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On homotopy P^5

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§1. Brieskorn involutions on 5-sphere.

We consider the Brieskorn 5-sphere Σ_d^5 $d \geqq 1$ odd integer. This is the submanifold of $\mathbb{C}^4$ described by equations,

$$z_0^d + z_1^2 + z_2^2 + z_3^2 = 0$$
$$z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = 1 .$$

Then Σ_d^5 is a homotopy 5-sphere and therefore diffeomorphic to the standard 5-sphere S^5. The involution $T_d : \Sigma_d^5 \longrightarrow \Sigma_d^5$ given by $T_d(z_0) = z_0$, $T_d(z_i) = -z_i$, $i > 0$, is a fixed point free involution on Σ_d^5. We denote an orbit space Σ_d^5/T_d by Π_d^5. Then Π_d is a homotopy projective 5-space. For $h\mathcal{S}(P^5)$ —— homotopy smoothing of P^5 ——, it is known that $h\mathcal{S}(P^5) = \{\Pi_d\}_{d=1,3,5,7}$ and Π_{d+8} is diffeomorphic to Π_d.

§2. A spinnable structure on Π_d.

Let T_0 be the standard involution on

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = 1\} ,$$

that is, $T_0(z_i) = - z_i$, $i = 1, 2, 3$. Define $f = z_1^2 + z_2^2 + z_3^2$

$$K_0 = \{(z_1, z_2, z_3) \in S^5 \mid f(z_1, z_2, z_3) = 0\}$$
$$F_0 = \{(z_1, z_2, z_3) \in S^5 \mid f(z_1, z_2, z_3) \leqq 0\}$$
$$K_d = \{(z_0, z_1, z_2, z_3) \in \Sigma_d \mid z_0 = 0\}$$
$$F_d = \{(z_0, z_1, z_2, z_3) \in \Sigma_d \mid z_0 \geqq 0\}$$

where for $z \in \mathbb{C}$ "$z \geqq 0$" means "the imaginary part of $z = 0$ and $z \geqq 0$".

Let $\psi_d : \Sigma_d \longrightarrow S^5$ be a map defined by $\psi_d(z_0, z_1, z_2, z_3) = (\frac{z_1}{1-z_0\bar{z}_0}, \frac{z_2}{1-z_0\bar{z}_0}, \frac{z_3}{1-z_0\bar{z}_0})$. Then ψ_d is a d-fold branched covering map with $K_0 = \psi_d(K_d)$ as branching locus. The restriction map $\psi_d | F_d$ give a diffeomorphism between F_d and F_0.

Define

$$\varphi_0 : F_0 \times [0, 1] \longrightarrow S^5$$

by

$$\varphi_0((z_1,z_2,z_3), t) = (z_1 \exp(\pi it), z_2 \exp(\pi it), z_3 \exp(\pi it))$$

$$\varphi_d : F_d \times [0, 1] \longrightarrow \Sigma_d$$

by

$$\varphi_d((z_0,z_1,z_2,z_3), t) = (z_0 \exp(2\pi it), z_1 \exp(d\pi it), z_2 \exp(d\pi it), z_3 \exp(d\pi it)).$$

It is easy to observe that ψ_d, φ_0, φ_d are equivariant with respect to T_0 and T_d, that is, the following diagrams commute.

$$\begin{array}{ccc} \Sigma_d & \xrightarrow{T_d} & \Sigma_d \\ \downarrow \psi_d & & \downarrow \psi_d \\ S^5 & \xrightarrow{T_0} & S^5 \end{array} \qquad \begin{array}{ccc} F_0 \times [0,1] & \xrightarrow{\varphi_0} & S^5 \\ \downarrow (T_0|F_0)\times id_{[0,1]} & & \downarrow T_0 \\ F_0 \times [0,1] & \xrightarrow{\varphi_0} & S^5 \end{array}$$

$$\begin{array}{ccc} F_d \times [0,1] & \xrightarrow{\varphi_d} & \Sigma_d \\ \downarrow (T_d|F_d)\times id_{[0,1]} & & \downarrow T_d \\ F_d \times [0,1] & \xrightarrow{\varphi_d} & \Sigma_d . \end{array}$$

Thus ψ_d, φ_0, φ_d induce maps $\bar{\psi}_d$, $\bar{\varphi}_0$, $\bar{\varphi}_d$ of orbit spaces, that is,

$$\bar{\psi}_d : \Sigma_d/T_d \longrightarrow S^5/T_0$$

$$\bar{\varphi}_0 : (F_0/T_0) \times [0, 1] \longrightarrow S^5/T_0$$

$$\bar{\varphi}_d : (F_d/T_d) \times [0, 1] \longrightarrow \Sigma_d/T_d .$$

We denote the class of $(z_0, z_1, z_2, z_3) \in \Sigma_d$ in Σ_d/T_d by $[z_0, z_1, z_2, z_3]$, also we use $[z_1, z_2, z_3] \in S^5/T_0$ similarly. Then

$$\bar{\varphi}_d([z_0, z_1, z_2, z_3], 1) = [z_0, -z_1, -z_2, -z_3]$$
$$= [z_0, z_1, z_2, z_3] = \bar{\varphi}_d([z_0, z_1, z_2, z_3], 0),$$

also

$$\bar{\varphi}_0([z_1, z_2, z_3], 1) = \bar{\varphi}_0([z_1, z_2, z_3], 0).$$

So, if we regard S^1 as $[0, 1]/0 \sim 1$,

$$f_0 : (F_0/T_0) \times S^1 \longrightarrow S^5/T_0$$
$$f_0([z_1, z_2, z_3], t) = [z_1 \exp(\pi it), z_2 \exp(\pi it), z_3 \exp(\pi it)]$$
$$f_d : (F_d/T_d) \times S^1 \longrightarrow \Sigma_d/T_d$$
$$f_d([z_0, z_1, z_2, z_3], t) = [z_0 \exp(2\pi it), z_1 \exp(d\pi it), z_2 \exp(d\pi it), z_3 \exp(d\pi it)]$$

are both well defined maps.

Simply we denote F_0/T_0, K_0/T_0 by F, K where $\partial F = K$. Easily we can observe that K is diffeomorphic to the 3-dimensional lens space of type $(4, 1)$ and $f_0 |(K \times S^1) : K \times S^1 \longrightarrow K \subset S^5/T_0$ is the standard free S^1-action on K. Thus we use a simple notation $\rho : K \times S^1 \longrightarrow K$ instead of $f_0 |(K \times S^1)$. Under the notation above, we obtain

$$S^5/T_0 = F \times S^1 \cup \{\text{mapping cylinder of } \rho : K \times S^1 \longrightarrow S^1\},$$

more exactly, S^5/T_0 is diffeomorphic to $F \times S^1 \cup_h K \times D^2$ where an attaching diffeomorphism $h : K \times S^1 \longrightarrow K \times S^1$ is defined by

$$h(x, t) = (\rho(x, t), t) \quad \text{for } x \in K, \ t \in S^1.$$

For Σ_d/T_d, there is a similar decomposition. It is described as follows. One can easily verify commutativity of the following diagrams,

$$\begin{array}{ccc}
(F_d/T_d) \times S^1 & \xrightarrow{f_d} & \Sigma_d/T_d = \Pi_d \\
\Big\downarrow \{\bar{\Psi}_d | (F_d/T_d)\} \times \bar{d} & & \Big\downarrow \bar{\Psi}_d \\
F \times S^1 & \xrightarrow{f_0} & S^5/T_0
\end{array}$$

$$\begin{array}{ccc}
(K_d/T_d) \times S^1 & \xrightarrow{f_d | \{(K_d/T_d) \times S^1\}} & K_d/T_d \\
\Big\downarrow \{\bar{\Psi}_d | (K_d/T_d)\} \times \bar{d} & & \cong \Big\downarrow \bar{\Psi}_d | (K_d/T_d) \\
K \times S^1 & \xrightarrow{\rho} & K
\end{array}$$

where $\bar{d} : S^1 \longrightarrow S^1$ is given by $\bar{d}(t) = dt$ for $t \in S^1$.

We define $h_d : K \times S^1 \longrightarrow K \times S^1$ by $h_d(x, t) = (\rho(x, dt), t)$. Then, observing that $\rho \circ (id_K \times \bar{d})(x, t) = \rho(x, dt)$, we have the following result.

Theorem 1. Π_d is diffeomorphic to

$$F \times S^1 \cup_{h_d} K \times D^2.$$

This theorem means that, for any odd integer $d > 0$, Π_d has a spinnable structure ~~[5]~~ with the same axis and generator, that is, K and F, and the trivial spinning bundle. Thus, for $d' \neq d$, the whole difference between $\Pi_{d'}$ and Π_d is due to attaching diffeomorphisms $h_{d'}$ and h_d.