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Theorems on the extension of solutions.

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Making Grusin's works [1], [2] as a starting point, we have hitherto studied the extendability of (regular) solutions in [3], [4], [5]. In this report we add something new in this direction. The details will be published in the forthcoming paper [7].

§1. Preliminaries. The case of hyperfunction solutions.

Take a convex compact set and its convex open neighborhood in the n-dimensional Euclidean space R^n, and denote by K, U the intersections of these sets with the open half space $H=\{x_n<0\}$, where $x=(x_1,\ldots,x_n)=(x',x_n)$ are the coordinates of R^n and their abbreviations. We put $L=\overline{K}$, the closure of K in R^n. Thus K is a locally closed bounded subset of R^n and L is compact.

Let p(D) be a partial differential operator with constant coefficients corresponding to the polynomial $p(\zeta)$, where $D=(D_1,\ldots,D_n)$, $D_1=\sqrt{-1}\,\frac{\partial}{\partial x_1}$ etc.. We denote by B the sheaf of germs of hyperfunctions and by B_p the sheaf of germs of hyperfunction solutions of $p(D)u=0$. We first note the following lemma which can be proved either by the Fourier transform and estimation of entire functions or step by step use of Holmgren's theorem.

Lemma 1.1 $H^0_K(U,B_p)=0$.

With use of the fundamental exact sequence of relative cohomology groups

$$(1)\qquad 0 \longrightarrow H^0_K(U,B_p) \longrightarrow H^0(U,B_p) \longrightarrow H^0(U\smallsetminus K,B_p)$$
$$\longrightarrow H^1_K(U,B_p) \longrightarrow H^1(U,B_p) \longrightarrow \cdots\cdots,$$

and the flabby resolution

$$(2)\qquad 0 \longrightarrow B_p \longrightarrow B \xrightarrow{p} B \longrightarrow 0,$$

which permits us the calculation of $H^1_K(U,B_p)$, and another exact sequence

$$(3)\qquad \cdots\cdots \longrightarrow H^0_K(U,B_p)$$
$$\longrightarrow H^1_{L\smallsetminus K}(R^n,B_p) \longrightarrow H^1_L(R^n,B_p) \longrightarrow H^1_K(U,B_p)$$
$$\longrightarrow H^2_{L\smallsetminus K}(R^n,B_p) \longrightarrow \cdots\cdots,$$

we obtain the following theorem, when we remenber that $H^1(U,B_p)=0$ due to the existence theorem of Harvey-Komatsu, $H^2_{L\smallsetminus K}(R^n,B_p)=0$ because B_p is of flabby dimension ≤ 1, and the statement of Lemma 1.1.

Theorem 1.2 $B_p(U\smallsetminus K)/B_p(U) \cong H^1_K(U,B_p)$

$$\cong H^0_K(U,B)/pH^0_K(U,B)$$

$$\cong H^1_L(R^n,B_p)/H^1_{L\setminus K}(R^n,B_p).$$

The method of arguements is the one frequently used in [5], so we omit here the details.

Now we consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{B[L\setminus K]} & \xrightarrow{p(\zeta)} & \widetilde{B[L\setminus K]} & \xrightarrow{d} & \widetilde{B[L\setminus K]}\{d,p\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{B[L]} & \xrightarrow{p(\zeta)} & \widetilde{B[L]} & \xrightarrow{d} & \widetilde{B[L]}\{d,p\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{B[L]}/\widetilde{B[L\setminus K]} & \xrightarrow{p(\zeta)} & \widetilde{B[L]}/\widetilde{B[L\setminus K]} & \xrightarrow{d} & \widetilde{B[L]/B[L\setminus K]}\{d,p\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here we employed the symbol $B[L]=H^0_L(R^n,B)$ and $B[L\setminus K]=H^0_{L\setminus K}(R^n,B)$. $\widetilde{B[L]}$ etc. denotes the Fourier transform of $B[L]$ etc.. d denotes a noetherian operator corresponding to $p(\zeta)$ (we can assume that each irreducible component of the associated algebraic variety $N(p)$ is normally placed with respect to ζ_1, and d consists of the composition of restriction to each irreducible component with the differentiations by ζ_1 up to the order equal to the multiplicity of the corresponding component minus 1). $\widetilde{B[L]}\{d,p\}$ etc. denotes the space of vectors of holomorphic functions on $N(p)$, which satisfy the same growth condition as the elements of $\widetilde{B[L]}$ etc. and which are locally in the image of the noetherian operator d.*) The first and the second rows are exact because of the so called Fundamental Principle (which is proved, in our case, in [5]). The last term of each column is defined as the quotient space in each sense. Therefore by the 9-lemma the diagram is exact when we define the last row in the natural way. We have:

Proposition 1.4 $B_p(U\setminus K)/B_p(U) \underset{\tilde{d}}{\cong} \widetilde{B[L]/B[L\setminus K]}\{d,p\}$. The isomorphism $\tilde{d}$ is given in the following way: For $u \in B_p(U\setminus K)$, let $[u] \in H^0(U,B)$ be an extension of u and let $[[p(D)[u]]] \in H^0_L(R^n,B)$ be an extension of $p(D)[u] \in H^0_K(U,B)$. Then, $\tilde{d}\cdot u = \widetilde{d[[p(D)[u]]]} \in \widetilde{B[L]}\{d,p\}$ mod. $\widetilde{B[L\setminus K]}\{d,p\}$.

Theorem 1.5 $B_p(U\setminus K)/B_p(U)=0$ if and only if for any $\varepsilon>0$, there exists some $C_\varepsilon>0$ such that the following inequality holds:**)

$$(4) \qquad H_L(\zeta) \leq \varepsilon|\zeta| + H_{L\setminus K}(\zeta) + C_\varepsilon, \qquad \zeta \in N(p).$$

Proof. The sufficiency follows directly from Proposition 1.4. In fact assuming the above inequality we have the inclusion $\widetilde{B[L]}\{d,p\} \subset \widetilde{B[L\setminus K]}\{d,p\}$. To prove the necessity choose a point $a \in L$ arbitrarily, and take a solution $E \in B(R^n)$ satisfying $p(D)E = \delta(x-a)$. Clearly $E \in B_p(U\setminus K)$. Therefore $\widetilde{B[L]}\{d,p\}$ contains a vector function $\tilde{d}\cdot p(D)E = d\cdot e^{\sqrt{-1}\langle a,\zeta\rangle}$, which contains the function $e^{\sqrt{-1}\langle a,\zeta\rangle}$ in its components.

Suppose that $B_p(U \setminus K)/B_p(U)=0$. Then by proposition 1.4 we have $\widetilde{B[L]}\{d,p\} \subset \widetilde{B[L \setminus K]}\{d,p\}$, so that the following inequality must hold for the function $e^{\sqrt{-1}<a,\zeta>}$:

$$(5) \quad |e^{\sqrt{-1}<a,\zeta>}| \leq C_\varepsilon e^{\varepsilon|\zeta|+H_{L\setminus K}(\zeta)}, \quad \forall \varepsilon>0,\ \exists C_\varepsilon>0.$$

The desired inequality (4) follows from this one by the absurdity. Since the arguement is elementary we omit the details.

Remark. From the condition of Theorem 1.5 we can easily conclude that p is hyperbolic (in the sense of hyperfunction) with respect to $(0,\dots,0,1)$. But mere hyperbolicity is not sufficient for $B_p(U \setminus K)/B_p(U)=0$. For example, assume that $n=2$ and $p(\zeta)=\zeta_1^2-\zeta_n^2$. Then $N(p)=\{\zeta_1+\zeta_n=0\} \cup \{\zeta_1-\zeta_n=0\}$, and the condition (4) is satisfied if and only if the projections of the two sets L, $L \setminus K$ to the planes $\{x_1+x_n=0\}$ and $\{x_1-x_n=0\}$ both agree. Thus for $K=\{(0,t);\ -1 \leq t<0\}$ we have a non-trivial element $u \in B_p(U \setminus K)/B_p(U)$ defined by

$$u(x_1,x_n) = \begin{cases} 1+x_1+x_n & \text{for } 1+x_1+x_n \geq 0,\ -1<x_1<0, \\ 1-x_1+x_n & \text{for } 1-x_1+x_n \geq 0,\ 0<x_1<1, \\ 0 & \text{otherwise.} \end{cases}$$

§2. Continuation of real analytic solutions.

Let A denote the sheaf of germs of real analytic functions; A_p the sheaf of real analytic solutions of $p(D)u=0$; $A(U)$, $A_p(U)$ the sections of these sheaves on U. We discuss when $A_p(U \smallsetminus K)/A_p(U)=0$. For this purpose we first quote a result on propagation of regularities.

Theorem 2.1 (T.Kawai) $B_p(U) \cap A(U \smallsetminus K) \subset A_p(U)$.

The proof is carried in a way similar to that of Lemma 1.1 mensioned first, but with a more delicate arguement using the Fourier hyperfunctions and the rapidly decreasing real analytic functions. See [9] Theorem 5.1.1.

Let $\chi \in C^\infty(U)$ be such that $\chi=1$ on a neighborhood of K, and $\overline{\mathrm{supp}\chi} \cap \partial U \subset L \smallsetminus K$, where the closure or the boundary is taken in R^n. Take $u \in A_p(U \smallsetminus K)$ arbitrarily. Then $\mathrm{supp}\, p(D)(\chi u) \cap K = \emptyset$, so that we can extend $p(D)(\chi u)$ to K by zero and obtain an element of $H^0_{\mathrm{supp}\chi}(U, C^\infty)$. Let $[\![p(D)(\chi u)]\!]_0$ be one of its extension to $H^0_{\mathrm{supp}\chi}(R^n, B)$.

Lemma 2.2 $\tilde{d} \cdot u = -d[\![p(D)(\chi u)]\!]_0 \mod. \widetilde{B[L \smallsetminus K]}\{d,p\}$.

Proof. Let $[\![\chi u]\!] \in H^0_{\mathrm{supp}\chi}(R^n, B)$ be an extension of χu. Then we have obviously

$$p(D)[\![\chi u]\!] \equiv [\![p(D)(\chi u)]\!]_0 + [\![p(D)[u]]\!] \mod. B[L \smallsetminus K].$$

Hence,

$$0 = dp(\zeta)\widetilde{[\![\chi u]\!]} \equiv \widetilde{[\![d\ p(D)(\chi u)]\!]_0} + d\widetilde{[\![p(D)[u]]\!]} \mod. \widetilde{B[L \smallsetminus K]}\{d,p\},$$

and we have proved

$$\begin{aligned} \tilde{d} \cdot u &= d\widetilde{[\![p(D)[u]]\!]} \mod. \widetilde{B[L \smallsetminus K]}\{d,p\} \\ &= -d[\![p(D)(\chi u)]\!]_0 \mod. \widetilde{B[L \smallsetminus K]}\{d,p\}. \quad \text{q.e.d.} \end{aligned}$$

Proposition 2.3 For $u \in A_p(U \setminus K)$, each representative $F(\zeta)$ of $\widetilde{d}\cdot u$ has the following property: for any entire infra-exponential function $J(\zeta)$ and any $\varepsilon > 0$, we have a decomposition $J(\zeta)F(\zeta) = f(\zeta) + g(\zeta)$, where f, g are vectors of holomorphic functions satisfying the estimates

$$|f(\zeta)| \le C_\eta \exp(\eta|\zeta| + \varepsilon|\operatorname{Im}\zeta| + H_{L\setminus K}(\zeta)), \quad \forall \eta > 0, \exists C_\eta > 0, \quad \zeta \in N(p),$$

$$|g(\zeta)| \le C\exp(\varepsilon|\operatorname{Im}\zeta'| + \tfrac{\varepsilon}{2}\operatorname{Im}\zeta_n + H_L(\zeta)), \quad \zeta \in N(p).$$

Proof. First remark that $J(D)u \in A_p(U \setminus K)$ also, where $J(D)$ is the local operator corresponding to $J(\zeta)$ (see [4] §2). Thus by Lemma 2.2 we have $\widetilde{d}\cdot J(D)u = -d\,\widetilde{[\![p(D)(\chi J(D)u)]\!]_0}$ mod. $\widetilde{B[L\setminus K]}\{d,p\}$. By a C^∞ cut-off function we decompose $[\![p(D)(\chi J(D)u)]\!]_0 = v + w$, where $\operatorname{supp} v \subset \{x_n > -\varepsilon\}$, and $\operatorname{supp} w \subset \{x_n < -\frac{\varepsilon}{2}\}$, $w \in C_0^\infty(R^n)$. Thus we have

$$\widetilde{d}\cdot J(D)u = -d\cdot\widetilde{v} - d\cdot\widetilde{w} \quad \text{mod. } \widetilde{B[L\setminus K]}\{d,p\},$$

and by the Paley-Wiener theorem the two terms in the right hand side satisfies the desired estimates. Adjusting by elements of $\widetilde{B[L\setminus K]}\{d,p\}$, we have obtained the desired decomposition for $d\,J(D)u$. But we have

$$\begin{aligned}
\widetilde{d}\cdot J(D)u &= d\,\widetilde{[\![p(D)[J(D)u]]\!]} \quad \text{mod. } \widetilde{B[L\setminus K]}\{d,p\} \\
&= d\,\widetilde{[\![J(D)p(D)[u]]\!]} \quad \text{mod. } \widetilde{B[L\setminus K]}\{d,p\} \\
&= d\cdot\widetilde{J(D)[\![p(D)[u]]\!]} \quad \text{mod. } \widetilde{B[L\setminus K]}\{d,p\} \\
&= d\cdot J(\zeta)\cdot\widetilde{[\![p(D)[u]]\!]} \quad \text{mod. } \widetilde{B[L\setminus K]}\{d,p\}.
\end{aligned}$$

Recall that the last element can be expressed by $J(\zeta)d\widetilde{[[p(D)[u]]]}$ as the multiplication with a matrix whose elements are derivatives of $J(\zeta)$ with ζ_1. See [4] p.573. Therefore just in the same way as there, we obtain the desired property for $J(\zeta)\tilde{d}\cdot u$.

Now we are ready to present our main result.

Theorem 2.4 Assume that each irreducible component p_λ of p satisfies either of the following two conditions:

1) p_λ is hyperbolic with respect to $(0,\dots,0,1)$,

2) there is a vector $\vartheta \in R^{n-1}$ for which the polynomial $p_\lambda(\tau\vartheta+\zeta', \zeta_n)$ on τ satisfies the following conditions: a) the coefficient of the highest term on τ is a constant independent of ζ' or ζ_n, b)*) for any fixed $\zeta' \in C^{n-1}$, the roots $\tau_j = \tau_j(\zeta_n)$ of $p_\lambda(\tau\vartheta+\zeta', \zeta_n)=0$ satisfy $|\tau_j(\zeta_n)/\zeta_n| \leq C_j$ with some constants C_j, and $|(\mathrm{Im}\,\tau_j(\zeta_n))/\zeta_n| \longrightarrow 0$ when $|\mathrm{Im}\,\zeta_n|$ is bounded and $\zeta_n \longrightarrow \infty$.

Then $A_p(U\setminus K)/A_p(U)=0$.

Proof. Due to Proposition 1.4 and Theorem 2.1 we only have to show that every element in $\widetilde{B[L]}\{d,p\}$ having the property stated in Proposition 2.3 necessarily belongs to $\widetilde{B[L\setminus K]}\{d,p\}$. For each hyperbolic factor, we have $\widetilde{B[L]}\{d_\lambda,p_\lambda^{\ell_\lambda}\} \subset \widetilde{B[L\setminus K]}\{d_\lambda,p_\lambda^{\ell_\lambda}\}$ by Theorem 1.5. (If U, K are not in the situation that

*) See the Errata at the end.

Theorem 1.5 can be applied directly, we can replace U by $U\cap\{x_n<c\}$ for a suitable c, and replace K by another suitable set similar to K, and apply this theorem little by little to obtain $u\in A_p(U)$, since there is a unique way of continuation for a real analytic solution.)

Next, suppose that the irreducible component p_λ satisfies the condition 2) of the theorem. Hereafter, we denote p for p_λ for simplicity. By a suitable coordinate transform for x'-variables, we can assume that the polynomial $p(\zeta_1,\zeta'',\zeta_n)$ for ζ_1 has the highest term with the coefficient independent of ζ'', ζ_n, and the roots $\zeta_1=\tau_j(\zeta_n)$ of $p(\zeta_1,\zeta'',\zeta_n)=0$ for fixed ζ'' satisfy the condition stated in 2)-b) of the theorem, where we put $\zeta=(\zeta',\zeta_n)=(\zeta_1,\zeta'',\zeta_n)$. Now fix ζ'' arbitrarily. The variety $N(p)\cap\{\zeta''=\text{constant}\}$ is covered by the following ~~open~~ sets:

$$\{|\mathrm{Im}\,\zeta_n|<C\}\cup\{\zeta_1=\tau_1(\zeta_n),\ \mathrm{Im}\,\zeta_n\geq C\}\cup\cdots\cup\{\zeta_1=\tau_m(\zeta_n),\ \mathrm{Im}\,\zeta_n\leq -C\}.$$

Choosing C large enough, we can assume that each τ_j can be expressed in the puiseux series

$$\tau_j(\zeta_n)=\sum_{k=k_0}^{-\infty}a_k\,\zeta_n^{k/q}.$$

On $|\mathrm{Im}\,\zeta_n|<C$ the estimation is easy. We are going to study on each of the remaining sets. Therefore, from now on we consider a fixed τ_j and omit the suffix j. Let $u\in A_p(U\setminus K)$ and

let $F(\zeta) = \tilde{d}\cdot u(\zeta)$. Then F satisfies the condition in Proposition 2.3. Thus the holomorphic function $G(\zeta_n) = F(\tau(\zeta_n),\zeta'',\zeta_n)$ of one variable ζ_n satisfies, for any choice of the branch of τ and the domain of definition $\mathrm{Im}\zeta_n>C$ (or similarly $\mathrm{Im}\zeta_n<-C$), the following condition: for any $\varepsilon>0$, and for any entire infra-exponential function $J(\zeta_n)$, we have a decomposition $J(\zeta_n)G(\zeta_n) = f(\zeta_n) + g(\zeta_n)$, where f, g are holomorphic in $\mathrm{Im}\zeta_n>C$ and satisfy

$$|f(\zeta_n)| \le C\exp(\varepsilon|\zeta_n| + H_{L\setminus K}(\tau(\zeta_n),\zeta'',\zeta_n)),$$

$$|g(\zeta_n)| \le C\exp(\varepsilon|\mathrm{Im}\zeta_n| + \varepsilon|\mathrm{Im}\tau(\zeta_n)| + H_L(\tau(\zeta_n),\zeta'',\zeta_n)).$$

We prepare a few lemmas.

Lemma 2.5 Assume that the function $u(x)$ of $x\geq 0$ satisfies the following estimate

$$|u(x)| \le C_\varphi \exp(ax - \frac{x}{\varphi(x)})$$

where $\varphi(x)$ is an arbitrarily chosen function so as to satisfy $\varphi(x)\nearrow\infty$ when $x\nearrow\infty$, and C_φ is a positive constant depending on φ. Then, there is a constant $a'<a$ for which the following inequality holds

$$|u(x)| \le C\exp(a'x).$$

This lemma can be proved in elementary way using the technique which is often used in [6], so we omit it.

Proposition 2.6 Assume that the holomorphic function $G(z)$

of one variable for $\operatorname{Im} z \geq 0$ satisfies the following condition: for any entire infra-exponential function $J(z)$, we have a decomposition $JG=f+g$, where f, g are holomorphic in $\operatorname{Im} z \geq 0$ and satisfy

$$|f(z)| \leq C\exp(\varepsilon|z|),$$

$$|g(z)| \leq C\exp(a|\operatorname{Re} z|^q + b|\operatorname{Im} z|),$$

where $q<1$ and all the constants except b and ε may depend on J. Then G satisfies

$$|G(z)| \leq C\exp(\varepsilon|z|).$$

This is, so to speak, a relative form of the theorems of Phragmén-Lindelöf type and the outline of the proof is the following. Put $z=x+\sqrt{-1}y$. Choosing $J=1$, we have for $y \geq 0$

$$|G(\sqrt{-1}y)| \leq C\exp(\operatorname{Max}\{\varepsilon y, by\}).$$

Now choose J so that $|J(\sqrt{-1}y)| \geq C\exp(y/\varphi(y))$ for $y \geq 0$ for given φ which increases monotonely to infinity. (For the construction of such J see [4], Lemma 6.) Assume that $\varepsilon<b$. Dividing the both sides by J we have for $y \geq 0$,

$$|G(\sqrt{-1}y)| \leq C_\varphi \exp(by - \frac{y}{\varphi(y)}).$$

Thus by Lemma 2.5 we have for some $b'<b$

$$|G(\sqrt{-1}y)| \leq C\exp(b'y).$$

Therefore, for any J, the function g appearing in the decomposition $JG=f+g$ has the following two estimates

$$|g(z)| \leq C\exp(a|\mathrm{Re}\, z|^q + b|\mathrm{Im}\, z|),$$

$$|g(\sqrt{-1}y)| \leq |J(\sqrt{-1}y)G(\sqrt{-1}y)| + |f(\sqrt{-1}y)|$$

$$\leq C_\eta \exp(b'y + \eta y), \quad \forall \eta > 0, \quad \exists C_\eta > 0.$$

Hence we can apply the usual Phragmén-Lindelöf theorem to the function $h(z) = g(z)\exp(b'\sqrt{-1}z - a'(\sqrt{-1}z)^q)$ with $a' = a/\cos\frac{q}{2}\pi$, and conclude that g satisfies the estimate in our proposition with b replaced by b'. In this way, we can replace b by a decreasing sequence of numbers b_k. To assure that we can finally replace by ε, assume that b_k converge to some b_0. Then, another use of the usual Phragmén-Lindelöf theorem shows that we can replace by b_0. Thus, by the absurdity, we can prove the assertion.

Now continue the proof of our theorem. By our assumption, we have $|\mathrm{Im}\,\tau(\zeta_n)| \leq a|\mathrm{Re}\,\zeta_n|^q + b|\mathrm{Im}\,\zeta_n| + C$ for some $a>0$, $b>0$, q, where $q<1$ due to Seidenberg's theorem, and b is independent of J since it is determined by K. Thus Proposition 2.6 can be applied to the function $G(\zeta_n)$ and we obtain

$$|G(\zeta_n)| \leq C\exp(\varepsilon|\zeta_n|),$$

where we can take $\varepsilon>0$ arbitrarily small. Thus we have

(11) $|\widetilde{d}\cdot u(\zeta)| \leq C_{\zeta'',\varepsilon} \exp(\varepsilon|\zeta|), \quad \forall\varepsilon>0, \forall\zeta'', \exists C_{\zeta'',\varepsilon}>0,$

(12) $|\widetilde{d}\cdot u(\zeta)| \leq C_\varepsilon \exp(\varepsilon|\zeta| + H_L(\zeta)), \quad \forall\varepsilon>0, \exists C_\varepsilon>0, \zeta\in N(p).$

By the caluculation of plurisubharmonic minorant, we can obtain from these two estimates the desired one for $\zeta \in N(p)$

(13) $|\tilde{d}\cdot u(\zeta)| \leq C_\varepsilon \exp(\varepsilon|\zeta| + H_{L\setminus K}(\zeta))$, $\forall \varepsilon > 0, \exists C_\varepsilon > 0$,

which proves the theorem. The last step is tedious, so we omit it.

§3. Examples.

1). In [8] M. Kashiwara introduced the notion of $\mathcal{C}$-hyperbolicity, and gave a characterization for it. An operator $p(D)$ with constant coefficient is called $\mathcal{C}$-hyperbolic with respect to ϑ when it has a fundamental solution with the singular support contained in a cone properly contained in $\{\langle x,\vartheta\rangle \geq 0\}$. Thus assume that $L\setminus K$ has a non-void interior in $\{x_n=0\}$. Then, for any $p(D)$ which is $\mathcal{C}$-hyperbolic but not hyperbolic w.r.t. $(0,\ldots,0,1)$, we can construct a non-trivial element of $A_p(U\setminus K)/A_p(U)$ using the fundamental solution.

2). But the above condition is far from necessary, since we have the following curious example. Let $p(D) = D_2^2+D_n^2$. $(n=3)$. Then $p(D)u=0$ has the following solution

$$u(x_1,x_2,x_n) = \log\left\{(x_2-x_1^2)^2+(x_n+1-\tfrac{1}{\varepsilon}x_1^2)^2\right\}$$

whose singularity in $H=\{x_n<0\}$ is ~~arbitrarily~~ close to the line segment $\{(0,0,t);\ -1\leq t<0\}$. This operator is not $\mathcal{C}$-hyperbolic by the criterion of M. Kashiwara.

3). In the preceding example, it seems that we cannot construct a solution whose singularity just agrees with the line segment given there. On this point we have a conjecture that when K is the line segment, the second condition in 2)-b) of Theorem 2.4 will be unnecessary. Clearly our operator satisfies the first condition but does not the second one in 2)-b).

4). There are no inclusion relations in the two conditions 1), 2) in Theorem 2.4. In fact, $p(D) = D_1^2 - D_n^2$ $(n=2)$ satisfies both 1) and 2). On the otherhand, $p(D) = D_1^2 - D_n^3$ $(n=2)$ satisfies 1), but none of the condition 2)-b).

5). Assume that the principal part of p does not contain ζ_n. Then, as is easily seen by an elementary consideration of algebraic equations, we have for a suitable $\vartheta \in \mathbb{R}^{n-1}$ $|\tau_j(\zeta_n)/\zeta_n| \longrightarrow 0$ when $|\zeta_n| \longrightarrow \infty$ for the roots τ_j of the equation $p(\tau\vartheta + \zeta', \zeta_n) = 0$ with fixed $\zeta' \in C^{n-1}$. Thus the condition 2) of Theorem 2.4 is trivially satisfied. Of cource these operators are not hyperbolic in the direction $(0,..,0,1)$.

Notes.

*) a vector of holomorphic functions satisfying such local property is called a holomorphic p-function after Palamodov.

**) $H_L(\zeta)$ etc. denotes the supporting function of $\underbrace{L,}_{\text{etc.}}$ i.e. $\sup_{x \in L} \mathrm{Re}\langle x, \sqrt{-1}\zeta\rangle$, sice we use the Fourier transform $\tilde{u} = \langle u, \exp(\sqrt{-1}\zeta)\rangle$.

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Errata (added on July 7, 1972).

The claim mentioned in p.12,↑ 1~p.13,↓ 4 does not hold in general. In fact we have a counter-example to Theorem 2.4 itself: For $p(D) = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_n^2}$ $(n=3)$, we have the following solution (with $0 \leq k < 1$)

$$u(x_1, x_2, x_n) = \frac{1}{\sqrt{(x_1^2 - x_2^2 - x_n^2)}} \log\left\{\left(\frac{x_2}{x_1^2 - x_2^2 - x_n^2}\right)^2 + \left(\frac{x_n + kx_1}{x_1^2 - x_2^2 - x_n^2} - k\right)^2\right\}.$$

Therefore, for the present we must content ourselves with the case $n=2$, where the management with the remaining variables ζ'' is not necessary. For the general n the corrected result will be given in a forthcoming paper.

講究録
No.162 追加

<u>Correction</u> (added on October 5, 1972)

We can prove the following result. The method of proof is just given in the body of this report, and the last claim mensioned there really holds under our new assumption.

<u>Theorem.</u> Assume that each irreducible component p_λ of p satisfies either of the following two conditions.

1) p_λ is hyperbolic with respect to a sequence of directions ϑ_k, $k=1,2,\ldots,$ converging to $(0,\ldots,0,1)$.

2) There exists a non-characteristic direction $(\vartheta,0)$ $R^{n-1}\times R$ such that $K\subset\{<\vartheta,x>=0\}$ and that for the roots τ of $p(\zeta'+\tau\vartheta,\zeta_n)=0$, the estimate

$$|\mathrm{Im}\,\tau|\leq\varepsilon|\zeta_n|+b|\mathrm{Im}\,\zeta_n|+C_{\zeta',\varepsilon},\quad \forall\varepsilon>0,\quad \exists C_{\zeta',\varepsilon}>0,$$

holds for $\mathrm{Im}\,\zeta_n\geq 0$.

Then $A_p(U\setminus K)/A_p(U)=0$.

P Hence we have, instead of Example 5),

<u>Corollary.</u> Assume that the principal part of p does not contain ζ_n and that $K=\{(0,\ldots,0,x_n);\ -c_n\leq x_n<0\}$. Then we have $A_p(U\setminus K)/A_p(U)=0$.

For the details and for other results see the following paper submitted to J. Math. Soc. Japan.

"On continuation of regular solutions of partial differential equations with constant coefficients."