \dots, z_s, 0, \dots, 0]\in P_n\mathbb{C}\}$.

Now let $j: P_s\mathbb{C} \to P_n\mathbb{C}$ be the inclusion and let N be the normal bundle of $P_s\mathbb{C}$ in $P_n\mathbb{C}$. Then clearly $y = j^* x$ is a generator of $H^2(P_s\mathbb{C})$.

We study the action of g on N. Since

$$g[z_0, z_1, \dots, z_s, z_{s+1}, \dots, z_n] = [\zeta_0 z_0, \zeta_1 z_1, \dots, \zeta_s z_s, \zeta_{s+1} z_{s+1}, \dots, \zeta_n z_n]$$
$$= [\zeta z_0, \zeta z_1, \dots, \zeta z_s, \zeta_{s+1} z_{s+1} \dots \zeta_n z_n]$$

$$=[z_0,\ z_1,\ldots,z_s,\ \zeta^{-1}\zeta_{s+1}z_{s+1},\ldots,\ \zeta^{-1}\zeta_n z_n],$$

we have

$$N = \sum_\theta N_\theta,$$

where $N_\theta = 0$ unless $\theta = \zeta^{-1}\zeta_j$ for some $j=s+1, \ldots n$ and $N_{\zeta^{-1}\zeta_j}$ is a complex line bundle over $P_s\mathbb{C}$. We obtain

$$(4)\quad L'(g,\ P_n\mathbb{C})_\zeta = L(P_s\mathbb{C}) \prod_\theta L_\theta(N_\theta) = \left(\frac{x}{\tanh y}\right)^{s+1} \prod_{j=s+1}^{n} \frac{\zeta^{-1}\zeta_j e^{2y}+1}{\zeta^{-1}\zeta_j e^{2y}-1} .$$

Therefore

$$(5)\quad L(g,\ P_n\mathbb{C})_\zeta = j^!_\circ\, L'(g,\ P_n\mathbb{C})_\zeta = \left(\frac{x}{\tanh x}\right)^{s+1} \cdot \prod_{j=s+1}^{n} \frac{\zeta^{-1}\zeta_j e^{2x}+1}{\zeta^{-1}\zeta_j e^{2x}-1} \cdot x^{n-s}$$

$$= \prod_{j=0}^{n} \left(x \frac{\zeta^{-1}\zeta_j e^{2x}+1}{\zeta^{-1}\zeta_j e^{2x}-1} \right) .$$

Observe that the right hand side of (5) is equal to zero unless $\zeta \in \{\zeta_0,\ \zeta_1,\ \ldots,\ \zeta_n\}$.

Now we can show (2) by using the trigonometric identity

$$\sum_{\lambda^a=1} \frac{\lambda z + 1}{\lambda z - 1} = a\,\frac{z^a + 1}{z^a - 1} .$$

Thus

$$(6)\quad \sum_{g\in G_a} L(g,\ P_n\mathbb{C}) = \sum_{\zeta_0,\ldots,\zeta_n} \sum_{\zeta\in S^1} \prod_{j=0}^{n} \left(x \frac{\zeta^{-1}\zeta_j e^{2x}+1}{\zeta^{-1}\zeta_j e^{2x}-1} \right)$$

$$= \sum_{\zeta\in S^1} \prod_{j=0}^{n} \left(x \cdot \sum_{\zeta_j^{a_j}=1} \frac{\zeta^{-1}\zeta_j e^{2x}+1}{\zeta^{-1}\zeta_j e^{2x}-1} \right)$$

$$= \sum_{\zeta\in S^1} \prod_{j=0}^{n} \left(a_j x \frac{\zeta^{-a_j} e^{2a_j x}+1}{\zeta^{-a_j} e^{2a_j x}-1} \right)$$

$$= \sum_{0\le\xi<\pi} \prod_{j=0}^{n} \frac{a_j x}{\tanh(a_j(x+i\xi))} . \qquad \text{(Q.E.D.)}$$

Now suppose $a_0,\ a_1,\ \ldots,\ a_n$ are mutually relatively prime numbers. Then by Theroem 2, we have

$$(7)\quad \pi^* L(P_n\mathbb{C}/G_a) = \prod_{j=0}^{n} \frac{a_j x}{\tanh a_j x} \bmod x^n .$$

Therefore, in terms of the total Pntrjagin class p, we have

(8) $\pi^* p(P_n\mathbb{C}/G_a) = \prod_{j=0}^{n} (1 + a_j^2 x^2) \bmod x^n$

Suppose n is even, say $n=2k$, then there arises a natural question;

Question. Are there values of $a_o, a_1, \ldots, a_{2k}$ such that (8) holds also in the highest term?

Now suppose $\{a_0, a_1, \ldots, a_{2k}\}$ satisfies the requirement of the Question. Then

(9) $\pi^* p(P_n\mathbb{C}/G_a) = \prod_{j=0}^{n} (1 + a_j^2 x^2).$

Since the action of G_a extends to an action of the torus T^{n+1}, we have

(10) $\pi^* : H^*(P_n\mathbb{C}/G_a;\ \mathbb{Q}) \cong H^*(P_n\mathbb{C};\ \mathbb{Q}).$

Hence

(11) $\operatorname{sign} P_n\mathbb{C}/G_a = 1.$

On the other hand, $P_n\mathbb{C}/G_a$ is a rational homology manifold. Therefore its signature is equal to the L-genus. From (9) and (11), we obtain

(12) $L_k(p_1, \ldots, p_k) = a_0 a_1 \ldots a_{2k}$

where p_j is the j-th elementary symmetric polynomial in a_j^2's. Conversely assume that (12) holds. Then it is easy to see that $\{a_0, a_1, \cdots, a_{2k}\}$ satisfies the requirement of the Question. Thus we have obtained

Proposition 3. Let $a_0, a_1, \cdots, a_{2k}$ be mutually relatively prime natural numbers $\geqq 1$. Then

$$\pi^* p(P_n\mathbb{C}/G_a) = \prod_{j=0}^{n} (1 + a_j^2 x^2)$$

if and only if $\{a_j\}$ satisfies the Diophantine equation

$$L_k(p_1, \cdots, p_k) = a_0 a_1 \cdots a_{2k}.$$

For $k = 1$, the equation (12) is

$$a_0^2 + a_1^2 + a_2^2 = 3a_0 a_1 a_2$$

and all solutions are known (see [2]). For k=2, the equation is

$$7(a_0^2a_1^2 + a_0^2a_2^2 + \dots + a_3^2a_4^2) - (a_0^2 + \dots + a_4^2)^2 = 45a_0a_1\dots a_4 .$$

Are there infinitely many solutions?

It is easy to check that (1, 1, 1, 1, 1) and (2, 1, 1, 1, 1) are solutions. Recently Zagier has found a solution (2, 7, 19, 47, 59) using a computer. Up to permutation, these are the only solutions in mutually relatively prime natural numbers ≤ 100.

3. L-classes of symmetric products.

Let X be a closed oriented differentiable manifold and let X^n be the n-th Cartesian product of X. Then the symmetric group of degree n, S_n, acts on X^n by permuting the factors.

Now if $\dim X$ is even, say $2s$, then this action is orientation preserving. Thus we can apply the result of §1.

Let $X(n) = X^n/S_n$ be the n-th symmetric product of X. If we choose a fixed point $x_0 \in X$, we have natural inclusions

$$X = X(1) \subset X(2) \subset \dots \subset X(\infty)$$

where $X(\infty) = \varinjlim_n X(n)$. We will write j for any inclusion map $j: X(n) \longrightarrow X(m)$, $\infty \geq m \geq n$.

Now if we use $\mathbb{Q}$ for the coefficient of the cohomology,we have

(13) $\quad H^*(X(n)) \cong H^*(X^n)^{S_n}$

where the right hand side is the S_n-invariant subgroup of $H^*(X^n)$. Henceforth we will identify these two groups by the above isomorphism.

It is rather easy to calculate $H^*(X(n))$. Let $\{f_0, f_1, \dots, f_b\}$ be a homogeneous basis for $H^*(X)$ with $f_0 = z \in H^{2s}(X)$, the cohomology fundamental class and $f_b = 1$. Let $n_0, \dots, n_b$ be non-negative integers with $n_0 + n_1 + \dots + n_b = n$. We define an element

$\langle n_0f_0\ldots n_bf_b\rangle \in H^*(X(n))$ as follows.

Let $\sigma \in S_n$. Then σ acts on X^n by

$$\sigma(x_1,\ldots, x_n) = (x_{\sigma(1)},\ldots, x_{\sigma(n)}).$$

We define an element

$$\langle u_1, u_2,\ldots, u_n\rangle \in H^*(X(n)) \quad \text{for } u_j \in H^*(X)$$

by $$\langle u_1, u_2,\ldots, u_n\rangle = \sum_{\sigma\in S_n}\sigma^*(u_1\times\ldots\times u_n)\in H^*(X^n)^{S_n} = H^*(X(n))$$

and we put

$$\langle n_0f_0\ldots n_bf_b\rangle = \langle \underbrace{f_0,\ldots, f_0}_{n_0},\ldots, \underbrace{f_b,\ldots, f_b}_{n_b}\rangle .$$

Then it can be shown that

Proposition 4. The elements $\langle n_0f_0\ldots n_0f_0\rangle$ with $n_0+\ldots+n_b = n$ and $n_i \leqq 1$ if degree f_i is odd, form a basis for $H^*(X(n))$.

Now we define an element $[n_0f_0\ldots n_{b-1}f_{b-1}]_n \in H^*(X(n))$ by

$$[n_0f_0\ldots n_{b-1}f_{b-1}] = \begin{cases} 0 & \text{if } n < n_0+\ldots+n_{b-1} \\ (n_b!)^{-1}\langle n_0f_0\ldots n_bf_b\rangle & \text{if } n_b = n-(n_0+\ldots+n_{b-1}) \geqq 0 \end{cases}$$

Then it can be seen that

(14) $$j^*[n_0f_0\ldots n_{b-1}f_{b-1}]_{n+1} = [n_0f_0\ldots n_{b-1}f_{b-1}]_n$$

Thus the elements $[n_0f_0\ldots n_{b-1}f_{b-1}]_n$ $(n = 1, 2, \ldots)$ defines an element $[n_0f_0\ldots n_{b-1}f_{b-1}] \in H^*(X(\infty))$ so that

(15) $$j^*[n_0f_0\ldots n_{b-1}f_{n-1}] = [n_0f_0\ldots n_{b-1}] = [n_0f_0\ldots n_{b-1}f_{b-1}]_n$$

where $j\colon X(n) \longrightarrow X(\infty)$. We write η for the element $[1f_0] \in H^{2s}(X(\infty))$. Then $\eta_n = [1f_0]_n = \sum_{i=1}^{n}\pi_i^*z$, where $\pi_i\colon X^n \longrightarrow X$ is the projection on the i-th factor. Then it can be shown that

(16) $$[n_0f_0\ldots n_{b-1}f_{b-1}] = \eta^{n_0}[n_1f_1\ldots n_{b-1}f_{b-1}]$$

and

Proposition 5. The elements

$\eta^{n_0}[n_1f_1\ldots n_{b-1}f_{b-1}]$ form a basis for $H^*(X(\infty))$ and

$$j^*(\eta^{n_0}[n_1f_1\ldots n_{b-1}f_{b-1}]) = \eta_n^{n_0}[n_1f_1\ldots n_{b-1}f_{b-1}]_n,$$

where $j: X(n) \longrightarrow X(\infty)$.

In terms of these elements of $H^*(X(\infty))$, we can write the second main result of Zagier.

Theorem 6. Let X be a connected closed oriented differentiable manifold of dimension $2s$. Let $j: X(n) \longrightarrow X(\infty)$ be the inclusion. Then there is a class $G \in H^{**}(H(\infty))$ such that

$L(X(n)) = j^*(Q_s(\eta)^{n+1} G)$ where $Q_s(t)$ is a power series defined by

$$Q_s(t) = \frac{t}{f_s(t)},$$

$$f_s(t) = g_s^{-1}(t), \qquad g_s(t) = t + \frac{t^3}{3^s} + \frac{t^5}{5^s} + \frac{t^7}{7^s} + \dots .$$

Equivalently, let $j: X(n) \longrightarrow X(n+1)$ be the inclusion, Then

$$j^* L(X(n+1)) = Q_s(\eta_n) \cdot L(X(n)).$$

The proof consists of a rather long and complicated calculation applying Theorem 1. Here we concentrate on the cases when $X = S^{2s}$ and $S = 1$ and make some remarks.

Thus assume first that $X = S^{2s}$. Then the basis for $H^*(X)$ is just $\{z, 1\}$ and the class G that appeared in Theorem 6 can be simply expressed and the result is

Proposition 7. Let $X = S^{2s}$. Then

$$L(X(n)) = \frac{f_s'(\eta)}{1 - f_s(\eta)^2} \left(\frac{\eta}{\tanh \eta}\right)^{n+1}$$

where f' denotes the derivative of f.

Now if $S = 1$, then $X(n)$ can be naturally identified with $P_n\mathbb{C}$ and $\eta_n \in H^2(X(n))$ is the standard generator. In this case Prop. 7 simply says the well-known result

$$L(P_n\mathbb{C}) = \left(\frac{\eta}{\tanh \eta}\right)^{n+1} .$$

Next assume that $S = 1$. Thus let X be a Riemann surface of genus g. We choose a basis $\{\alpha_1, \dots, \alpha_g, \alpha_1', \dots, \alpha_g'\}$ for $H'(X)$ such that

$$\alpha_i \alpha_j = \alpha_i' \alpha_j' = 0 \quad (\forall_{i,j})$$
$$\alpha_i \alpha_j' = \alpha_i' \alpha_j \quad (i \neq j)$$
$$\alpha_i \alpha_i' = -\alpha_i' \alpha_i = z.$$

We define elements γ_i $(i = 1, \dots, g)$ by,

$\gamma_i = [1\alpha_i' \; 1\alpha_i] \in H^2(X(\infty))$.

Then we can show

Theorem (Macdonald)

Let X be a Riemann surface of genus g. Then

$$L(X(n)) = \left(\frac{\eta}{\tanh \eta}\right)^{n-2g+1} \prod_{i=1}^{g} \frac{\gamma_i}{\tanh \gamma_i}.$$

This theorem had been proved by Macdonald by a different method.

Finally we mention that Zagier has also calculated the equvariant L-classes $L(g, X(n))$ for the actions on $X(n)$ which are induced from actions on X.

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