数理解析研究所講究録
170巻 1973年 143-154

A classification of simple spinnable structures on S^{2n+1}

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§1. Introduction

The notion of a spinnable structure on a closed manifold has been introduced by I. Tamura [5] and independently by Winkelnkemper [6] ("open books" in his term), who obtained a necessary and sufficient condition for existence of it on at least a simply connected closed manifold.

The purpose of the paper is to classify "simple" spinnable structures on an odd dimensional sphere S^{2n+1} in terms of their Seifert matrices.

Definition. A closed manifold M is *spinnable*, if there is a compact manifold F, called *generator*, a diffeomorphism $h : F \longrightarrow F$, called *characteristic diffeomorphism*, such that $h \mid \partial F = \text{id.}$, and a diffeomorphism $g : T(F, h) \longrightarrow M$, where $T(F, h)$ is a closed manifold obtained from $F \times [0, 1]$ by identifying $(x, 0)$ with $(h(x), 1)$ for all $x \in F$ and (x, t) with (x, t') for all $x \in \partial F$ and $t, t' \in [0, 1]$. A triple $\mathcal{S} = \{F, h, g\}$ will be called a spinnable structure on M. A second spinnable structure $\mathcal{S}' = \{F', h', g'\}$ on M is isomorphic with $\mathcal{S}$, if there is a diffeomorphism $f : M \longrightarrow M$ such that $f \circ g(F \times t) = g'(F' \times t)$ for all $t \in [0, 1]$. A spinnable structure $\mathcal{S} = \{F, h, g\}$ on M is simple if its generator is of the homotopy type of a finite CW-complex of dimension $\leq \left[\frac{\dim M}{2}\right]$.

We are interested in simple spinnable structures on S^{2n+1}. In the case, a generator F is (n-1)-connected and a characteristic diffeomorphism $h : F \longrightarrow F$ induces an isomorphism $h_* : H_n(F, \mathbb{Z}) \longrightarrow H_n(F, \mathbb{Z})$ of the integral n-dimensional homology group of F, which will be called the monodromy of the spinnable structure. In § 2, we shall define a Seifert matrix $\Gamma(\mathcal{S})$ of a simple spinnable structure $\mathcal{S}$ on S^{2n+1} so that it is unimodular and determines the intersection matrix of F and the monodromy.

Theorem A. For a unimodular $m \times m$ matrix A, there is a simple spinnable structure $\mathcal{S}$ on S^{2n+1} with $\Gamma(\mathcal{S}) = A$, provided that $n \geqq 3$.

Theorem B. If $\mathcal{S}$ and $\mathcal{S}'$ are simple spinnable structures on S^{2n+1} with congruent Seifert matrices, then $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, provided that $n \geqq 3$.

These theorems imply that there is a one to one correspondence of isomorphism classes of simple spinnable structures on S^{2n+1} with congruence classes of unimodular matrices via the Seifert matrix.

§ 2. Seifert matrices of simple spinnable structures on an Alexander manifolds.

First of all we prove:

Proposition 2.1. If $\mathcal{S} = \{F, h, g\}$ is a simple spinnable structure on a closed orientable (2n+1)-manifold M, then $g | F \times t : F \times t \longrightarrow M$ is n-connected, in particular, if $M = S^{2n+1}$, then F is (n-1)-connected and hence is of the homotopy type of a bouquet of n-spheres;

$$F \simeq \bigvee_{i=1}^{m} S_i^n .$$

Proof. For the proof, putting $F_t = g(F \times t)$, it suffices to show that (M, F_0) is n-connected. We put $W = g(F \times [0, \frac{1}{2}])$ and $W' = g(F \times [\frac{1}{2}, 1])$. Since $\mathcal{S}$ is simple, it follows from the general position that there is a PL embedding $f : K \longrightarrow \text{Int } W'$ from an n-dimensional compact polyhedron K into $\text{Int } W'$ which is a homotopy equivalence. Since $\partial W' = \partial W$ is a deformation retract of $W' - f(K)$, we have that

$$\begin{aligned} \pi_i(M, F_0) &\cong \pi_i(M, W) = \pi_i(M, M - W') \\ &\cong \pi_i(M, M - f(K)) \\ &= 0 \qquad \text{for } i \leqq n, \end{aligned}$$

completing the proof.

We shall call a closed orientable (2n+1)-manifold M is an Alexander manifold, if $H_n(M) = H_{n+1}(M) = 0$. By the Poincaré duality, then $H_{n-1}(M)$ is torsion free and hence if $\mathscr{S}$ is a simple spinnable structure on M, then $H_{n-1}(F)$ and $H_n(F)$ are torsion free. Then a bilinear form

$$\gamma : H_n(F) \otimes H_n(F) \longrightarrow \mathbb{Z}$$

is defined by

$$\gamma(\alpha \otimes \beta) = L(g_{\#}(\alpha \times t_0), g_{\#}(\alpha \times t_1)),$$

where $0 \leqq t_0 < \frac{1}{2}$, $\frac{1}{2} \leqq t_1 < 1$, and $L(\xi, \eta)$ stands for the linking number of cycles ξ and η in M so that $L(\xi, \eta)$ = intersection number $\langle \lambda, \eta \rangle$ of chains λ and η in M for some λ with $\partial\lambda = \xi$.

For a basis $\alpha_1, \cdots, \alpha_m$ of a free abelian group $H_n(F)$, a square matrix $(\gamma(\alpha_i \otimes \alpha_j)) = (\gamma_{ij})$ will be called a Seifert matrix of $\mathscr{S}$ and denoted by $\Gamma(\mathscr{S})$. It is a routine work to make sure that the congruence class of $\Gamma(\mathscr{S})$ is invariant under the isomorphism class of $(M, \mathscr{S})$.

We have an alternative expression of $\Gamma(\mathscr{S})$ in terms of an isomorphism

$$a : H_n(W) \overset{\partial^{-1}}{\cong} H_{n+1}(M, W) \overset{exc^{-1}}{\cong} H_{n+1}(W', \partial W') \overset{\text{Poincaré dual}}{\cong} H^n(W') \overset{\text{dual space}}{\cong} H_n(W')$$

which will be called the Alexander isomorphism.

We have homomorphisms

$$\varphi : H_n(W) \overset{\partial^{-1}}{\cong} H_{n+1}(M, W) \overset{exc^{-1}}{\cong} H_{n+1}(W', \partial W) \overset{\partial}{\longrightarrow} H_n(\partial W)$$

and

$$\varphi' : H_n(W') \cong H_{n+1}(M, W') \cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)$$

so that $i_* \circ \varphi$ = id. and $i_*^! \circ \varphi_*^!$ = id. and the following sequences are exact:

$$0 \longrightarrow H_n(W') \xrightarrow{\varphi'} H_n(\partial W) \xrightarrow{i_*} H_n(W) \longrightarrow 0 ,$$

$$0 \longrightarrow H_n(W) \xrightarrow{\varphi} H_n(\partial W) \xrightarrow{i'_*} H_n(W') \longrightarrow 0 ,$$

where $i_* : H_n(\partial W) \longrightarrow H_n(W)$ and $i'_* : H_n(\partial W) \longrightarrow H_n(W')$ are homomorphisms induced from the inclusion maps. Let $\alpha_1, \cdots, \alpha_m$ be a basis of $H_n(W)$. Then, putting $\beta_i = a(\alpha_i)$, $i = 1, \cdots, m$, we have a basis $\beta_1, \cdots, \beta_m$ of $H_n(W')$. By the definition of the Alexander isomorphism, if we put $\bar{\alpha}_i = \varphi(\alpha_i)$ and $\bar{\beta}_i = \varphi'(\beta_i)$, $i = 1, \cdots, m$, then we have that the intersection number in ∂W

$$\langle \bar{\alpha}_i, \beta_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

Let $g_t : F \longrightarrow M$ be an embedding defined by

$$g_t(x) = g(x, t) \quad \text{for all } x \in F, \ t \in [0, 1].$$

For a subspace X of M with $g_t(F) \subset X$, we denote the range restriction of g_t to X by $X \mid g_t : F \longrightarrow X$;

$$X \mid g_t(x) = g_t(x) \quad \text{for all } x \in F.$$

We identify a basis $\alpha_1, \cdots, \alpha_m$ of $H_n(W)$ with that of $H_n(F)$ via $(W \mid g_{1/3})_*$ and a basis $\beta_1, \cdots, \beta_m$ of $H_n(W)$ with that of $H_n(F)$ via $(W \mid g_{2/3})_*$.

Again by the definition of the Alexander isomorphism, we have that

$$L(\alpha_i, \beta_j) = \delta_{ij} \quad \text{for } i, j = 1, \cdots, m.$$

Since $W \mid g_{1/3}$ and $W \mid g_{1/2} = i \circ (\partial W \mid g_{1/2})$ are homotopic in W and $W' \mid g_{2/3}$ and $W' \mid g_{1/2} = i' \circ (\partial W \mid g_{1/2})$ are homotopic in W', it follows that $(\partial W \mid g_{1/2})_*(\alpha_i)$ is of a form

$$(\partial W \mid g_{1/2})_*(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij} \bar{\beta}_j \; .$$

and hence that $(W' | g_{\frac{2}{3}})_*(\alpha_i) = \sum_{j=1}^{m} a_{ij}\beta_j = \sum_{j=1}^{m} a_{ij}a(\alpha_j)$.

Therefore, we have that $\gamma_{ij} = L((g_{\frac{1}{3}})_\# \alpha_i, (g_{\frac{2}{3}})_\# \alpha_j) = L(\alpha_i, \Sigma a_{jk}\beta_k) = a_{ji}$ for $i, j = 1, \cdots, m$. Thus we conclude as follows:

<u>Proposition</u> 2.2. For a basis $\alpha_1, \cdots, \alpha_m$ of $H_n(F) \overset{(W|g_{\frac{1}{3}})_*}{\cong} H_n(W)$, the following (1), (2) and (3) are equivalent.

(1) $(\partial W | g_{\frac{1}{2}})_*(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij}\bar{\beta}_j$,

(2) $a^{-1}\circ (W' | g_{\frac{2}{3}})_*(\alpha_i) = \sum_{j=1}^{m} a_{ij}\alpha_j$

and

(3) $\Gamma^t = (a_{ij})$.

In particular, the Seifert matrix Γ is unimodular.

Now we determine algebraic structures of simple spinnable structures on an Alexander manifold.

<u>Theorem</u> 2.3. Let $\mathcal{S} = \{F, h, g\}$ be a simple spinnable structure on an Alexander manifold M^{2n+1}.

(1) The intersection matrix $I = I(F)$ of F and the Seifert matrix $\Gamma = \Gamma(\mathcal{S})$ of $\mathcal{S}$ are related in a formula:

$$- I = \Gamma + (-1)^n \Gamma^t ,$$

where Γ^t is the transposed matrix of Γ.

(2) The n-th monodromy $h_* : H_n(F) \longrightarrow H_n(F)$ is given by a formula:

$$h_* = (-1)^{n+1}\Gamma^t \cdot \Gamma^{-1} ,$$

or

$$h_* - E = I \cdot \Gamma^{-1} .$$

<u>Proof</u>. For the proof of (1), we follow Levine [3], p.542. We

take chains $d = g_{\#}(\alpha_i \times [\frac{1}{3}, \frac{2}{3}])$, e_1 and e_2 in M such that

$$\partial d = g_{\#}(\alpha_i \times \tfrac{2}{3}) - g_{\#}(\alpha_i \times \tfrac{1}{3}) = (g_{\frac{2}{3}})_{\#}(\alpha_i) - (g_{\frac{1}{3}})_{\#}(\alpha_i),$$

$$\partial e_1 = -(g_{\frac{2}{3}})_{\#}(\alpha_i)$$

and

$$\partial e_2 = (g_{\frac{1}{3}})_{\#}(\alpha_i).$$

Since $d + e_1 + e_2$ is a cycle, we have that

$$\begin{aligned} 0 &= \langle d+e_1+e_2, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle \\ &= \langle d, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle + \langle e_1, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle + \langle e_2, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle \\ &= \langle \alpha_i, \alpha_j\rangle + (-1)L((g_{\frac{2}{3}})_{\#}(\alpha_i), (g_{\frac{1}{2}})_{\#}\alpha_j) + L((g_{\frac{1}{3}})_{\#}(\alpha_i), (g_{\frac{1}{2}})_{\#}\alpha_j) \end{aligned}$$

Since

$$L((g_{\frac{2}{3}})_{\#}(\alpha_i), (g_{\frac{1}{2}})_{\#}(\alpha_j)) = (-1)^{n+1}L((g_{\frac{1}{2}})_{\#}(\alpha_j), (g_{\frac{2}{3}})_{\#}(\alpha_i))$$
$$= (-1)^{n+1}\gamma(\alpha_j \otimes \alpha_i)$$

and

$$L((g_{\frac{1}{3}})_{\#}(\alpha_i), (g_{\frac{1}{2}})_{\#}(\alpha_j)) = \gamma(\alpha_i \otimes \alpha_j),$$

we have that

$$-I = \Gamma + (-1)^n \Gamma^t,$$

completing the proof of (1). To prove (2), we take chains $d = g_{\#}(\alpha_i \times [0, 1])$, e_0 and e_1 in M so that $\partial d = g_{1\#}(\alpha_i) - g_{0\#}(\alpha_i)$, $\partial e_0 = g_{0\#}(\alpha_i)$ and $\partial e_1 = -g_{1\#}(\alpha_i) = -g_{0\#}(h_*(\alpha_i))$. Since $d + e_0 + e_1$ is an (n+1)-cycle in M, we have that

$$\begin{aligned} 0 &= \langle d+e_0+e_1, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle \\ &= \langle d, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle + \langle e_0, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle + \langle e_1, (g_{\frac{1}{2}})_{\#}(\alpha_j)\rangle \\ &= \langle \alpha_i, \alpha_j\rangle + L(g_{0\#}(\alpha_i), (g_{\frac{1}{2}})_{\#}(\alpha_j)) + (-1)L(g_{0\#}(h_*(\alpha_i)), (g_{\frac{1}{2}})_{\#}(\alpha_j \\ &= \langle \alpha_i, \alpha_j\rangle + \gamma(\alpha_i \otimes \alpha_j) - \gamma(h_*(\alpha_i) \otimes \alpha_j) \\ &= \langle \alpha_i, \alpha_j\rangle + \gamma((id - h_*)(\alpha_i) \otimes \alpha_j) \end{aligned}$$

and hence that

$$- I = (E - h_*)\cdot\Gamma ,$$

where E is the identity matrix (δ_{ij}). Therefore, by making use of (1), we have that

$$(h_* - E) = I\cdot\Gamma^{-1}$$
$$= -E + (-1)^{n+1}\Gamma^t\Gamma^{-1} ,$$

or

$$h_* = (-1)^{n+1}\Gamma^t\Gamma^{-1} ,$$

completing the proof.

§ 3. Proof of Theorem A.

Suppose that we are given an $m\times m$ unimodular matrix $A = (a_{ij})$. Let K denote a bouquet of m n-dimensional spheres; $K = \bigvee_{i=1}^{m} S_i^n$. We have a PL embedding $f : K \longrightarrow S^{2n+1}$. Let W be a smooth regular neighborhood of $f(K)$ in $S^{2n+1} = S$ and $W' = S - \mathrm{Int}\, W$. We denote the Alexander isomorphism

$$H_n(W) \cong H^n(S - \mathrm{Int}\, W) = H^n(W') = \mathrm{Hom}(H_n(W')) \cong H_n(W')$$

by $a : H_n(W) \cong H_n(W')$. Thus we have that W, W' and ∂W are $(n-1)$-connected, and there are splittings

$$\varphi : H_n(W) \cong H_{n+1}(S, W) \cong H_{n+1}(W', \partial W) \longrightarrow H_n(\partial W) ,$$
$$\varphi' : H_n(W) \cong H_{n+1}(S, W') \cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)$$

of $i_* : H_n(\partial W) \longrightarrow H_n(W)$ and $i'_* : H_n(\partial W) \longrightarrow H_n(W')$, respectively. Note that the following sequences are exact.

$$0 \longrightarrow H_n(W) \xrightarrow{\varphi} H_n(\partial W) \xrightarrow{i'_*} H_n(W') \longrightarrow 0$$

and

$$0 \longrightarrow H_n(W') \xrightarrow{\varphi} H_n(\partial W) \longrightarrow H_n(W) \longrightarrow 0 .$$

If $\alpha_1, \cdots, \alpha_m$ is a basis of $H_n(K) \cong H_n(W)$ and we put $a'(\alpha_i) = \beta_i$, $\varphi(\alpha_i) = \bar{\alpha}_i$, and $\varphi(\beta_i) = \bar{\beta}_i$, $i = 1, \cdots, m$, then we have that the intersection numbers in ∂W $\langle \alpha_i, \bar{\alpha}_j \rangle = 0$, $\langle \bar{\beta}_i, \bar{\beta}_j \rangle = 0$ and $\langle \bar{\alpha}_i, \bar{\beta}_j \rangle = \delta_{ij}$ for $i, j = 1, \cdots, m$, and the linking numbers in S $L(\alpha_i, \beta_j) = \delta_{ij}$, $i, j = 1, \cdots, m$.

A splitting $s : H_n(W) \longrightarrow H_n(\partial W)$ of $i_* : H_n(\partial W) \longrightarrow H_n(W)$ will be called a <u>non-singular section</u>, if $i'_* \circ s : H_n(W) \longrightarrow H_n(W')$ is an isomorphism. Indeed, a section $s : H_n(W) \longrightarrow H_n(\partial W)$ has to be of a form

$$s(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij} \bar{\beta}_j$$

and hence $i'_* \circ s(\alpha_i) = \sum_{j=1}^{m} a_{ij} \beta_j$. Thus the correspondence $s \longmapsto (a_{ij})$ gives rise to a one to one correspondence of non-singular sections $H_n(W) \longrightarrow H_n(\partial W)$ with unimodular $m \times m$ matrices (a_{ij}). As is found by Winkelnkemper [6] and also Tamura [4] for a non-singular section $s : H_n(W) \longrightarrow H_n(\partial W)$, there is a PL embedding $f' : K^n \longrightarrow \partial W$, provided that $n \geq 3$, which is homotopic to $f : K \longrightarrow W$ and $f'_*(\alpha_i) = s(\alpha_i)$ in ∂W. Moreover, if F is a regular neighborhood of $f'(K)$ in ∂W and $F' = \partial W - \text{Int } F$, then $(W; F, F')$ and $(W'; F', F)$ are relative h-cobordisms, since $s(\alpha_1), \cdots, s(\alpha_m)$ is a basis of $H_n(F)$ as a subgroup of $H_n(\partial W)$ and the inclusion maps induce isomorphisms

$$j_* : H_n(F) \cong H_n(W) \; ; \quad j_*(s(\alpha_i)) = \alpha_i$$

and

$$j_* : H_n(F) \cong H_n(W'); \quad j'_*(s(\alpha_i)) = i'_* \circ s(\alpha_i) = \sum_{j=1}^{m} a_{ij} \beta_j$$

and W, W', F, F' are 1-connected.

It follows that by the h-cobordism theorem, S^{2n+1} admits a spinnable structure $\mathcal{S}_A = \{F, h, g\}$ for a given unimodular matrix $A = (a_{ij})$ such that

$$g(F \times [0, \tfrac{1}{2}]) = W ,$$

$$g(F \times [\tfrac{1}{2}, 1]) = W'$$

and $$g(x, \tfrac{1}{2}) \qquad \text{for all } x \in F.$$

We would like to show that $\Gamma(\mathcal{S}_A) = A^t$. We have seen that $(\partial W \mid g_{\frac{1}{2}})_*(\alpha_i) = s(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij}\beta_j$. It follows from Proposition 2.2 that $\Gamma(\mathcal{S}_A) = A^t$. Therefore, for a given unimodular matrix A, $\mathcal{S}_{A^t}$ is the required spinnable structure on S^{2n+1}, completing the proof.

§ 4. Proof of Theorem B.

The crux of the proof of Theorem B is due to J. Levine [2], who proved essentially the following:

Proposition 4.1 (Levine). Let $\mathcal{S} = \{F, h, g\}$ and $\mathcal{S}' = \{F', h', g'\}$ be spinnable strcutures on S^{2n+1}. Suppose that $n \geq 3$. Then two generators F_0 and F_0' are ambient isotopic in S^{2n+2} if $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S}')$ are congruent.

Proof. By a suitable change of bases, we may assume that $\Gamma(\mathcal{S}) = \Gamma(\mathcal{S}')$. The rest of the proof is what Levine has done in his classification of simple knots (Lemma 3, [2], §14- §16, pp.191-192). His arguments work equally well in our case, completing the proof.

Thus we have a diffeomorphism $f : S^{2n+1} \longrightarrow S^{2n+1}$ such that $f(F_0) = F_0'$, and f is diffeotopic to the identity. By opening out the spinnable structure, we have a diffeomorphism $H : F \times [0, 1] \longrightarrow F' \times [0, 1]$ such that

$$H(x, 0) = (k(x), t) \quad \text{for all} \quad (x, t) \in \partial F \times [0, 1]$$

$$H(x, 0) = (k(x), 0) \quad \text{for all} \quad x \in F \quad \text{and}$$

$$H(x, 1) = (h'^{-1} \circ k \circ h(x), 1) \quad \text{for all} \quad x \in F,$$

where

$$(k(x), 0) = (g')^{-1} \circ f \circ g(x, 0) \quad \text{for all} \quad x \in F .$$

This implies that $(k^{-1} \times id)\ H : F \times [0, 1] \longrightarrow F \times [0, 1]$ is an pseudo-diffeotopy from id to $k^{-1} \circ j'^{-1} \circ k \circ h$ keeping ∂F fixed. Since $n \geq 3$, F and ∂F are 1-connected, it follows from Cerf [1] that the pseudo-diffeotopy is diffeotopic to a diffeotopy $G : F \times I \longrightarrow F \times I$ keeping $\partial(F \times I)$ fixed. This implies that f is diffeotopic to an isomorphism $(S^{2n+1}, \mathcal{S}) \longrightarrow (S^{2n+1}, \mathcal{S}')$ keeping F_0 fixed. Therefore, $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, completing the proof.

Remark. As is known from the proof, $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic by an ambient diffeotopy.

References.

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