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Embedding manifolds in codimension two

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Let W^{m+2} be a connected compact oriented PL-manifold with boundary of dimension $m+2$. Suppose that W^{m+2} is an oriented Poincaré complex of formal dimension m. By definition, there is a fundamental class $\mu \in H_m(W; Z)$ such that

$$\mu \cap : H^r(W; Z) \to H_{n-r}(W; Z)$$

is an isomorphism for all r. We assume $m \geq 5$ and we aim at finding a closed PL-submanifolds M^m of dimension m such that the inclusion $M \to W$ is an simple homotopy equivalence. We call such submanifold a spine.

Since W is a PL-manifold, there exists a fundamental class $\eta \in H_{m+2}(W, \partial W; Z)$ such that

$$\eta \cap : H^r(W, \partial W; Z) \to H_{m+2-r}(W; Z)$$

is an isomorphism for all r.

We define the Thom isomorphism

$$\phi : H^r(W; Z) \to H^{r+2}(W, \partial W; Z)$$

by $\phi = (\eta\cap)^{-1}(\mu\cap)$ and the second Whitney class $s_2 \in H^2(W; Z)$ by

$$s_2 = \phi^{-1}(\phi(1) \cup \phi(1))$$

where 1 denote the generator of $H^0(W; Z)$.

Let $N(W)$ be the stable normal bundle of W and $s_2(W)$ be the 2-dimensional PL block bundle over W which represents s_2. Then the spherical fibre homotopy type of the stable PL-bundle $N(W) - s_2(W)$ is the Spivak fibration of the Poincaré complex W. Then as in [8], we obtain the abstract surgery obstruction element $\theta(W) \in L_m(\pi_1(W))$ in the oriented Wall group.

We have an homomorphism of $L_m(1)$ into $L_m(\pi_1(W))$ by regarding $L_m(1)$ as $\pi_k(V_{k,k})$ if $m = 2k$, which is an isomorphism if $\pi_1(W) = 1$. We obtain a factor group $L_m(\pi_1(W))/L_m(1)$ and the coset class $[\theta(W)] \in L_m(\pi_1(W))/L_m(1)$.

We obtain the following which extends results of Kato-Matumoto [1], the technique of the proof being similar.

<u>Theorem 1.</u> If m is odd and $\theta(W) \in L_m(\pi_1(W))$ is zero, we have a locally flat spine. If m is even and $[\theta(W)] \in L_m(\pi_1(W))/L_m(1)$ is zero, we have a spine which is locally flat except at one point.

Now we want to find a spine which is not necessary locally flat on higher skelton. Noguchi [6] defined a PL-knot cobordism groups G^n as the obstruction to the locally flatness, which Kervaire [2] had shown that $G^{2k} = 0$ by a smooth method. Levine [3] [4] gave algebraic calculations for G^{2k-1}. By a geometrical method, Matsumoto [5] showed that there exists a canonical homomorphism

$$b : G^{n-1} \longrightarrow \pi_n(F/PL) = L_n(1)$$

which is surjective for $n \geq 5$. For $n = 4k$, the homomorphism b is given by the 1/8 signature of the Witt group $W(Z) = Z$ defined by the Seifert matrix. For $n = 4$ the image $b(G^4)$ is $2Z \subset Z \cong L_4(1)$.

Let us denote by $(F/PL)^{m-1}_{(2)}$ the truncated space

$$(F/PL)^{m-1}_{(2)} = K(Z_2, 2) \underset{\delta Sq^2}{\times} K(Z_{(2)}, 4) \times \prod_{m>4i+2\geq 6} K(Z_2, 4i+2) \times \prod_{m>4i\geq 8} K(Z_{(2)}, 4i)$$

consisting of the Postnikov system of $F/PL_{(2)}$ with degrees smaller then m. (See Sullivan [7]). We have the inclusion

$$i : (F/PL)^{m-1}_{(2)} \rightarrow F/PL_{(2)} \rightarrow F/PL$$

Remark.that there is an fibering

$$\Omega^4 : F/PL \rightarrow \Omega^4 F/PL$$

with fiber $K(Z_2, 3)$ and that

$$\Omega^4 F/PL_{(2)} = \prod_{i\geq 1} K(Z_{(2)}, 4i) \times K(Z_2, 4i-2)$$

Let $g : W \longrightarrow (F/PL)^{m-1}_{(2)}$ be a map, then we have $i_* g : W \longrightarrow F/PL$.

We can twist the bundle $N(W) - s_2(W)$ by this map which gives a new surgery obstruction

$$\psi(g) \in L_m(\pi_1(W)).$$

Since it is homotopy invariant, it defines a map

$$\psi : [W, (F/PL)^{m-1}_{(2)}] \longrightarrow L_m(\pi_1(W)),$$

which satisfies

$$\psi(0) = \theta(W).$$

We first represent the fundamental class μ by a submanifold by the Thom's theorem and we do the surgery until the middle dimension. Then we break the locally flatness inductively from the neighborhood of the higher dimentional simplexes by stuffing the Seifert surface of a knot , and after by doing the surgery until the middle dimension. If its Noguchi obstruction to the locally flatness lies in

$$\zeta \in H^j(W, G^{j-1}),$$

the normal bundle is changed by an element of $[W, F/PL]$ which is mapped by Ω^4 to

$$b\xi \in H^j(W, L_j(1)) \subset [W, \Omega^4 F/PL_{(2)}] .$$

We can change the local structure which corresponds to the kernel of Ω^4.

Consequently we obtain

Theorem 2. If the class

$$[\theta(W)] \in [W, (F/PL)^{m-1}_{(2)}] \setminus L_m(\pi_1(W)) / L_m(1)$$

is zero, we have a spine.

Let M and W be (simple) homotopy equivalent PL-manifolds. Suppose M is closed and let $f : M \to W$ be a homotopy equivalence. Then f induces an normal map $g \in [M, F/PL]$. We call f mod 2 K-theory equivalent if $p_* g = 0 \in [M, (F/PL)_{(odd)}] \cong [M, BO_{(odd)}]$ where $p : F/PL \to (F/PL)_{(odd)}$ is the localization.

Corollary.

If W^{m+2} is mod 2 K-theory equivalent to a PL-manifold M^m, we can embed M^m in W^{m+2} such that M^m is a spine.

The author does not know if the condition of mod 2 K-theory equivalence is really necessary or not.

Reference

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