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On the number of moduli
of certain algebraic surfaces
of general type

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0. Introduction. Let \( \mathbb{P}^3 \) denote the projective
3-space defined over the field of complex numbers, \( S \) an
irreducible hypersurface of degree \( n = 2r \) in \( \mathbb{P}^3 \), defined
by the equation

\[
g^2 + Agh + Bh^2 = 0
\]

where \( g, h, A, \) and \( B \) are homogeneous polynomials of
degree \( r, s, r-s, \) and \( 2(r-s) \), respectively, with two
positive integers \( r > s \). Clearly, the curve \( \Delta \), defined by
\( g = h = 0 \), is contained in the singular locus of \( S \).

We assume that \( S \) is generic in the following sense:
1) \( S \) has only ordinary singularities (see [4]) and
is non-singular outside of \( \Delta \).
2) \( \Delta \) is non-singular.
3) The normalization \( X \) of \( S \) (which is non-singular
by 1)) is a surface of general type.

In [4], Kodaira studied families of surfaces with
ordinary singularities in $\mathbb{P}^3$. In particular, he proved that above $S$ belongs to an effectively parametrized family $\mathcal{F}$ of surfaces $S_t$, $t \in M_1$, with ordinary singularities in $\mathbb{P}^3$ whose characteristic system on each $S_t$ is complete (see [14], Theorem 8 and §5.4). In our case, the number $\mu(S)$ of effective parameters of the family $\mathcal{F}$ is given by

$$\mu(S) = C(r) + C(s) + C(2r-2s) - C(r-2s) - 2$$

where

$$C(m) = \begin{cases} \frac{(m+3)(m+2)(m+1)}{6} & \text{for } m \geq 0 \\ 0 & \text{for } m < 0 \end{cases}$$

On the other hand, Kodaira-Spencer introduced the concept of the number of moduli $m(X)$ of a compact complex manifold $X$ (see [5], Definition 11.1).

Main Theorem. Let $S$ be a generic hypersurface in $\mathbb{P}^3$ defined by the equation (1), $X$ the normalization of $S$. Then, the number of moduli $m(X)$ is defined, and we have

$$m(X) = \dim H^1(X, \mathcal{O}_X) = \mu(S) - 15 - 4 \mathcal{S}_{r,s+1}$$

where $\mathcal{O}_X$ denotes the sheaf of germs of holomorphic vector fields on $X$, and $\mathcal{S}_{r,s+1}$ is Kronecker's delta.

Let $f: X \to \mathbb{P}^3$ denote the composition of normalization and the embedding. Then the difference of $\mu(S)$ and
m(X) is the contribution of the number of parameters on which the holomorphic map \( f \) depends.

For \((r, s) = (3,1)\) or \((4,3)\), \( S \) is one of the examples of M. Noether [6].

For \((r, s) = (3,1)\), \( X \) is a minimal algebraic surface with \( p_g = 4, \ q = 0, \) and \( c_1^2 = 6 \), where \( p_g \), \( q \), and \( c_1^2 \) denote, respectively, the geometric genus, the irregularity, and the Chern number. We have

\[
m(X) = 10(p_g - q + 1) - 2c_1^2 = 38,
\]

\[
H^2(X, \mathcal{O}_X) = 0
\]

(cf. Kodaira [8]).

For \((r, s) = (4,3)\), \( X \) is a complete intersection of two hypersurfaces of degree 2 and 4 in \( \mathbb{P}^4 \). We have also \( H^2(X, \mathcal{O}_X) = 0 \).

1. Preliminaries. Let \( E \) be a hyperplane section of \( S, \ \tilde{E} = f^*E, \ \tilde{\Delta} = f^{-1}(\Delta) \). From the equation (1), we infer

Lemma 1. \( \tilde{\Delta} \) is linearly equivalent to \( s\tilde{E} \) on \( X \).

We note that \( (n-4)\tilde{E} - \tilde{\Delta} \) is a canonical divisor on \( X \), and that

\[
H^\nu(X, \mathcal{O}(m\tilde{E} - \tilde{\Delta})) \subseteq H^\nu(S, \mathcal{O}(mE - \Delta)) \quad \text{for} \ \nu = 0, 1, 2
\]

(see [4]). By a standard computation (cf. [7]), we get

Lemma 2. \( \dim H^0(X, \mathcal{O}(\tilde{E})) = 4 + \delta_{r,s+1}, \ H^3(X, \mathcal{O}(\tilde{E})) = 0 \).
Lemma 3. 1) The canonical bundle $K$ of $X$ is ample. In particular, $X$ is minimal.

2) \[ p_g = \binom{2r-1}{3} - (1/2)rs(3r - s - 4), \]
\[ q = 0, \]
\[ c_1^2 = 2r(2r - s - 4)^2. \]

Remark. If $r = s + 1$, then the complete linear system $|\widetilde{E}|$ is very ample, and $X$ is a complete intersection of two hypersurfaces of degree 2 and $r$ in $\mathbb{P}^4$.

2. Relation between deformations of $S$ and $X$. Let \( \{ S_t \}_{t \in M} \) be a family of surfaces of degree $n$ in $\mathbb{P}^3$ with ordinary singularities, $S = S_0$, $0 \in M$. Letting $T_0(M)$ denote the tangent space of $M$ at $0$, we have the characteristic map

$$
\sigma : T_0(M) \longrightarrow H^0(S, \mathcal{N}_S^0)
$$

where $\mathcal{N}_S^0$ denotes the sheaf $\mathcal{O}_{\mathbb{P}^3}(-\Sigma B_i)$ in the notation of [4]. We note that we have an exact sequence

$$
0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{\mathbb{P}^3}|_S \longrightarrow \mathcal{N}_S^0 \longrightarrow 0.
$$

On the other hand, the normalization $X_t$ of $S_t$ form a family $\mathcal{X} = \{ X_t \}_{t \in M}$ of deformations of $X = X_0$ and the holomorphic map $f : X \rightarrow \mathbb{P}^3$ extends to a holomorphic map $\Phi : \mathcal{X} \rightarrow \mathbb{P}^3 \times M$ over $M$. Let $\mathcal{O}_X$ and $\mathcal{O}_{\mathbb{P}^3}$ denote the sheaves of germs of holomorphic vector fields on $X$ and $\mathbb{P}^3$ respectively, and let $\mathcal{F}_{X/\mathbb{P}^3}$ denote the cokernel.
of the canonical homomorphism \( F: \Phi_X \rightarrow f^* \otimes_{\mathbb{P}^3} \). Then we have the characteristic map

\[
\tau: T_0(M) \longrightarrow H^0(X, \mathcal{J}_{X/\mathbb{P}^3})
\]

(see [3], §1).

**Lemma 4.** There is a canonical isomorphism

\[
\mathcal{J}: \mathcal{N}^0_S \longrightarrow f_* \mathcal{J}_{X/\mathbb{P}^3}
\]

which induces an isomorphism

\[
\mathcal{J}: H^0(S, \mathcal{N}^0_S) \longrightarrow H^0(X, \mathcal{J}_{X/\mathbb{P}^3})
\]

such that \(-\tau = \mathcal{J} \circ \sigma\).

We have two exact sequences

\[
0 \longrightarrow \Theta_S \longrightarrow \Theta_{\mathbb{P}^3}|_S \longrightarrow \mathcal{N}^0_S \longrightarrow 0,
\]

\[
0 \longrightarrow f_* \Theta_X \longrightarrow f_* f^* \otimes_{\mathbb{P}^3} \rightarrow f_* \mathcal{J}_{X/\mathbb{P}^3} \longrightarrow 0.
\]

Moreover, there exists a canonical homomorphism

\[
f^*: \Theta_{\mathbb{P}^3}|_S \longrightarrow f_* f^* \otimes_{\mathbb{P}^3}.
\]

One can easily see that \( f^* \) induces a desired isomorphism.


**Lemma 5.** The coboundary map

\[
\mathcal{S}: H^0(X, \mathcal{J}_{X/\mathbb{P}^3}) \longrightarrow H^1(X, \Phi_X)
\]
is surjective.

Proof. Let $\mathcal{P} \in H^1(X, \mathcal{O}_X)$. Then $\mathcal{P}$ corresponds to a deformation $X_\mathcal{P}$ of $X$ over $I = \text{Spec } \mathcal{O}[t]/(t^2)$.

By Lemma 1, we have $K = (n - s - 4)[E]$. It follows that the line bundle $[E]$ extends to a line bundle on $X_\mathcal{P}$. Then, by Lemma 2, the holomorphic map $f$ extends to a holomorphic map $X_\mathcal{P} \to P^3 \times I$ over $I$. This means that $\mathcal{P}$ is surjective.

By the result of Kodaira cited in Introduction, and by Lemma 4, we obtain a family $\mathcal{F}_1 \leftarrow \{X_t\}_{t \in M_1}$ of deformations of $X = X_0$ such that

$$\tau: T_0(M_1) \longrightarrow H^0(X, \mathcal{F}_X/P^3)$$

is surjective. By Lemma 5 and [13], Proposition 1.4, the infinitesimal deformation map

$$\mathcal{P}: T_0(M_1) \longrightarrow H^1(X, \mathcal{O}_X)$$

is surjective.

This implies the existence of an effectively parametrized complete family of deformations of $X$ and the equality $m(X) = \dim H^1(X, \mathcal{O}_X)$.

On the other hand, we have

$$\dim H^0(X, f^*\mathcal{O}_{P^3}) = 15 + 4 \cdot \delta_{r,s+1}$$

by Lemmas 2 and 3. Finally it follows that
\[ \dim H^1(X, \Theta_X) = \mu(S) - 15 - 4 \delta_{r,s+1} \]

by Lemma 5.

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