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Vanishing theorems with algebraic growth and algebraic divisible properties.

(Complex analytic De Rham cohomology 1)

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The purpose of the present note is to announce certain quantitative properties of coherent sheaves and analytic varieties. Results given here are originally and primarily intended for applications to differential forms on complex analytic varieties with arbitrary singularities (c.f. the end of this note). Results stated here are, however, of their own interests. Our basic purpose is to discuss vanishing theorems of certain types where quantitative properties (of objects considered) appear. Details of this note will appear elsewhere.
Quantitative properties examined here are as follows : (I) Asymptotic behavoors w.r.t pole loci. (II) Divisible properties w.r.t. subvarieties. Our arguements will be divided to two steps : (i) A step in which the asymptotic behavior only enters. (ii) A step where both asymptotic behaviors and divisible properties appear.

Notational remarks : We write linear functions and monomials as L and M. A couple, denoted by $\sigma = (\sigma_1, \sigma_2)$, is a couple of positive numbers. For a set $\{\sigma^1, \ldots\ldots, \sigma^s\}$ of couples

Maps (L): $\{\sigma^1,\ldots,\sigma^S\} \longrightarrow \sigma'$ and (M): $\{\sigma^1,\ldots,\sigma^s\} \mapsto \xi \in \mathbb{R}$ are said to be of <u>exponential -algebraic type</u> (<u>(e.a)- type</u>) if

$$\sigma' = \{ M_1(\sigma_1^1,\ldots,\sigma_1^s) \times \exp M_2(\sigma_2^1,\ldots,\sigma_2^s) ,\ L(\sigma_2^1+\cdots+\sigma_2^s) \},\ \mathcal{E} = M_1(\sigma_1^1,\ldots,\sigma_1^s) \times \exp M_2(\sigma_2^1,\ldots,\sigma_2^s).$$

(I) We start with a datum $(\Delta(r; P_0)\ ,\ X,\ D\ )$ of a polydisc Δ with the center P_0 of radius r in $\mathbb{C}^n$, a variety* $V \ni P_0$ in Δ and a divisor $D \ni P_0$ in Δ. We write irreducible decom-positions of X and D at P_0 as $X_{P_0} = \bigcup_j X_{P_0 j}$ and $D_{P_0} = \bigcup_j D_{P_0 j}$.

Assume that D contains the singular locus of X and that $X_{P_0 j} \not\subset D_{P_0 j'}$ for any pair (j,j'). Moreover , consider a coherent sheaf (F) admitting a resolution of the following form.

$$0 \longrightarrow \mathcal{O}^{d_s} \longrightarrow \cdots\cdots\cdots \xrightarrow{K_2} \mathcal{O}^{d_1} \xrightarrow{K_1} F(\subset \mathcal{O}^d) \rightarrow ,$$

where K ' s are matrices whose coefficients are meromorphic functions on X with the pole $D' = D \cap X$. A point P is near P_0 if P is is a small neighbourhood of P_0. For a point near P_0, the

* A variety and a function are always complex analytic ones in this note.

intersection $\Delta(r; P) \cap X$ is denoted by $\Delta(r; P, X)$. Moreover, for a point $Q \in \Delta(r; P, X) - D$, we mean by $N_\sigma(Q, D)$ the neighbour-hood of Q in X defined by $N_\sigma(Q, D) = \{ Q' : d(Q, Q') \leq \sigma_1 d(Q,D)^{\sigma_2} \}$.

Define a (non locally finite) covering $(A)_\sigma(r;P,D)$ of $\Delta(r;P,X) - D$ by $(A)_\sigma(r; P,D) = \{ N_\sigma(Q,D) : Q \in \Delta(r; P, X) - D \}$. We formulate our problem in terms of such covrrings (A). A q- cochain $\varphi \in C^q(N((A)_\sigma(r;P,D)), (F))$; N= nerve ; is of algebrqic growth $\alpha = (\alpha_1, \alpha_2)$ if

$$|\varphi(Q)| \leq \alpha_1 d(Q,D)^{-\alpha_2}$$

Then our first result is as follows.

Lemma 1. A vanishing theorem with algebraic growth.

There exists a datum (L_1, L_2, M) depending on (X,D,F) only such that the following is valid .

For a cocycle $\varphi \in Z^q(N((A)_\sigma(r;P,D), F))$ of algebraic growth α, there exists a cochain $\varphi' \in C^{q-1}(N(A_\sigma(r;P,D), F)$ of growth (α_1', α_2') so that

(1) $\delta(\varphi') = \varphi$,

(2) $(\alpha_1', \alpha_2') = (\alpha_1'' \times M(r)^{-1}, \alpha_2'')$, $\sigma' = L_2(\sigma)$

and $r' = M(r)$ with $(\alpha_1'', \alpha_2'') = L_1(\alpha)$.

60 In the equation (1) φ is regarded as an element in $C^q(N((A)_\sigma(r;P,D)),F)$ by taking a refinement and a restriction suitably.

Remark 1. For a domain $\Sigma = \Delta(r;P) \times C^N$ and for a coherent sheaf F' over Σ, a similar result to the lemma 1 is valid (by changing the distance to D by $\Sigma_j |x_j|$: (x_j) are cooodinates of C^N).

Remark 2. That the datum $(\mathcal{U}_1, \mathcal{U}_2, M)$ in the lemma 1 is independent of points P shows that the lemma 1 is of semi-global nature.

Remark 3. Cohomology theories with growth conditions have recently been studied by various persons for various purposes (c.f. [1], [3], [4], [5], [6]). Our methods depend on examinations of Cousin integrals and of combinatorial arguements. We procced along standard methods of discussing vanishing theorems on Stein varieties and have many similarities with works cited above. However our situation as well as our statement are, to the author's knowledge, new. Our notion of 'algebraic growth' was inspired the notion of 'polynomial growth' due to R. Narasimahn (His result is that of $\bar{\partial}$-estimation of L. Hörmander (see [5])).

(   ) Here we state our basic problem in this note .

We consider a proper subvariety V of X and a set of analytic functions $(f) = (f_1,\dots,f_t)$. Let $V_{P_0} = \cup V_{P_0 j}$ be the irreducible decomposition of V at P_0. We assume conditions : $V_{P_0 j} \neq X_{P_0 j'}$ for any pair (j,j') , $V_{P_0 j} \neq D_{P_0 j''}$ for any pair (j, j'') and $f_i \not\equiv 0$ on $X_{P_0 j}$ for any pair (i,j). Moreover we assume that V is the zero locus of (f) on X . Our problem is formulated in terms of (f).

Let $\mathcal{X}^{m,0}$ be the subsheaf of $\mathcal{O}_X$ defined to be $\mathcal{O}_X(f_1^m,\dots,f_t^m)$. This sheaf $\mathcal{X}^{m,0}$ is our basic subject . We associate sheaves $\mathcal{X}^{m,s}$ $(s=1,\dots,t-1)$ to $\mathcal{X}^{m,0}$ in the following manner : For a multiindex $I_s=(i_1,\dots,i_s)$ define an element $\mathfrak{f}(I_s,m) \in \mathcal{O}^{\binom{t}{s-1}}(X)$ by

$$\mathfrak{f}(I_s,m)(J_{s-1}) = \begin{cases} 0, & \text{if } J_{s-1} \not\subset I_s , \\ (-1)^{k-1} f_{i_k}^m & \text{if } J_{s-1} = (i_1,\dots,\hat{i}_k,\dots,i_s), \end{cases}$$

(for a vector $g \in \mathcal{O}^{\binom{t}{s}}$, $g(J_s)$; $J_s=(j_1,\dots,j_s)$ is J_s-component of g.)

Using the above elements $\mathfrak{f}(I_s,m)$, define sheaf homomorphisms $K(s,m)$ $(s=1,\dots,t)$: $\mathcal{O}^{\binom{t}{s}} \longrightarrow \mathcal{O}^{\binom{t}{s-1}}$ by

$$K(s,m)(g^s) = \sum_{I_s} g^s(I_s)\, \mathfrak{f}(I_s,m).$$

Note that $\mathcal{X}^{m,0} = K(s,m)\mathcal{O}^t$. Define $\mathcal{X}^{m,s}$ $(s=1,\dots,t-1)$ by $\mathcal{X}^{m,s} = K(s,m)(\mathcal{O}^{\binom{t}{s+1}})$.

If the jacobian condition : $\det \frac{\partial(f_1,\dots,f_t)}{\partial(x_1,\dots,x_t)} \neq 0$ holds at each point on X ($(x_1,\dots,x_t)$ are coordinates of X) then the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{K(t,m)} \mathcal{O}^{\binom{t}{t-1}} \longrightarrow \dots\dots \longrightarrow \mathcal{O}^t \xrightarrow{K(1,m)} \mathcal{X}^{m,0} \longrightarrow 0$$

holds. In general the above sequence fails. It is , however, found that associating $\mathcal{X}^{m,s}$ to $\mathcal{X}^{m,0}$ is meaningful in our discussion concerned with divisible properties : We formulate two problems in terms of sheaves $\mathcal{X}^{m,s}$ rather than $\mathcal{X}^{m,0}$ only: (i) A cochain $\psi \in C^q(\mathcal{A}_\sigma(r;P,D),\mathcal{X}^{m,s})$ is of algebraic growth $(\alpha_1,\alpha_2,\alpha_3)$ if ψ has the following property.

If $\bigcap_{i=1}^{q+1} N_\sigma(Q_i) \neq \phi$, the element $\psi(\bigcap_{i=1}^{q+1} N_\sigma(Q_i) ; Q)$ is written as

$$(1) \quad \psi(Q) = \sum_{I_s} g^s \cdot \sigma(I_s,m) \; ; \quad g^s \in \mathcal{O}^{\binom{t}{s}}(\bigcap N_\sigma(Q_i))$$

with the estimation

$$(2) \quad |g^s| \leq \alpha_1 d(Q,D)^{-\alpha_2} d(Q,V)^{/\alpha_3}$$

Now our first assertion is as follows .

Theorem 1 . There exists a datum $(L, L_i, M_i ; i=1,2,3)$ depending on (X,V, D,(f)) only with which the following are valid.

For a cocycle $\varphi \in Z^q(N(\mathcal{A}_\sigma(r,P,D), \mathcal{X}^{m,s})$ of growth $(\alpha_1, \alpha_2, \alpha_3)$

there exists a cochain $\psi' \in C^{q-1}(\cdot N(A)(r, P, D)), X^{m,s})$ of growth $(\alpha'_1, \alpha'_2, \alpha'_3)$ such that the equation

$$(1) \qquad \delta(\psi') = \psi ,$$

and

$$(2) \qquad r' = M_1(r),\ \sigma' = L(\sigma), \quad (\alpha'_1, \alpha'_2, \alpha'_3) = (M_2(\alpha_1, \sigma_1, r) \cdot \exp M_3(\alpha_2, \sigma_2, \alpha_3),\ L_1(\alpha_2 + \sigma_2),\ L_2(\alpha_3)),$$

so far as $\alpha_3 \geq L_3(m)$.

In the above statement an emphasize is put on a ' middle point ' of the asymptotic behavior and the divisible property : ' The independenceness of the order of the pole α'_2 from the divisible index m ' is a key point . This is possible by diminishing the given α_3 to α'_{3} $(= L_3(\alpha_3))$ and will be made an essential use of in our application of Theorem 1 to differential forms :

The second problem is as follows : Given an element $g^s \in (0)^{(s)}_t$ so that $K(s,m)(g^s) = 0$ ($s \leq 1$, if $s = 0$ we do not consider an algebraic condition) , find $g^{s+1} \in 0^{(s+1)}$: $K(s+1, m)(g^{s+1}) = g^s$.

Precisely let us consider a proper subvariety V' of V . Instead we do not consider the divisor D in this case . Take a point $P \in V - V'$. $N_r(P, V')$ is a neighbourhood of P in X defined to be $\{Q: d(Q, \mathcal{Q}') \leq r\}$.

An element $(g^s\ (s=0,\dots,t-1) \in \mathcal{O}^{\binom{t}{s}}(N_r(P,V'))$ is a testifying datum with quantitative property (b,α_3) ie the following are valid.

$(A)_1$ $K(s,m)(g^s) = 0 \quad (s \geq 1)$,

$(A)_2$ $|g^s| \leq b d(Q,V)^{/\alpha_3}$, $Q \in N_r(P,V')$. $(s=0,1,\dots,t-1)$.

Then our second assertion is follows .

Theorem 2 . ' Weak syzygy with quantities .

There exists a datum $(M, M_i\ (i=1,2,3),\ L_i(i=1,2,3),\ \sigma)$ depending on $(X,V,V',(f))$ only with which the following is true.

(B) For a testifying datum g^s with quantitative property (b,α_3), there exists an element $g^{s+1} \in \mathcal{O}^{\binom{t}{s+1}}(N_r(P,V'))$ so that

$(B)_1$ $K(s,m+1)\,g^{s+1} = g^s$,

$(B)_2$ g^{s+1} is of quantitative property (b', α'_3),

where $b' = M_1(r)^{/-1} \cdot M_2(b) \cdot \exp M_3(\alpha_3)$, $\alpha'_3 = L_1(\alpha_3)$ and $r' = M(r)$, holds so far as $\alpha_3 \geq L_2(m)$, $r \leq \sigma_1 d(P,V')^{/\sigma_2}$.

Remark . Problems of differential forms which we consider are as follows . Detailed arguements will be given elsewhere . Here we do a sketch of our problems : Start with a datum (X,V,D,P_0) defined in previous arguements : For differential sheaves $\Omega = \Omega_X$ and $\Omega(*D) = \Omega_X(*D')$ = sheaf of meromorphic forms with the pole $D' = D \cap X$, $\hat{\Omega}$ and $\hat{\Omega}(*D')$ are completions of Ω and $\Omega(*D')$

along V . Our problems are spoken in terms of the above two rings $\hat{\Omega}$, $\hat{\Omega}(*D)$. Concerning the ring $\hat{\Omega}(*D)$ our problem is to show the isomorphism

$$(R_i^* \mathbb{C})_{P_0} \cong H^*(\hat{\Omega}(*D)_{P_0}) \; ; \; i \text{ is the inclusion;}$$

$i : V-D \hookrightarrow V$.

This is a generalization of a well known theorem of A. Grothendieck [ ].

Concerning the ring $\hat{\Omega}$ we ask , under the assumption of X = smooth variety , the exact sequence : $0 \to \hat{\Omega}^0 \xrightarrow{d} \hat{\Omega}^1 \to \cdots$, and a divisible property of the integration of differential forms .

Precise meaning of the above problems will be discussed , the author plans , in an another announcement . Roughly the theorem 1 and the lemma 1 are analytic keys to our problems on differential forms .

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Quantitative properties of analytic varieties in C^∞- differentiable aspects

(Complex analytic De Rham Cohomology 2.)

This is a continuation of 1°. We state certain (elementary) quantitative properties of real analytic varieties. Our results stated here are intended for differential forms as in 1°. As in 1° two properties ... asymptotic behavior w.r.t. a subvariety and divisible properties w.r.t a subvariety --- will be studied.

Let V be a real analytic manifold, and let W be a real analytic subvariety of W. We assume that $V-W$ is covered by a finite coordinate neighbourhoods U_j. A C^∞- function φ in $V-W$ will be said to have asymptotic behavior w.r.t. W if the following is true.

For any differential operator $D^i = \frac{\partial^{m_1+\cdots+m_k}}{\partial x_{i,1}^{m_1} \cdots \partial x_{i,k}^{m_k}}$. $D^i\varphi_i$ is estimated as

$$|D^i(\varphi(Q))| \leqq \gamma_1^i(D)\, d(Q:W)^{-\gamma_2^i(D)},$$

where (x^i) = coordinates of U^i.

A C^∞- form φ in $V-W$ has asymptotic

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behavior w.r.t. W if each coefficient ϑ_J has such a property .

Let (U, V, P) be a datum composed of a domain U in R^n a subvariety V in U and a point P in V . This datum is fixed throughout this note . Our results are two . We state at first our results in an 'intrinsic' manner ((i.e) in terms of varieties in question and of coordinates (x)) . The first problem is as follows :

(I) C^∞ thickings and their quantitative properties .

In this problem we consider a proper subvariety V' of V in addition to the datum (U,V,D). For a couple σ, we mean by $N_\sigma(V:V')$ the neighbourhood of V- V' defined by $N_\sigma(V:V') = \{ N(Q : V') : Q \in V-V' \}$. A neighbourhood N of V- V' is a C^∞-thicking of V-V' if $H^*(V-V' : R) \cong H^*(N : R)$. Let $\{N_j ; j \in Z^+\}$ be a direct system (w.r.t. the inclusion relation) of C^∞ thickings of V-V'. For a direct system $\{N_j ; j \in Z^+\}$ the following conditions are always assumed .

(1) For any N_j there exists a couple σ_j such that $N_j \supset N_{\sigma_j}(V:V')$.

(2) For an arbitrarily given σ, $N_j \subset N_\sigma(V:V')$ for a sufficiently large j.

For a neighbourhood N of V- V', $\Omega(N)$ stands for the ring of C^∞-differential forms in N . Moreover, we understand by $\Omega(N: V')$ the subring of $\Omega(N)$ composed of those C^∞-forms having asymptotic behaviors w.r.t. V'. Given a direct system $\{N_j ; j\in Z^+\}$ of C^∞ thickings of V-V', we let $\widehat{\Omega}(N:V')$ be the direct limit: lim. dir.$_j\ \Omega(N_j ; V')$. This ring $\widehat{\Omega}$ is a differential ring in an obvious way. Our first result, which is essentially of elementary nature, is as follows:

Lemma 1. For a fixed datum (U,V,V',P), there exist a neighbourhood U' of P : U⊃U', and a direct system $\{N_j\}$ of C^∞-thickings of V-V' (in U') in such a manner that

$$H^*(V-V' : R) = (H)^*(\widehat{\Omega})$$

holds.

The second problem is as follows.

(I I) Quantitative properties of retraction maps.

In this second situation we start with the datum (U,V,P). Consider a subvariety D' of U such that $D' \not\supset V$. Let I be the interval [0, 1]. A continious map $\tau : I\times U \hookrightarrow U$ is a retraction of (U, V, D') (to P) if the following are satisfied : (i) $\tau(1, Q) = Q : Q\in U$, (i i) $\tau(0, Q) = P ; Q\in U$, (i i i) $\tau : I\times V$ (or $I\times D'$) $\hookrightarrow$ V (D'). Moreover, τ is C^∞ map outside D' if ψ is C^∞-map in $(0,1]\times(U-D')$. For a fixed datum (U,V,D',P), a retraction map τ is assumed to satisfy the above conditions (i),(i i), (i i i) and the differentiable property mentioned just above . A retraction map τ satisfies quantitative conditions w.r.t. (V, D') if the following conditions are satisfied .

(1) There exist triples* $(\beta) = (\beta_1, \beta_2, \beta_3)$ and $(\beta') = (\beta'_1, \beta'_2, \beta'_3)$ in such a manner that the following ' distance preserving property ' to V

$$\beta_1 \cdot d(Q, V)^{\beta_2} \cdot \rho^{\beta_3} \leqq d(Q_\rho, V) \leqq \beta'_1 \cdot d(Q, V)^{\beta'_2} \cdot \rho^{\beta'_3},$$

holds for a point $Q \in U$. Here ρ is in $(0, 1]$ and $Q_\rho = \tau(\rho, Q)$.

(2) For each pair (k, K) ; $k \in Z^+$, $K \in (Z^+)^n$, there exists a triple $\gamma(k, K)$ with which the inequality

$$\left(\partial^k / \partial \rho^k\right) \cdot D_K \left(x_j(Q_\rho)\right) \leqq \gamma(k,K)_1 \, d(Q, D')^{-\gamma(k,K)_2} \cdot \rho^{-\gamma(k,K)_3}$$

holds for each point $Q \in U - D'$.

Then our second assertion, which is also elementary, is as follows.

Lemma 2. Quantitative properties of retractions .

For a given datum (U, V, P) we find a neighbourhood

-* ' Triple' is a triple of positive numbers.

U' of P such that the following is valid .

( ) There exist varieties D_j $(j=1,\ldots,m)$ so that

( ) $\bigcap_j D_j$ is a proper subvariety of V ,

and

( ) for each D'_j there exists a retraction τ_j of (U, V, D_j) satisfying quantitative conditions w.r.t. (V, D_j).

Remark 1 . In both lemmas 1 and 2 neighbourhoods U' with which assertions in lemmas are valid , are cofinal in the set of all the neighbourhoods of P .

Remark 2 . In the lemma 2 it seems quite likely that a ~~divisor~~ subvariety D' is chosen to be the singular locus of V. In our treatment of lemmas we take a ' relative version ' which will be explained soon later . Varieties D'_j appear because of the existence of singular loci of maps considered besides the singular locus of V itself .

We quickly indicate in what a manner the above lemmas are related to our original proof of the complex analytic De Rham cohomology[1] : The lemma 1 is a C^∞ help as well as a C^∞ analogue of the isomorphism : $(R_{i*}\mathbb{C})_P \cong \mathcal{H}^j(\hat{\Omega}(*D))$ (c.f. [3]).

The lemma 2 is used to show a ' divisible property ' of integrations of differential forms : Start with a datum (U,V,P). Let V be the zero locus of an analytic function f . An r-times sifferentiable form φ is m-times differentiable by f if each coefficient φ_J of φ is divisible m-times by f in the category of c^r- functions. Given a c^r closed form φ in U which is divisible by f m-times $(m \geqslant 0)$. Our problem , whose precise formulation will be given in an another publishment , is of the following type.

To find a domain U' : $U \supset U' \ni P$,and a $c^{r'}$-form φ' in U' in such a manner that

$$d\varphi' = \varphi \quad , \text{ and } \varphi' \text{ is } m'\text{-times divisible by } f \ (m' \geqslant 0).$$

It is not difficult to see that the lemma 2 ,combined with a standard method of proving the Poincaré's lemma (c.f. De Rham [2]), plays a key roll to the above explained problem .

Lemmas 1 and 2 are of intrinsic nature to the given data (U,V,P) and (U,V,V',P). In our discussions of these lemmas some other materials are introduced . Materials introduced will be explained soon later . Several interesting (to the author) problems arise through our approach though it is an another thing if our approach gives 'short ways' to the above lemmas. We attach a series of varieties $\{V^1, V^2, \cdots, V^s = V\}$, where $V^1 \subset U^1 \subset \mathbb{R}^1, \cdots, V^i \subset U^i \subset \mathbb{R}^i \cdots$ to V. Also we attach a series of stratification $\mathcal{S}^1, \cdots, \mathcal{S}^s$ of $V^1, \cdots, V^s$. We do not assume the basic conditions A, B of H. Whitney by some technical reasons. We assume that $\pi_{i+1,i} : V^{i+1} \to V^i$ is (not necessarily surjective) finite ramified map

We will state our problem in terms of these strata. Roughly we impose 'compatiblity conditions' between objects considered and stratifications $\{\mathcal{S}^i\}$. In doing our arguements the following condition will be considered for $\{\mathcal{S}^i\}$.

(i). If $S^{j'} \in \mathfrak{S}^{j'}$ then $\pi_{jj'}(S^{j'}) \in \mathfrak{S}^{j}$ ($j < j'$), and conversely if S^j is in $\mathfrak{S}^j$ then $\pi^{-1}_{jj'}(S^j) \cap U^j$ is a union of strata in $\mathfrak{S}^{j'}$.

In addition to the series S, sets of analytic functions $\mathfrak{F} = \{ f^j_t(S) \ ; \ j = 1,\ldots,n, \ S \in \mathfrak{S}, \ t = 1,\ldots, j\text{-dim}(S^j)\}$ are considere

The following conditions for $\mathfrak{F}$ are assumed.

$(\quad)_1$ $f^j_t = f^j_t(x_1,\ldots,x_{n_j} : x_{n_j+t})$ is a monic polynomial in x_{n_j+t} with variables $(x_1,\ldots,x_{n_j})$: $n_j = \dim(S)$.

(i $)_2$ $f^j_t(S^j_t)$'s vasish on S^j and the 'jacobian condition:

$$\det\left[\frac{\partial(g_1,\ldots\ldots,f_{n_j})}{\partial(x_{n_j+1},\ldots,x_j)}\right] \neq 0 \text{ holds.}$$

(i $)_3$ If $S^j_2 > S^j_1$ then the order of $f^j_t(S^j_2)$ = the minimal non-negative integer so that $\partial f^j_t/\partial x_{n_j+t} = 0$: is constant on S^j_1.

The condition (i) is impored in order to controll behaviors of $S^{j'}_2$ (aroun S^j_1) and is important in our quantitative discussions. We work with the data $\{U, V, S, \mathfrak{F}\}$. Then corresponding facts to lemmas 1,2 are formulated in term of S^j's and behaviors of concerned subjects under the projections $\pi_{jj'}$ are examined *. Concerning the lemma 2 we impose conditions to retractions τ of (U^j, V^j) : For each stratium S^j, $\tau_j : U S^j \to S^j$ and if $j' > j$ then $\pi_{jj'} \cdot \tau_{j'} = \tau_j$.

Our situation in the lemma 1 is as follows : A c-thicking N^{8} of S^{j} is an assignment N'^{j} : S^{j} $N'(S^{j})$ = neighbourhood of S^{j} . Basic properties of N' are : For S''^{j} , $N_{N(S'^{j})} = S''^{j}$, (i i) For $S'^{j}_{1}, \ldots$ S'^{j}_{t}, H ($N(S'^{j}_{i})$) H($N(S'^{j})$ V^{j}) . Consider direct systems N'^{j} o c-thickings of S^{j} . A compatible condition with $_{j\,j'}$ similar to (i) is imposed to N'^{j} . Subvarieties V'^{J} ($V'^{n} = V'$) are determined by V' and S . For a series $S^{j}_{1}, \ldots S^{j}_{t}$ rings ($_{=1}^{t}N^{j}(S^{j})$: V'^{j}) and their direct limit ($S^{j}_{1}, \ldots, S^{j}_{t}$; V'j),, are defined in a similar way to V- V': V') , V-V' : V') . Our first key point is to reduce ( ) to the following 'local version '.

( ) For each series $S^{j}_{1}, \ldots S^{j}_{t}$, H*($S^{j}_{1}, \ldots, S^{j}_{ttt}; V'^{j}$) dir.lim.H*(N

The above procedure is similar to that of the residue operator.

Our another key is to associate a finite simple covering A $(N(S^{j}))$ to each series $S^{j}_{11} \ldots S^{j}_{t}$ so that a compatible condition similar to $(i)_{1}$

and and a quantitative condition isare satisfied. (For the definition of the simple covering , see A. Weil [ ]) .

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§ 2-1 C^∞- thickings

n. 1. In this numero, our arguements will be concentrated to problems of expressing the (topological) cohomology group by C^∞ - differentiable forms with suitable asymptotic behavior. We shall fix some notations used here. Let U be a domain in $\mathbb{K}^N$, and let W be a closed set in U. (we do not assume that W is a $\mathbb{K}$ - analytic variety in general.) Moreover, assume that a finite set $\mathcal{S}^*$ of $\mathbb{K}$ - analytic manifolds is given. We assume the following two conditions.

(2.1.1)$_1$ Each element $S_\lambda^* \in \mathcal{S}^*$ is equidimensional (of dimension $n(\lambda) \leqq \dim(U)$).

(2.1.1)$_2$ W is a disjoint union of S_λ^*'s in $\mathcal{S}^*$: $W = \bigcup_\lambda S_\lambda^*$; $S_\lambda^* \in \mathcal{S}$

Let $\bar{S}^*$ be the set theoretical closure of S^* in U. We call $\bar{S}^*$ the closure of S^* (as usual. On the otherhand, $\bar{S}^* - S^*$ is called the

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*. This numero is independent of the other parts of this section.

(set - theoretical) frontier of S^*. Concerning the set $\mathcal{S}^*$, we shall impose the following (frontier condition) only.

(2.1.1)′ For each $S^* \in \mathcal{S}^*$, Fron (S^*) is a disjoint union of lower dimensional strata in $\mathcal{S}^*$.

The following follows easily from $(2.1.1)_2$ and (2.1.1)′

$(2.1.1)'_1$ If S^* has a common point with $S (\neq S')$ then $S^{*\prime} \subset$ Fron (S) holds.

The set $\mathcal{S}^*$ satisfying (2.1.1) and (2.1.1)′ will be called a pre- stratifisation of W, and W is called a pre stratified set.

(Remark) In this numero, we shall not impose basic conditions of R.Thom [ ] other than (2.1.1)′. Especially we do not assume the basic conditions concerning the existence of retraction maps (c.f. R. Thom [ ]). By weakening conditions as explained in the above, we are led to work with differential forms with weaker conditions.

2 - 1 - 2

In this numero also, we consider not only strata S^{*}'s but also certain R-analytic functions related to strata S^{*}'s. First we consider the following condition.

(2.2.1)$''_{1}$ For each $S^{*}_{\lambda} \in \mathcal{S}^{*}$, there exists a real analytic functions $g(S^{*})$, $h(S^{*})$, defined in U, in such a manner that the following relations are valid.

$$(2.2.1)''_{1.1} \qquad g(S^{*}, Q) \sim d(Q, \overline{S}^{*})$$

for any points Q in a neighbourhood $N_{\delta}(S^{*}, \mathrm{Fron}(S^{*}))$ with a suitable couple (δ).

$$(2.2.1)''_{1.2} \qquad h(S^{*}, P) \sim d(P, \mathrm{Fron}(S^{*}))$$

for any points P in S^{*}.

In order to impose a further condition, define a subset $W^{+}(S^{*})$ of W attached to each S^{*}, as follows

$(W^{+})_{2.1}$ $W^{+}(S^{*}) = \bigcup_{S^{*\prime}} S^{*\prime}$, where $S^{*\prime}$'s are elements in $\mathcal{S}$

so that condition: $S^{*\prime} \succ S^{*}_{,}$ $\dim(S^{*\prime}) < \dim(U)$ hold.

A further condition which we consider is formulated in the following manner.

$(2.1.1)''_{2}$ For any $S^{*} \in \mathcal{S}^{*}$, there exists a real analytic function $g(W^{+}(S^{*}))$ in U in such a way that the relation

$$g(W^{+}(S^{*}) : Q)) \sim d(Q, W^{+}(S^{*}))$$

is valid in a neighbourhood $N_{\delta}(S^{*}, \mathrm{Fron}(S^{*}))$ of S^{*} with a suitable (δ').

Besides the above conditions $(2.2.1)_{1}'$, $(2.2.1)''$, we consider the following two quantitative conditions on $\mathcal{S}^{*}$.

$(2.1.1)'''_{1}$ $d(P, \bar{S}^{*\prime}) \sim d(P, \bar{S}^{*\prime} \cap \bar{S}^{*})$ if $P \in S^{*}$, and $\bar{S}^{*\prime} \cap \bar{S}^{*} \neq \phi$

Note that the conditions $(2.1.1)'''_{1}$, $(2.1.1)$ and $(2.1.1)'$ imply the following

$(2.1.1)'''_{2}$ If $S^{*\prime} \cap S^{*} = \phi$ then $N_{\delta''}(S^{*}, \mathrm{Fron}(S^{*})) \cap S^{*\prime} = \phi$ for a small (δ'').

A $\mathbb{K}$-analytic pre-stratified set W satisfying $(2.2.1)''$ and endowed with real analytic functions $\mathfrak{F}^*(\mathcal{S}^*) = \{ g (s^*),$ $h (s^*),$ $g (W^+(s^*)) \; s^* \in \mathcal{S}^* \}$ with which the conditions (2.1.1)″, (2.1.19)″$_1$ hold, will be called

<u>(Q− D) − admissible K − pre- stratified set</u>

Henceforth, when we speak of (Q − D)− pre stratified set W, <u>we assume that a set of functions $\mathfrak{F}^*(\mathcal{S}^*)$ are fixed.</u> Our arguement in this section is mainly related to reduce our problems spoken in terms of W to corresponding problems of pre - stratification $\mathcal{S}^*$ We shall begin by introducing certain notations, which will be used in the later arguements.

Let $\mathcal{N}: S^* \in \mathcal{S}^* \to N(S^*)$ be an assignment which assignes to each stratum $S^* \in \mathcal{S}^*$ a neighborhood $N(S^*)$ of S^* in the ambient space of $W^*(=U^*)$. We assume that $N(S^*)$ is of N (N = dimension of the ambient space) dimensional manifold ($\subset U^*$) and moreover, that the following disjoint conditions are valid.

(1) Unless $S^*_{\lambda'} \succ S^*_{\lambda}$ $N(S^*_{\lambda}) \cap S^*_{\lambda'} = \emptyset$,

(2) If $N(S^*_{\lambda}) \cap N(S^*_{\lambda'}) \neq \emptyset$ then one of the inclusion relations $S^*_{\lambda} \succ S^*_{\lambda'}$ or $S^*_{\lambda'} \succ S^*_{\lambda}$ holds.

We let $N_{W^*}(S^*_{\lambda})$ be the intersection : $N(S^*_{\lambda}) \cap W^*$. The set $N_{W^*}(S^*)$ is a neighbourhood of S^* in W^*. The disjoint conditions (1), (2) lead to the disjoint conditions imposed on $N_{W^*}(S^*)$ obtained from (1) and (2) by changing $N(S^*)$, $N_{W^*}(S^*)$ to $N_{W^*}(S^*)$, $N_{W^*}(S)$. For a given series of strata $S_1, \cdots S_{\ell}$, we define sets $N(S_1, \ldots, S_{\ell})$ (resp. $N_{W^*}(S_1, \ldots, S_{\ell})$) by $N(S_1, \ldots, S_{\ell}) = \bigcap_{\lambda=1}^{\ell} N(S_{\lambda})$ (resp. $N_{W^*}(S_1, \ldots, S_{\ell}) = \bigcap_{\lambda=1}^{\ell} N_{W^*}(S_{\lambda})$).

We assume the following basic isomorphisms for the assignment $\mathcal{N}: S \to N(S)$.

$(2.1.2)_1$ $\quad \mathcal{H}^*(N(S^*)) \cong \mathcal{H}^*(N_{W^*}(S^*)) \cong \mathcal{H}^*(S^*)$ for each stratum S.

$(2.1.2)_2$ $\quad \mathcal{H}^*(N(S^*_1, \ldots, S^*_{\ell})) \cong \mathcal{H}^*(N_{W^*}(S^*_1, \ldots, S^*_{\ell}))$

for any series of strata $S^*_1 \prec \cdots \prec S^*_{\ell}$.

If the above two conditions are satisfied, we say that the assignment $\mathcal{N}: S^* \to N(S^*)$ is a <u>C^∞- thicking of the stratification $\mathcal{S}^*$</u>: The first condition is a desired one in order that the assignment $\mathcal{N}: S \to N(S)$: is called a C^∞-thicking of $\mathcal{S}$. On the otherhand the second condition, which we call <u>Mayer-Vietoris condition</u>, is imposed as a homological version of incidence conditions through which the results obtained around each stratum can be pieced together. We strongly remark <u>that the second one</u> :

Mayer-Vietoris condition is of algebraic nature and that this condition might be possibly taken up as a suitable homological incidence relations for more abstract situations(*) where the notion of the stratification is well understood. Given a closed set $\overset{*}{W}'$ of $\overset{*}{W}$ which can be expressed as a union of strata of $\mathcal{S}^*$. For a given assignment $\mathcal{N}: \overset{*}{S} \longrightarrow N(\overset{*}{S})$, we denote by $N(\overset{*}{W}': \mathcal{S}^*)$ and $N(\overset{*}{W}-\overset{*}{W}': \mathcal{S}^*)$ the unions $\bigcup_{\overset{*}{S}} N(\overset{*}{S})$ $(\overset{*}{S} \subset \overset{*}{W}')$ and $\bigcup_{\overset{*}{S}} N(\overset{*}{S})$ $(\overset{*}{S} \subset \overset{*}{W}-\overset{*}{W}')$ respectively. It is clear that these two sets $N(\overset{*}{W}': \mathcal{S}^*)$ and $N(\overset{*}{W}-\overset{*}{W}': \mathcal{S}^*)$ are neighbourhoods of $\overset{*}{W}'$ and $\overset{*}{W}-\overset{*}{W}'$ respectively. We also remark that $N(\overset{*}{W}-\overset{*}{W}': \mathcal{S}^*) \cap \overset{*}{W}' = \emptyset$ holds. Now we show the following

Proposition 2.1.1 Given a c^∞-thicking $\mathcal{N}$ of the stratification $\mathcal{S}^*$ of $\overset{*}{W}$ and a subvariety $\overset{*}{W}'$ (expressed as an union of strata of $\mathcal{S}^*$.), the pair $(N(\overset{*}{W}, \mathcal{S}^*), N(\overset{*}{V}', \mathcal{S}^*))$ is a c^∞-thicking of the pair $(\overset{*}{W}, \overset{*}{W}')$ while $N(\overset{*}{W}-\overset{*}{W}', \mathcal{S}^*)$ is a c^∞-thicking of $\overset{*}{W}-\overset{*}{W}'$.

We paraphase the above result simply that a c^∞-thickings of the stratification $\mathcal{S}^*$ determines c^∞-thickings of $(\overset{*}{W}, \overset{*}{W}')$ as well as of $(\overset{*}{W}-\overset{*}{W}')$.

Proof. Let $\mathcal{S}^*_{\overset{*}{W}'} = \{\overset{*}{S} \in \mathcal{S}^* : \overset{*}{S} \subset \overset{*}{W}'\}$. From the condition imposed on the subvariety $\overset{*}{W}'$, it is clear that a collection $\mathcal{S}^*_{\overset{*}{W}'}$ determines a stratification of $\overset{*}{W}'$. It is obvious that conditions $(2.1.2)_1$ and $(2.1.2)_2$ are valid for the strata in $\mathcal{S}^*_{\overset{*}{W}'}$. Therefore the assignment

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(*) For algebraic varieties with arbitrary singularities over a field of characteristic p, notions of -adic completion have been developped recently by S. Lubkin and D. Meredoth. It would be meaningful to try the possiblities of translations of proposition 2 to such abstract (and atractively opend) subjects especially with connection of finiteness property of -adic cohomology theory.

$N_{W'}: S^* \in \mathcal{S}^*_{W'} \to N(S^*)$: determines a c^∞-thicking of the stratification $\mathcal{S}^*_{W'}$ of W'^*. Including the case where $W'^* = \emptyset$, we know that it is sufficient for our assertion to show that $N(W^*-W'^*: \mathcal{S}^*) = \bigcup_{S^*} N(S^*)$ $(S^* \in W-W')$ is a c^∞-thicking of $W-W'$. We also remark that the case of the neighbourhood $N(W-W', \mathcal{S}^*)$ of $W^*-W'^*$ is the unique case which we shall make use of later. Now the verification of the above explained fact is a direct consequence of proposition 2.1.1' which will be stated soon after:

Given a topological space X and two sets of open subsets of X: $\mathfrak{U} = \{U_n\}$, $\mathfrak{U}' = \{U_{n'}\}$ $(n \in \mathbb{Z}^+)$. We assume the inclusion relation $U_n \supset U_n'$ for each n. Moreover, we assume that the coverings $\mathfrak{U}$ and $\mathfrak{U}'$ are locally finite coverings of the sets $\bigcup_n U_n$ and $\bigcup_{n'} U_{n'}$ respectively. Finally we assume that the Mayer-Vietoris condition

$$(2.1.3)_1 \qquad \mathcal{H}^*(U_{n_1} \cap \cdots \cap U_{n_\ell}) \cong \mathcal{H}^*(U'_{n_1} \cap \cdots \cap U'_{n})$$

is valid for any indices $(n_1, \ldots, n_\ell)$,

and that

Under the above asumptions we show the following.

Proposition 2.1.1' $\mathcal{H}^*(\bigcup_n U_n) = \mathcal{H}^*(\bigcup_n U_n')$.

Proof. Given indices $(n_1, \ldots, n_\ell)$, we let $U_{n_1,\ldots,n_\ell}$ and $U'_{n_1,\ldots,n_\ell}$ be intersections $\bigcap_{i=1}^{\ell} U_{n_i}$ and $\bigcap_{i=1}^{\ell} U'_{n_i}$ respectively. For each open set U_μ (resp. U'_μ), we define $U_{n_1\ldots n_\ell}(\mu)$ $(U'_{n_1\ldots n_\ell}(\mu))$ by

$$U_{n_1,\ldots,n_\ell}(\mu) \text{ (resp. } U'_{n_1,\ldots,n_\ell}(\mu)) = U_{n_1\ldots n_\ell} \cap U_\mu \text{ (resp. } U'_{n_1\ldots n_\ell} \cap U'_\mu).$$

We let $\mathfrak{N}_{n_1 \cdots n_\ell}$ and $\mathfrak{N}'_{n_1 \cdots n_\ell}$ be finite open coverings of $U_{n_1 \cdots n_\ell}$ and $U'_{n_1 \cdots n_\ell}$ composed of open sets of the form $U_{n_1 \cdots n_\ell}(\mu)$ $(\neq \emptyset)$ and $U'_{n_1 \cdots n_\ell}(\mu)$ $(\neq \emptyset)$. It is clear that, for indices $\mu_1, \ldots, \mu_t$, the Mayer-Vietoris isomorphism of the following form follows from the Mayer-Vietoris isomorphism $(2.1.2)_2$

$$(2.1.3)_2 \qquad \mathcal{H}^*(U_{n_1 \cdots n_\ell}(\mu_1) \cap \cdots \cap U_{n_1 \cdots n_\ell}(\mu_t))$$

$$\cong \mathcal{H}^*(U'_{n_1 \cdots n_\ell}(\mu_1) \cap \cdots \cap U'_{n_1 \cdots n_\ell}(\mu_t)).$$

We first show the following statement:

$(2.1.4)^m_{n_1 \cdots n}$ For any indices $(n_1, \ldots, n_\ell)$ and $U_{n_1 \cdots n_\ell}(\mu_s)(U'_{n_1 \cdots n_\ell}(\mu_s)) \in \mathfrak{N}_{n_1 \cdots n_\ell}(\mathfrak{N}'_{n_1 \cdots n_\ell})$ $(s=1, \ldots, m)$, the following isomorphism

$$(2.1.3)' \qquad \mathcal{H}^*(\cup_{s=1}^m U_{n_1 \cdots n_\ell}(\mu_s)) = \mathcal{H}^*(\cup_{s=1}^m U'_{n_1 \cdots n_\ell}(\mu_s))$$

holds.

The above assertion is proven inductively on m: If $m=1$ then the condition (2.1.3) provides directly an answer. Fix indices $(n_1, \ldots, n_\ell)$ and $(\mu_1, \ldots, \mu_{m+1})$. Then from the inductive hypothesis it follows that $\mathcal{H}^*(\cup_{s=1}^m U_{n_1 \cdots n_\ell}(\mu_s)) \cong \mathcal{H}^*(\cup_{s=1}^m U'_{n_1 \cdots n_\ell}(\mu_s))$ holds. On the otherhand we have the following relations immediately.

$$\left(\cup_{s=1}^m U_{n_1 \cdots n_\ell}(\mu_s)\right) \cap U_{n_1 \cdots n_\ell}(\mu_{m+1}) = \cup_{s=1}^m U_{n_1 \cdots n_\ell, \mu_{m+1}}(\mu_s),$$

$$\left(\left(\cup_{s=1}^m U'_{n_1 \cdots n_\ell}(\mu_s)\right) \cap U'_{n_1 \cdots n_\ell}(\mu_{m+1}) = \cup_{s=1}^m U'_{n_1 \cdots n_\ell, \mu_{m+1}}(\mu_s)\right).$$

Therefore the induction hypothesis (for m) implies the following isomorphism

$$(2.1.3)^{m+1} \quad H^*\left(\bigcup_{s=1}^{m} U_{n_1,\dots n_\ell,\mu_{m+1}}(\mu_s)\right) \cong H^*\left(\bigcup_{s=1}^{m} U'_{n_1,\dots n_\ell,\mu_{m+1}}(\mu_s)\right)$$

(The Meyer- Vietoris sequence as well as the five lemmas applied to the couples ($U_{n_1,\dots,n_\ell}(\mu_s)$, $U_{n_1\dots n_\ell}(\mu_{m+1})$) and ($\bigcup_{s=1}^{m} U'_{n_1\dots n_\ell}(\mu_s)$, $U'_{n_1\dots n_\ell}(\mu_{m+1})$) leads easily to the is isomorphism $(2.1.3)^{m+1}$).

Now the isomorphism $(2.1.3)^{m+1}$, the Meyer – Vietoris sequence and the five lemma applied to couples ($\bigcup_{s=1}^{m} U$, U_{m+1}), ($\bigcup U'^{m}_{s=1}$, U'_{m+1}) (m=1,....,) gives the proof of our proposition.

Now we return to our original problem : Given two c – thickings $\mathcal{N}_i : S^* \to N(S_i^*)$ (i = 1,2) , we understand by $\mathcal{N}_1 > \mathcal{N}_2$ the inclusion relation of the following form.

$$N_1(S^*) \supset N_2(S^*) \quad \text{for each} \quad S^* \in \mathcal{S}^*.$$

It is obvious that the above relation implies the following relation : $N_1(W : \mathcal{S}^*) \supset N_2(W : \mathcal{S}^*)$ between c^∞-thickings of W . For a given c^∞-thicking $\mathcal{N}$, d define differential graded rings $\Omega^\infty(N(S^*))$, $\Omega^\infty(N(S_1^*,\dots S_\ell^*))$, $\Omega^\infty(\bigcup_{\mu=1} N(S_\mu^*))$ to be the ones composed of c- differential forms in $N(S^*)$, $N(S_1^*,\dots S_\ell^*)$ and $\bigcup_\mu N(S_\mu^*)$.

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Let $\mathfrak{F}(\bigcup S^*_\mu)$ be the sub-closed set of $\mathbb{W}$ defined by

$\mathfrak{F}(\bigcup_\mu S^*_\mu) = \bigcup_{\mu'} S^*_{\mu'} - \bigcup_{\mu'} S^*_{\mu''}$; where $S^*_{\mu'}$ exhaust all the strata contained in Fron (S^*_μ)'s while $S^*_{\mu''}$ are at the same time appearing as elements in $\{S^*_\mu\}$.

Define sub rings $\Omega^\infty_{asy}(S^*, \mathfrak{F}ron(S^*))$ of $\Omega(S^*)$, $\Omega_{asy}(S^*_1, S^*_2, \mathfrak{F}r(S^*_2))$ of $\Omega(S^*_1 - S^*_2)$ and $\Omega^\infty_{asy}(\bigcup_\mu S^*_\mu, \mathfrak{F}(\bigcup S^*_\mu))$ of $\Omega^\infty(\bigcup S^*_\mu)$ by the following asymptotic behaviors (for notations used here, see)

$$(1.1.5)\quad\begin{cases}(1.1.5)_1 & |\varphi(Q)| \leqq b_1\cdot d(Q, \mathfrak{F}ron(S^*))^{-b_2}, & b_1, b_2 \in \mathbb{R}^+,\\ (1.1.5)_2 & |\varphi(Q)| \leqq b_1'\cdot d(Q, \mathfrak{F}ron(S^*_2))^{-b_2'}, & b_1, b_2 \in \mathbb{R}^+\\ (1.1.5)_3 & |\varphi(Q)| \leqq b_1''\cdot d(Q, \mathfrak{F}(\bigcup_\mu S^*_\mu))^{-b_2''}, & b_1, b_2 \in \mathbb{R}^+\end{cases}$$

Moreover, we define , for a given closed set $\mathbb{W}'$, a graded ring $\Omega(\mathbb{W} - \mathbb{W}')$ and its subring $\Omega_{asy}(\mathbb{W}-\mathbb{W}')$ to be the ring of C^∞-forms in $N(\mathbb{W} - \mathbb{W}')$ and the ring of elements $\varphi \in \Omega(N(\mathbb{W}-\mathbb{W}'))$ characterized by

$$(1.1.5)'. \qquad |\varphi(Q)| \leq b_1\cdot d(Q: \mathbb{W}')^{-b_2} \ ; \ b_1, b_2 \in \mathbb{R}^+$$

Let us assume that a directed set $\{\mathcal{N}_n\}_{n\in\mathbb{Z}}$ of C^∞-thickings of $\mathcal{S}^*$ is given. In this section when we say a directed set $\{\mathcal{N}_n\}_{n\in\mathbb{Z}}$, we always assume the following conditions .

$(2.1.6)_1$ For a given $(\mathcal{S})$, the relation $\mathcal{N}_n(S^*) \subset \mathcal{N}_s(S^*, \mathfrak{F}ron(S^*))$ holds for each $S^* \in \mathcal{S}^*$ for a suitable integer n.

$(2.1.6)_2$ For a given integer n, there exists a couple $(\mathcal{S})$

in such a manner that the following condition

$$\mathcal{N}_n(S^*) \supset \mathcal{N}_s(S^*, \mathfrak{F}_{ron}(S^*))\ .$$

is valid.

For sets of strata S^*_1, $S^*_2, \ldots, S^*_t$ $\{S^*_1, \ldots, S^*_\ell\}$ define ring homomorphisms $\rho^*_{n,n_1}: \Omega^*(\mathcal{N}_2(S)) \to \Omega^*_{asy}$ by the restrictions of C^∞ forms from $\mathcal{N}(S)$ to $\mathcal{N}(S), \ldots$. Define rings $\hat{\Omega}^\infty_{asy}(\mathcal{N}_n(S^*), \mathfrak{F}_{ron}(S^*))$, $\hat{\Omega}^\infty_{asy}(S^*_1, \ldots, S^*_t, \mathfrak{F}_{ron}(S^*_t))$, and $\hat{\Omega}^\infty_{asy}(\bigcup_{i=1}^{\ell} S^*_i, \mathfrak{F}(\bigcup S^*_i))$ by $\text{proj.}\lim_n \Omega_{asy}(\mathcal{N}(S^*), \mathfrak{F}_{ron}(S^*))$, $\text{proj.}\lim_n \Omega^\infty_{asy}(S^*_1, \ldots, S^*_t, \mathfrak{F}_{ron}(S^*_t))$ and $\text{proj.}\lim_n \Omega^\infty_{asy}(\bigcup_{i=1}^{\ell} S^*_i, \mathfrak{F}(\bigcup S^*_i))$ respectively).

(Strictly speaking, the limit should be understood to be for the fixed set $\{\mathcal{N}_n\}_{n\in Z^+}$. Because there is no confusion, we omit the chosen $\{\mathcal{N}_n\}_{n\in Z^+}$ in the sequel)

The differential operator d is compatible with the limit process, Therefore the rings $\hat{\Omega}^\infty_{asy}$'s are graded differential rings with the operator d . We use the symbols $\hat{\Phi}^\infty_{asy}$'s to mention subrings of $\hat{\Omega}^\infty_{asy}$'s composed of closed forms in $\hat{\Omega}^\infty_{asy}$.

We ask in what a manner the considerations of the ring $\tilde{\Omega}^\infty_{asy}(W-V, W)$ is reduced to the problems of $\hat{\Omega}^\infty_{asy}(S^*_1, \ldots, S^*_t, \mathfrak{F}_{ron}(S^*_t))$. Let us assume that a directed set $\{\mathcal{N}_n\}_{n\in Z^+}$ is given and is fixed.

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Now we shew the following

Proposition. 2.1.2 . Assume the following conditions (2.1.7)$_{1,2}$ are

(2.1.7)$_1$ For each series $S_1^* < \cdots < S_t^*$: $S_i^* \in W - W'$,

and for any integers (n, m), the isomorphisms

$$H^*(\mathcal{N}_n(S_1^*, \cdots S_t^*)) \cong \mathcal{H}^*(\mathcal{N}_{n'}(S_1^*, \cdots, S_t^*))$$

hold,

(2.1.7)$_2$ The incidence isomorphisms

$$\text{Proj.}\lim_n H^*(\mathcal{N}_n(S_1^*, \cdots, S_t^*)) \cong \frac{\hat{\Phi}_{asy}(S_1^* \cdots S_t^*)}{d\hat{\Omega}_{asy}(S_1^*, \cdots S_t^*)}$$

holds for each $S_1^* < \cdots < S_t^*$.

Then the isomorphism

$$H(W - W' : \mathbb{C}) \cong \frac{\hat{\Phi}_{asy}(\mathcal{S}(W - W'), W')}{d\hat{\Omega}(\mathcal{S}(W - W'), W')}$$

holds.

Proof . Given a series of strata : $S_1^* < \cdots < S_t^*$; $S_k^* \in W - W'$, we define a set $\mathcal{S}^M\langle S_1^*, \ldots\ldots, S_t^*\rangle$ of strata S^*'s by the following requirements.

(2.1.8)$_1$ A stratum $S^* \in {}^m\mathcal{S}^*\langle S_1^*, \ldots, S_t^*\rangle$ if and only if the conditions (i) $S^* > S_t^*$ and (ii) $\ell(S^*) \geqq m$ hold. (In the sequel we assume the inequality : $m \leqq \ell(S_t) - 1$.)

Similarly we define a set $\mathcal{S}^{*m}\langle W - W'\rangle$ of strata S's in $W - W'$ by

(2.1.8)$'_1$ a stratum S is in $\mathcal{S}^m\langle W - W'\rangle$ if and only if $S \subset W - W'$ and $\ell(S) \geqq m$ holds. (We assume that $m \leqq n(W-W') = \max_{S \subset W-W'} \ell(S)$ holds in the sequel .)

Let N_n be a C^∞- thicking of $^*\mathcal{S}^*(W-W')$. Denote by $\Omega^\infty(N_n(\mathcal{S}_m\langle S_1^*, \ldots, S_t^*\rangle))$ and $\Omega^\infty(N_n(\mathcal{S}_m(W - W')))$ the rings of C^∞- differentiable forms in $N(\mathcal{S}^{*m}\langle S_1^*, \ldots, S_t^*\rangle) = N_n(S_1^*, \ldots, S_t^*) \cap \left(\bigcup_S N(S^*)\right) : S \in \mathcal{S}^{*m}(S_1^*, \ldots, S_t^*)$ and in $\bigcup_S N_n(S^*)$ ($S^* < \mathcal{S}^{*m}\langle W - W'\rangle$) respectively. Define subclosed set $\mathfrak{F}_m = \mathfrak{F}_m(W)$ of W and $\mathfrak{F}'_m(W - W')$ of $W-W'$ by the following equations.

$$\mathfrak{F}_m(W) = \bigcup_{S^* \subset W} \bar{S}^* \; ; \; \ell(S) \geqq m+1 ,$$

$$= \qquad W,$$

$$\mathfrak{F}_m(W-W') = \bigcup_{S^* \subset W-W'} \bar{S}^* \; : \; S \in W - W' \quad v .$$

Define also sub closed set $\mathfrak{F}_m\langle S_1, \ldots, S_t\rangle$ and $\mathfrak{F}'_m\langle W - W'\rangle$ by

$\mathfrak{F}_m\langle S_1^*, \ldots\ldots, S_t^*\rangle = \bigcup_{S^*} \bar{S}^*$: $S^*\in \mathfrak{F}_m\langle W\rangle$ and $S^* \succ S_t^*$,

$\mathfrak{F}'_m\langle W - W'\rangle = \bigcup_{S^*\subset W} S^* \cup W'$: $S^* \subset W - W'$ and $\ell(S^*) \geq m + 1.$

Denote by $\Omega^{\infty}_{asy}(N_n(\mathcal{S}^{*m}\langle S_1^*, \ldots\ldots, S_t^*\rangle), \mathfrak{F}_m)$ and $\Omega^{\infty}_{asy}(N_n(\mathcal{S}^{*m}\langle W - W'\rangle), \mathfrak{F}'_m\langle W-W'\rangle)$ the subrings of $\Omega^{\infty}(N_n(\mathcal{S}^{*m}\langle S_1^*, \ldots, S_t^*\rangle))$ and $\Omega^{\infty}(N_n(\mathcal{S}^{*n}(W - W')))$ whose elements are characterized by the following asymptotic behaviors respect-ively .

$$(2.1.9)_1 \qquad |\varphi| \leqq b_1 \cdot d(Q, \mathfrak{F}_n)^{-b_2} ; b_1, b_2 \in \mathbb{R}^+,$$

$$(2.1.9)_2 \qquad |\varphi| \leqq b'_1 d(Q, \mathfrak{F}_m)^{-b'_2} ; b'_1, b'_2 \in \mathbb{R}^+.$$

In view of (2.1.1), it is clear that $d(Q, \mathfrak{F}_m) = d(Q, \mathfrak{F}_m\langle S_1^*, \ldots\ldots, S_t^*\rangle)$ holds if Q is in $N_m(\mathcal{S}^{*m}\langle S_1^*, \ldots\ldots, S_t^*\rangle)$. On the otherhand it is also clear that, if Q is in $N_S(S_1^*, \ldots S_t^*, S_{m'}^*)$; $S_{m'}^*$ is of length m' ($< m$) , then $d(Q, \mathrm{Fron}(S_{m'}^*)) \leqq d(Q, \bar{S}_t^* \cap (\mathfrak{F}_n \cup W'))$ holds for $Q \in N_n(S_{m'}, \mathrm{Fron}(S_{m'}))$. Thus the relation $d(Q, \mathfrak{F}_m \cup W') \sim d(Q, \bar{S}_{m'}^* \cap (\mathfrak{F}_m \cup W'))$ also holds for $Q \in N_n(S_{m'}, \mathrm{Fron}(S_{m'}))$.

An another obvious fact is that the relation

$d(\quad , \quad) \quad d(\quad , W - W'))$ holds.

Therefore varieties $\mathfrak{F}_n$ and $\mathfrak{F}_m\langle W - W'\rangle$ in $(2.1.9)_{1,2}$ can be replaced by $\mathfrak{F}_m\langle S_1^* \ldots\ldots S_t^*\rangle$ and by $\mathfrak{F}'_n\langle W - W'\rangle$

respectively. Now we take up sets

$$\mathcal{S}^*_m < S_1^*, S_2^*, \ldots S_t^* >, \quad \mathcal{S}^*_{m+1} < S_1^*, \ldots\ldots, S_t^* > (\, m+1 \leq \ell(S_k^*)-1),$$

and

$$\mathcal{S}^*_m(W - W'), \qquad \mathcal{S}_{m+1} < W - W' > (\, m + 1 \leq n(W-\emptyset)-1).$$

Then the following intersection relations

$$N_n(\mathcal{S}^m < S_1^*, \ldots\ldots, S_t^* >) \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} \{\cup_\mu N_n(S_1^*, \ldots, S_t^*, S'^*_{m+1,\mu})\}$$
$$= \begin{cases} N_n(\mathcal{S}^{m+1} < S_1^*, \ldots, S_t^* >), \\ \cup_\mu N_n(\mathcal{S}^m < S_1^*, \ldots, S_t^*, S'^*_{m+1,\mu} >) \end{cases}$$

$$N_n(\mathcal{S}^m < W - W' >) \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} \{\cup_\mu N_n(S'^*_{m+1,\mu})\} = \begin{cases} N_n(\mathcal{S}^{m+1} < W - W' >), \\ \cup_\mu N_n(\mathcal{S}^m < S_{m+1,\mu} >), \end{cases}$$

hold, where $S^*_{m+1,\mu}$'s and $S'^*_{m+1,\mu}$'s exhaust strata of exactly length $(m+1)$ in $\mathcal{S} < S_1^* \ldots\ldots S_t^* >$ and $\mathcal{S}^{m+1} < W - W' >$ respectively.

Then we obtain the resulting Meyer – Vietoris sequences for every $n \in Z$ in usual manners:

$$(2.1.10) \qquad 0 \to \Omega^\infty(N_n(\mathcal{S}^{m+1} < S_1^*, \ldots, S_t^* >)) \xrightarrow{i_{n,*}} \Omega^\infty(N_n(\mathcal{S}^m < S_1^*, \ldots, S_t^* >))$$
$$\oplus\ \Sigma_\mu \Omega^\infty(N_n(S_1^*, \ldots, S_t^*, S^*_{m+1,\mu})) \xrightarrow{\mathcal{I}_{n,*}} \Sigma_\mu \Omega^\infty(N_n(\mathcal{S} < S_1^*, \ldots, S_t^*, S^*_{m+1,\mu} >))$$
$$\to 0,$$

$$(2.1.10)' \qquad 0 \to \Omega^\infty(N_n(\mathcal{S}^{m+1} < W - W' >) \xrightarrow{i_{n,*}} \Omega^\infty(N_n(\mathcal{S}^m < W - W' >)) \oplus \Sigma_\mu \Omega^\infty(N_n(S'^*_{m+1,\mu}))$$
$$\to \Sigma_\mu \Omega^\infty(N_n(\mathcal{S}^m < S'^*_{m+1,\mu} >)) \to 0$$

Consider subrings $\Omega^\infty_{asy}(N_n(\mathcal{S}^{m+1} < S_1^*, \ldots, S_t^* >), \mathcal{F}_{m+1}(W))$, $\Omega^\infty_{asy}(N_n(\mathcal{S}^m < S_1^*, \ldots, S_t^* >), \mathcal{F}_m(W))$, $\Omega^\infty_{asy}(N_n(S_1^*, \ldots, S_t^*, S^*_{m+1,\mu}), \mathcal{F}_{m+1}(W))$ and $\Omega^\infty_{asy}(N_n(\mathcal{S} < S_1^*, \ldots\ldots S_t^*, S^*_{m+1,\mu} > \mathcal{F}_{m+1}(W))$ of above four rings in (2.1.10). Also consider subrings

$\Omega^{\infty}_{asy}(N_n(\mathcal{S}^{m+1}\langle W-W'\rangle, \mathcal{F}'^{\langle W-W'\rangle}_{m+1}))$, $\Omega^{\infty}_{asy}(N_n(\mathcal{S}^{m}(W-W)), \mathcal{F}'^{\langle W-W\rangle}_{m})$, $\Omega^{\infty}_{asy}(N_n(S_{m+1,\mu}), \mathcal{F}'^{\langle W-W\rangle}_{m+1})$ and $\Omega^{\infty}_{asy}(N_n(\mathcal{S}^{m}\langle S_{m+1,\mu}\rangle), \mathcal{F}_m^{\langle W-W'\rangle})$ of four rings defined in the sequence (2.1.10)′. We remark that the following isomorphisms are valid.

$$\Omega^{\infty}_{asy}(N_n\langle S_1,\dots,S_t, S_{m+1,\mu}\rangle, \mathcal{F}_{m+1}(W)) \cong \Omega^{\infty}_{asy}(N_n\langle S_1,\dots,S_t,S_{m+1,\mu}\rangle, \mathcal{F}_{con}(S_{m+1,\mu})),$$
$$\Omega^{\infty}_{asy}(N_n(S_{m+1,\mu}), \mathcal{F}_{m+1}(W)) \cong \Omega^{\infty}_{asy}(N_n\langle S_{m+1,\mu}\rangle, \mathcal{F}_{con}(S_{m+1,\mu})),$$
$$\Omega^{\infty}_{asy}(N_n(\mathcal{S}^m S_{m+1,\mu}), \mathcal{F}_m(W)) \cong \Omega^{\infty}_{asy}(N_n(\mathcal{S}^m S_{m+1,\mu}), \mathcal{F}(S_{m+1,\mu})),$$
$$\Omega^{\infty}_{asy}(N_n(\mathcal{S}^m\langle S_{m+1,\mu}\rangle), \mathcal{F}_m(W)) \cong \Omega^{\infty}_{asy}(N_n(\mathcal{S}^m(S_{m+1,\mu}), S_{m+1,\mu})$$

Taking projective limits of the above rings, we obtain graded differential rings $\hat{\Omega}_{asy} = \hat{\Omega}_{asy}(\mathcal{S}^{m+1}\langle S_1,\dots S_t\rangle, \mathcal{F})$ ($=$ proj. $\lim_n$ $\Omega(N_n(\mathcal{S}_{m+1}\langle S_1,\dots, S_t\rangle, \mathcal{F}_{m+1}))$, ... etc.......,)

Now we show that the following (Mayer- Vietoris) sequence corresponding to (2. 1. 10) (2.1.10)′ are valid.

$$(2.1.11)_1 \quad 0 \to \hat{\Omega}_{asy}(\mathcal{S}^{m+1}\langle S_1,\dots,S_t\rangle, \mathcal{F}_{m+1}) \xrightarrow{\hat{i}_*} \hat{\Omega}_{asy}(\mathcal{S}^{m}\langle S_1,\dots,S_t\rangle, \mathcal{F}_m) \oplus \Sigma_\mu \hat{\Omega}_{asy}(\langle S_1,\dots,S_t, S_{m+1,\mu}\rangle, \mathcal{F}_{m+1}) \xrightarrow{\hat{r}_*} \Sigma_\mu \hat{\Omega}^*_{asy}(\mathcal{S}^m\langle S_1,\dots,S_t,S_{m+1,\mu}\rangle, \mathcal{F}_m$$

$$(2.1.11)_2 \quad 0 \to \hat{\Omega}_{asy}(\mathcal{S}^{m+1}\langle W-W'\rangle, \mathcal{F}'^{\langle W-W'\rangle}_{m+1}) \xrightarrow{\hat{i}'_*} \hat{\Omega}_{asy}(\mathcal{S}^m\langle W-W'\rangle, \mathcal{F}'^{\langle W-W'\rangle}_m) \oplus \Sigma_\mu \hat{\Omega}_{asy}(S_{m+1,\mu}, \mathcal{F}^{\langle W-W'\rangle}_{m+1}) \xrightarrow{\hat{r}_*} \Sigma_\mu \hat{\Omega}_{asy}(\mathcal{S}^m\langle S_{m+1,\mu}\rangle, \mathcal{F}^{\langle W-W'\rangle}_m) \to$$

Note that, in the above sequences , the 'polar loci' $\mathcal{F}_{m+1}, \dots\dots, \mathcal{F}'_n$ are outside of sets in which differential forms in question are defined .

In the above $\hat{i}_*$, $\hat{r}_*$, $\hat{i}'_*$, $\hat{r}'_*$ are defined by taking limits of $i_{n,*}, \ldots, r_{n,*}$. It is clear from (2.1.10), (2.1.10)' that $\hat{i}_*$ (resp. $\hat{i}'_*$) is injective and that $\hat{r}_* \hat{i}_* = 0$ ($\hat{r}'_* \hat{i}'_* = 0$) holds.

That Coker ($\hat{r}_*$) (resp. Coker $\hat{r}'_*$) = Ker ($\hat{i}_*$) (resp. Ker ($\hat{i}'_*$)) is shown quickly as follows : Given ($\hat{\varphi}$, $\Sigma_\mu \hat{\varphi}_\mu$) (resp. ($\hat{\varphi}'$, $\Sigma_\mu \hat{\varphi}'_\mu$)) in $\hat{\mathfrak{L}}_{asy}(\mathcal{S}^m\langle S_1^*, \ldots, S_t^* \rangle, \mathcal{F}_m(W))$ (resp. $\hat{\mathfrak{L}}_{asy}((W - W'), \mathcal{F}_m\langle W-W' \rangle)$, let us take a representative (φ_n, $\Sigma \varphi_{n,\mu}$) (resp. (φ'_n, $\Sigma \varphi'_{n,\mu}$)) in the ring

$\mathfrak{L}^\infty_{asy}(N_n(\mathcal{S}^m\langle S_1, \ldots, S_t \rangle), \mathcal{F}_m) \oplus \Sigma_\mu \mathfrak{L}^\infty_{asy}(N_n(S_1, \ldots, S_t, S_{m+1,\mu}), \mathcal{F}_{m+1})$
(resp. in $\mathfrak{L}(N_n(\mathcal{S}^m\langle W-W' \rangle), \mathcal{F}'_m(W))\oplus \Sigma_\mu \mathfrak{L}(N_n(S_{m+1,\mu}), \mathcal{F}'_{m+1}(W))$)
for a suitable n so that $r_{n,*}(\varphi_n, \Sigma \varphi_{n,\mu}) = 0$ (resp. $r'_{n,*}(\varphi'_n, \Sigma \varphi'_{n,\mu}) = 0$) holds.

From the sequances (2.1.10) (2.1.10)' there exists the unique element $\varphi_n \in \tilde{\mathfrak{L}}^\infty(N_n(\mathcal{S}^{m+1}\langle S_1, \ldots, S_t \rangle))$ (resp. $\varphi'_n \in \mathfrak{L}^\infty(N_n(\mathcal{S}^{m+1}(W-W')))$ in such a way that $i_n(\varphi_n) = (\varphi_n, \Sigma \varphi_{n,\mu})$ (resp. $i'_n(\varphi'_n) = (\varphi'_n, \varphi'_{n,\mu})$) holds. Examine asymptotic behaviors of elements φ_n (resp. φ'_n). (i) That $i_{n,*}(\varphi_n)$ (resp. $i'_{n,*}(\varphi'_n)$) is equal to $\Sigma \varphi_{n,\mu}$

(resp. $\sum_\mu \varphi_{n,\mu}$) $\in \sum_\mu \mathcal{SL}^\infty(N_n(\mathcal{S}^m\langle S_{m+1,\mu}\rangle, \mathcal{F}_{n+1})$ (resp. $\mathcal{SL}^\infty(N_n(\mathcal{S}^m\langle S_{m+1,\mu}\rangle, \mathcal{F}_{m+1})$)

shows that φ_n (resp. φ_n') have desired asymptotic behaviors

(in $\bigcup N_\mu$ (($S_1^*, \ldots, S_t^*, S_{m+1,\mu}^*$) (resp. $\bigcup N_n(S_{m+1,\mu}^*)$)) w.r.t. $\mathcal{F}_{m+1}$ (resp. $\mathcal{F}_{m+1}'\langle W-W'\rangle$)

(ii) On the otherhand, if Q (resp. Q') ($\in N_n(\mathcal{S}^m\langle S_1^*, \ldots, S_t^*\rangle$)

(resp. $N_n(\mathcal{S}^m\langle W-W'\rangle)$) is outside of $\bigcup_\mu N_n(S_1^*, \ldots, S_{m+1,\mu}^*)$

(resp. $\bigcup_\mu N_n(S_{m+1,\mu}^*)$) then inequalities of the forms

$$d(Q, \mathcal{F}_{m+1}\langle W\rangle) \leq c_1 \cdot d(Q, \mathcal{F}_m\langle W\rangle)^{c_2}$$

$$d(Q', \mathcal{F}_{m+1}'(W-W')) \leq c_1' \cdot d(Q, \mathcal{F}_m\langle W-W'\rangle)^{c_2'}$$

are valid with constants (c) , (c') 's .

The above inequalities are sufficient to assure the inequality of the following forms for $Q \in N_n(\mathcal{S}^n\langle S_1^*, \ldots, S_t^*\rangle)$

(resp. $Q' \in N_n(\mathcal{S}_m\langle V - V'\rangle)$)

$$|\varphi_n(Q)| < \tilde{b}_1 \cdot d(Q, \mathcal{F}_{m+1}\langle W\rangle)^{-\tilde{b}_2} \quad ; (\tilde{b}_1, \tilde{b}_2) \in \mathbb{R}^+,$$

$$|\varphi_n'(Q')| < \tilde{b}_1' \cdot d(Q, \mathcal{F}_{m+1}'\langle W-W'\rangle)^{-\tilde{b}_2'} \quad ; (b_1', b_2') \in \mathbb{R}^+.$$

It is clear that the above inequalities combined with ($\dot{i}$) assures that Coker ($\hat{r}$) = Ker ($\hat{i}$) (resp. Coker ($\hat{\mathcal{L}}^*$) = Ker ($\hat{\mathcal{L}}^*$)) hold.

Thus the remaining problem is to show that $\hat{r}$ (resp. $\hat{r}'$) is surjective : Recall that $W^+(S_{m+1,\mu}^*)$ is defined to be

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the closed set : $W^{\dagger}(S^{*}_{m+1,\mu}) = \bigcup \bar{S}^{*} : S^{*} \succ S^{*}_{m+1,\mu}$ ~~and that~~ $W^{\dagger}(S^{*}_{m+1,\mu})$ is defined to be the union $W^{\dagger}(S^{*}_{m+1,\mu}) = \bigcup S^{*} : S^{*} \succ S^{*}_{m+1,\mu}$. For the closed set $W^{\dagger}(S^{*}_{m+1,\mu})$, we consider a series of sub- closed set $W^{\dagger}_{s}(S^{*}_{m+1,\mu})$ $(1 \leq s \leq m+1)$ as follows : $W^{\dagger}_{1}(S'^{*}_{m+1,\mu}) = W^{\dagger}(\bar{S}^{*}_{1}) : \ell(\bar{S}^{*}_{1}) \geqq 1$, $S_{1} < W^{\dagger}(S'_{m+1,\mu})$

$W_{m+1}(S'^{*}_{m+1,\mu}) = \bigcup \bar{S}^{*} ; \ell(S^{*}) \geqq s, S^{*}_{s} \subset V(S'^{*}_{m+1,\mu})$

$(1 \leq s \leq m)$. $W(S^{*}) = \bar{S}^{*}_{m+1,\mu}$ $(1 \leqq s \leqq m)$

$W(S^{*})$ holds. It is clear that $W(S^{*}_{m+1,\mu})$

$W(S^{*}_{m+1,\mu}) \supsetneqq \cdots \supsetneqq W(S^{*}_{m+1,\mu})$ holds. Note that, if S^{*} $(\subset V(S^{*}_{m+1,\mu}))$ is of length $m' \leqq m$, then

$d(Q'_{m}, V_{m+1}(S^{*\prime}_{m+1,\mu})) \sim d(Q'_{m}, \mathrm{Fron}(S^{*}_{m+1,\mu}))$ holds

for $Q_{m'} \in S^{*\prime}_{m'} \cap N_{n}(S^{*}_{m+1,\mu})$. For $S^{*} \succ S^{*}_{m+1,\mu}$, $\mathrm{Fron}_{m+1,\mu}(S^{*})$ is defined to be the union $\bigcup S^{*}$ ($S'^{*} < \mathrm{Fron}(S^{*})$, $S'^{*} \succ S^{*}_{m+1,\mu}$). Then $\mathrm{Fron}_{m+1,\mu}(S^{*}_{m+1,\mu})$ is closed in $N_{\delta}(S^{*}_{m+1,\mu})$. Note that, in $N_{\delta}(S^{*}_{m+1,\mu}, \mathrm{Fron}(S^{*}_{m+1,\mu}))$,

$N_{\delta}(S^{*}_{m+1,\mu}, \mathrm{Fron}(S^{*}_{m+1,\mu})) \cap V(S^{*}_{m+1,\mu}) \fallingdotseq N_{\delta}(S^{*}_{m+1,\mu}, \mathrm{Fron}(S^{*}_{m+1,\mu})) \cap V(S^{*}_{m+1,\mu})$ holds. Also note that, for a set $S^{*} \succ S^{*}_{m+1,\mu}$ and $N_{\delta}(S^{*}_{m+1,\mu}, \mathrm{Fron}(S^{*}_{m+1,\mu}))$

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$N_n(S^* , \mathrm{Fron}(S^*)) \cap N_S(S^*_{m+1}, \mathrm{Fron}(S^*_{m+1,\mu}))$

$N_S(S^*_{m+1} , \mathrm{Fron}(S^*_{m+1,\mu}))$. Combining the above remark with (1,) , we obtain the following relation.

$$\{ N_n(S^*_{m+1,\mu}) \cap \{ \bigcup N_n(S^*_m, \mathrm{Fron}(S^*_m)) \} \}$$
$$\sim \{ N_S(V(S^*_{m+1,\mu}) , \overline{S}^*_{m+1,\mu}) \cap N_S(S^*_{m+1,\mu}) \} .$$

In the above the left and the right sides stand for all the open sets f in and in respectivwly. Choose an element $\sum_\mu \varphi_\mu \in \sum_\mu \Omega(\mathcal{S}^m < S_1, \dots, S_t, S'_{n+1,\mu} >, \mathfrak{F}_m)$ (resp. $\sum_\mu \varphi_\mu \in \sum_\mu \Omega(\mathcal{S}^m S_{m+1,\mu} >, \mathfrak{F}_m \nu)$) and take a representative $\sum_\mu \varphi_{n,\mu} \in \sum_\mu \Omega(N_n < \mathcal{S}^m < S_1 , \dots S_t, S_{m+1,\mu}> , \mathfrak{F}'_m)$ or $\sum \varphi_\mu$ (resp. $\sum_\mu \varphi_\mu$) for a suitable n . Take a couple (a) of positive numbers suitably. Then we can assume that the c^∞- function $\chi_a(V(S_{n+1,\mu}), \overline{S_{n+1,\mu}})$ ((c.f) definition) has the following properties.

(2.1.12) If $\chi_a = \chi_a(V(S_{m+1,\mu}), \overline{S}_{n+1,\mu}) \neq 0$ at a point Q in $S_{n+1,\mu} \vee V$, then the point Q is contained in N_n(S': Fron (S')) ($S' \in \mathcal{S}^2 < S_{m+1,\mu}' >$).

Thus any form φ_μ , defined in Ω(N_n($\mathcal{S}^m$ < $S_1, \ldots, S_t$, $S_{m+1,\mu}$))) (resp. Ω (N ($\mathcal{S}^m$(S_{m+1}), $\mathfrak{T}_m$(W- W')) can be extendable to a form φ_μ (resp. φ_μ) in N_n($S_1, \ldots, S_t$, $S_{m+1,\mu}$) - $S_{m+1,\mu}$ (resp. N ($S_{m+1,\mu}$) - $S_{m+1,\mu}$) by

$$\varphi_\mu = \mathcal{X}_a \cdot \varphi_\mu \quad (\text{resp. } \varphi_\mu = \mathcal{X}_a \varphi_\mu).$$

It is clear that this form φ (resp. φ) has an asymptotic behavior w.r.t. $\bar{S}_{m+1,\mu}'$ (in N($S_1 \ldots S_t$ $S_{m+1,\mu}$) - $\bar{S}_{m+1,\mu}$) (resp. N_n($S_{m+1,\mu}'$) - $\bar{S}_{m+1,\mu}'$) and so w.r.t. V_{n+1}'.

This form $\varphi = \varphi$ (resp. $\varphi = \varphi$) is zero outside of $\bigcup_S N_S$ (S) (resp.) and the relation

$$\varphi = \varphi \quad (\text{resp. } \varphi = \varphi) \quad \text{holds in } N_n(\mathcal{S}^m <$$

$S_1, \ldots, S_t, S_{m+1,\mu}$>) (resp. N_n($\mathcal{S}_m$< $S_{m+1,\mu}$>)) for a sufficiently large n . In this sense the form $\Sigma_\mu \varphi$ (resp. $\Sigma_\mu \varphi$) represents the limit $\Sigma \varphi$ (resp. Σ) . Next take a C^∞- function $\mathcal{X}_{a'} = \mathcal{L}_{a'}$($S_{m+1,\mu}'$. Fron ($S_{m+1,\mu}'$)) with a suitable couple (a') . Define C^∞- forms

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$\tilde{\varphi}$ (resp. $\tilde{\varphi}'$) (i = 1,2) by

$$\tilde{\varphi} = \chi_{a'} \cdot \varphi , \quad \tilde{\varphi}' = (1- \chi_{a'}) \cdot \varphi_{n}$$

Note that we can assume , by a suitable choice of (a') $\tilde{\varphi}$ ($\tilde{\varphi}'$) is extendable to $N_n(S_1, \dots, S_t) - \overline{S_{m+1}}$ (resp. $U - \overline{S_{m+1}}$) by defining $\tilde{\varphi}$ ($'$) to be zero outside of $N_n(S_{m+1,\mu})$. On the otherhand, φ can be extended to $N_{n'}(S_1, \dots, S_t, S_{m+1,\mu})$ ($N_{n'}(S_{m+1,\mu})$) by defining $\hat{\varphi}'$ to be zero on $S_{m+1,\mu}$. It is obvious that the following relation

$$\varphi + \tilde{\varphi} = \varphi \quad (\text{resp. } \tilde{\varphi}' + \hat{\varphi}' = \varphi')$$

holds in $N_{n''}(S_1, \dots S_t, S_{m+1}) - S_{m+1,\mu}$ (resp. in $N_{n''}(S_{m+1}) - S_{m+1}$) for a suitebly large n'' .

Recall that forms φ , φ are defined in $N_{n'}(S_1, \dots, S_t, S_{m+1,\mu})$ (and in $N_{n'}(S_{m+1,\mu})$) respective;y.

On the othernand, resticted forms φ , φ, defined in $N_{n''}(S_1, \dots, S_t, S'_{m+1,\mu}) - \overline{S_{m+1,\mu}}$ (resp. $\overline{U - S_{m+1,\mu}}$), to open sets $N_{h''}(\mathcal{S}^{m}(S_1, \dots, S_t)_{l=2,3})$ and $N_{n'}(\mathcal{S}^{m} \langle w - w' \rangle)$

respectively, we regard forms φ, φ elements in $\Omega(N_{n'}(\mathcal{S}^m\langle S_1,\ldots,S_t\rangle))$ and $\Omega(N_{n'}\langle\mathcal{S}^m(W - W')\rangle)$.

Define forms φ, φ, defined in $\Omega(\langle S_1,\ldots,S_t\rangle)$ (resp. in $N\,\mathcal{S}\langle V - V'\rangle$) by

$$\varphi_{n'} = \Sigma'_\mu \varphi_{\mu}, \quad (\varphi_{n''} = \Sigma_\mu \varphi_\mu)$$

it is clear that $\varphi_{n''} = \varphi_{n''}$ and that $\varphi_{n''} = \varphi_{n''}$ hold in $N_{n''}(S_1,\ldots S_t, S_{m+1,\mu}) - S_{m+1,\mu}$. Thus, for $(\varphi_{n'',1}, \Sigma_\mu \varphi_{\mu,n''}) \in \Omega(N_n(\mathcal{S}^m(S_1,\ldots,S_t)) \oplus \Sigma_\mu \Omega(N_n(S_1,\ldots,S_t,S_{m+1,\mu}))$ ($(\varphi_{n''}, \Sigma_\mu \varphi_{\mu}) \in \Omega(N_n(\mathcal{S}^m(W - W')) \oplus \Sigma_\mu \Omega(N_n(S_{m+1,\mu}))$), we obtain the equalities

$$(2.1.13) \qquad r_n(\varphi_{n''}, \Sigma_\mu \varphi_\mu) = \Sigma_\mu \varphi_{\mu,n''},$$

$$r(\varphi_{n''}, \varphi_{\mu,n''}) = \Sigma\, \varphi_{\mu,n'}.$$

Quantitative properties of $(\varphi, \Sigma_\mu \varphi_\mu)$ (resp. $(\varphi_{n''}, \Sigma_\mu \varphi_{\mu,n})$) are examined in the following way; Recall that forms φ_n have asyptotic behaviors w. r. t. Fron ( ) . Then

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from the facts (i) φ_μ = 0 (resp. φ'_μ = 0), outside of a neighbourhood N_δ($S_{m+1,\mu}$, Fron ($S_{m+1,\mu}$)) and (i i) φ_n and φ_n have asymptotic behaviour w.r.t. Fron ($S_{m+1,\mu}$) , it follows that φ and φ_μ have asymptotic behaviour w.r.t. Fron($S_{m+1,\mu}$)and (v - v) respectively. On the otherhand, because = (+) in each neighbourhood N (S ,.....,S , S), N ((v- v)) N (S) have asymptotic behavior w.r.t S in neighbourhood mentioned just above. Thus, in $\bigcup_\mu N_{n''}$ (S_1 ,...., S_t , $S_{m+1,\mu}$) ,$\bigcup_\mu N_n$($\mathcal{S}^m$ ($S_{m+1,\mu}$)) , $\varphi_{n''}$, $\varphi_{n''}$ have asymptotic behavior w.r.t. $\mathcal{F}_m$ $\mathcal{F}_m$(v-V) .

The surjectivity of the map r (resp.) follows from the above arguements immeadiately. q.e.

It is easy to deduce the assertion ( ) from (the exact sequences ') , ( ) .

WE leave details to the readers .

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n.2 From now on we take the field $\mathbb{K}$ to be the field of real numbers. By this restriction a quantiyative arguement (concerning a type of arguement related to extensions of a given map. (c.f. P.) becomes easy. Now we shall formulate our problem : Let U be a domain in R^N, and let V be a real analytic variety in U. (dim $V \leq N-1$). we also assume a regular series of stratification $\mathcal{R}$ and its nomalizing data $\mathcal{F}$ is attached to V. The series $\mathcal{R}$ is assumed to be reduced and is of type(I). our arguements in this section will be done for such a fixed pair $(\mathcal{R}, \mathcal{F})$. For the sake of simplicity, we assume that the domains U are of the form $U = \prod_{\nu=1}^{n} \square_\nu$ in $\mathbb{K}^N$. Recall that, for a pair $(\mathcal{R}, \mathcal{F})$, a series $\mathcal{R}'$ and a set of functions $\mathcal{F}'$ were fixed. we argue by fixing such data $(\mathcal{R}', \mathcal{F}')$ from now on. What we work with from now on is, howevwr, neither $\mathcal{R}$ nor $\mathcal{R}'$. We let $\mathcal{S}^{*\delta}$ be the set of all the

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connected components $S^{*}_{\lambda\mu}$ of $\mathcal{S}'^{j}$. Also we mean by $\mathcal{R}^{*}$ the series composed of sets $\{\mathcal{S}^{*j}\}_{j=1,\cdots,N}$. In this numero our concern is to investigate simple properties of such a series $\mathcal{R}^{*}$: It is clear that the following are valid :

$$(2.1.14)_1 \qquad \dim S^{*}_{\lambda_j,\mu_j} = \dim S^{*}_{\lambda_j}, \quad S'_{\lambda_j} \in \mathcal{S}', \quad S'_{\lambda_j} \supset S^{*},$$

$$(2.1.14)_2 \qquad U^{j} = \bigcup_{\lambda_j,\mu_j} S^{*}_{\lambda_j,\mu_j} \quad ; \quad S^{*}_{\lambda_j,\mu_j} \in \mathcal{S}^{*j}.$$

In the above (and in the sequel) the stratum S'_{λ_j} stands for the one which contains $S'^{*}_{\lambda_j,\mu_j}$. Several notations defined for $\mathcal{R}' = \{\mathcal{S}'^{j}\}_{j=1,\cdots,N}$ will be translated to the series $\mathcal{R}^{*} = \{\mathcal{S}^{*j}\}_{j=1,\cdots,N}$. An element $S'^{*}_{\lambda_j,\mu_j}$ is of first or of second type according to whether S'_{λ_j} is of first or of second type. Coordinates (x_{λ_j,μ_j}) and sets of functions $\{\mathfrak{f}(S^{*}_{\lambda_j,\mu_j})\}$ $h(S^{*j}_{\lambda_j,\mu_j})$, $g(\mathbb{V}^{+}(S^{*}_{\lambda_j,\mu_j}))$ are associated with $S^{*j}_{\lambda_j,\mu_j}$ by the following formula.

$$(x_{\lambda_j\mu_j}) = (x_{\lambda_j}), \qquad \{\mathfrak{f}(S^{*}_{\lambda_j\mu_j})\} = \{\mathfrak{f}(S'^{j}_{\lambda_j})\},$$

$$h(S^{*j}_{\lambda_j\mu_j}) = h(S'^{j}_{\lambda_j}), \qquad g(\mathbb{W}^{+}(S^{*}_{\lambda_j\mu_j})) = g(\mathbb{W}^{+}(S'^{j}_{\lambda_j})).$$

It is clear that, for the pair ($x_{\lambda'_\delta,\mu'_\delta}$), $\{f(S^{*\delta}_{\lambda_\delta,\mu_\delta})\}$ the condition (J) $\det\frac{\partial(f(S^{*\delta}_{\lambda_\delta\mu_\delta}))}{\partial(x_{\lambda_\delta\mu_\delta})}$ is true on $S^{*}_{\lambda_\delta,\mu_\delta}$.

In the sequel, when the necessity occurs to make clear the difference of the types of $S^{*\delta}_{\lambda_\delta\mu_\delta}$'s, we write an element $S^{*\delta}_{\lambda_\delta\mu_\delta}$ in $\mathcal{S}^{*\delta}$ as $S^{*\delta}_{\lambda'_\delta\mu'_\delta}$ or $S^{*\delta}_{\lambda''_\delta\mu''_\delta}$ according to whether $S^{*\delta}_{\lambda_\delta\mu_\delta}$ is of first or of secind type.

(A) We shall make several ovservations about the series $\mathcal{S}^*=\{\mathcal{S}^*_\delta\}_{\delta=1,\cdots,N}$: Let $S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ be an element in $\mathcal{S}^{*\delta-1}_{\lambda_\delta\mu_\delta}$.

Consider the inverse image $\pi^{-1}_{\delta-1,\delta}(S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}})$ of $S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ in U^δ.

Take an element $S^{*\delta}_{\lambda'_\delta\mu'_\delta}$ of first type so that $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda'_\delta,\mu'_\delta}) = S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ holds (Note that it may happen that there is no $S^{*\delta}_{\lambda'_\delta\mu'_\delta}$ of first type lying over $S^{\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$.

Take a point $P^{\delta-1}$ on $S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ and take a point $P^{*\delta}_{\lambda'_\delta\mu_\delta}{}'$ on $S^{*\delta}_{\lambda'_\delta,\mu'_\delta}$ so that $\pi_{\delta-1,\delta}(P^{*\delta}_{\lambda'_\delta\mu'_\delta}) = P^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ holds . We regard the coordinate $x_\delta(P^{*\delta}_{\lambda'_\delta\mu'_\delta})$ of $P^{*\delta}_{\lambda'_\delta\mu'_\delta}$ as real analytic function of $P^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$. Because of the condition (1.), $x_\delta(P^{*\delta}_{\lambda'_\delta\mu'_\delta}{}')$ is a (at least multivalued real analytic function on $S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$. Let $\{P^{*\delta}_{\lambda'_\delta\mu'_\delta,\sigma}\}_\sigma$

be all the points in $\bigcup_{\lambda'_j \mu'_j} S^{*}_{\lambda'_j \mu'_j}$ lying over $P^{*j-1}_{\lambda_{j-1},\mu_{j-1}}$.
Because of (1.), coordinates $x_j(P^{*j}_{\lambda'_j \mu'_j,\sigma}) \neq x_j(P^{*j}_{\lambda'_j \mu'_j,\sigma'})$ if $\sigma \neq \sigma'$. Order coordinates $x_j(P^{*j}_{\lambda'_j \mu'_j,\sigma})$'s as follows :

$$x_j(P^{*j}_{\lambda'_j \mu'_1}) < x(P^{*j}_{\lambda'_j \mu'_2}) < \cdots\cdots\cdots < x(P^{*j}_{\lambda'_j \mu'_\sigma}) < \cdots$$

Take a continious arc $\gamma^{*,j-1}$ in $S^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$ whose initial point is $P^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$. Denote by $P'^{*j-1}_{\lambda_{j-1},\mu_{j-1}}$ the end point of $\gamma^{*,j-1}$. Consider the analytic continuation of $x_j(P^{*j}_{\lambda'_j \mu'_\sigma})$ along $\gamma^{*,j-1}$. Note that the condition (1.) implies that $x_j(P'^{*j}_{\lambda'_j \mu'_j})$, the analytic continuation of $x_j(P^{*j}_{\lambda_j \mu_j})$ along $\gamma^{*,j-1}$, is a real valued. Also note that the condition (1.) shows that the order of $x_j(P^{*j}_{\lambda_j,\mu_j})$'s are unchanged by the analytic continuation along $\gamma^{*,j-1}$. Thus we know that coordinates $x_j(P^{*j}_{\lambda_j,\mu_j})$'s are single valued real analytic f fînctions on $S^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$.

(*) For arguements of similar types, see Margrange [ ].

Therefore we know that

($\mathcal{S}^{*j}_{1}$)$^{\sim}$ For each connected component $S^{*j-1}_{\lambda_{j-1},\mu_{j-1}}$

$\pi^{-1}_{j-1,j}$ ($S^{*j-1}_{\lambda_{j-1},\mu_{j-1}}$) is described by the following formula

($\mathcal{S}^{*j}$)$'^{\sim}$ $\pi_{\lambda_{j-1},\mu_{j-1}}$($S^{*j-1}_{\lambda_{j-1},\mu_{j-1}}$) = $S^{*j}_{\lambda_j',\mu_j'} \cup S^{*j}_{\lambda_j'',\mu_j''}$ s $\pi_{j-1,j}(S^{*j}_{\lambda_j',\mu_j'})$

$=\pi_{j-1,j}(S^{*j}_{\lambda_j'',\mu_j''})=S^{*j-1}_{\lambda_{j-1}',\mu_{j-1}'}$, where $S^{*j}_{\lambda_j',\mu_j'}$'s are real analytically biholo- -morphic onto $S^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$, while $S^{*j}_{\lambda_j'',\mu_j''}$'s are of second type .

It is clear that any elements $S^{*j}_{\lambda_j',\mu_j'}$ $S^{*j}_{\lambda_j'',\mu_j''}$ are connected components of $\pi^{-1}_{j-1,j}$($S^{*j-1}_{\lambda_{j-1},\mu_{j-1}}$) . On the otherhand, it is also clear that any connected component $S^{*j}_{\lambda_j',\mu_j'}$ $S^{*j}_{\lambda_j'',\mu_j''}$ of $S'^{j}_{\lambda_j}$ appears in the right side of ($\mathcal{S}^{*j}$)$'^{\sim}$.We give an order for elements $\{S^{*j}_{\lambda_j',\mu_j'}\}'_{s}$ of first type by

(2.1.15) $S^{*j}_{\lambda'_{j,1}\mu'_{j,1}} \prec S^{*j}_{\lambda'_{j,2},\mu'_{j,2}} \prec \cdots S^{*j}_{\lambda'_{j\sigma},\mu'_{j\sigma}} \prec \cdots$ if and only if , at one point $P^{*j-1}_{\lambda_{j-1}\mu_{j-1}} \in S^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$, coordinates x_j($P^{*j}_{\lambda_{j-1}\mu_{j-1}}$) ; $P^{j-1}_{\lambda_{j-1}\mu_{j-1}} \in S^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$ are ordered in the manner

$$x_j(\ P^{*j}_{\lambda_{j1}\mu_{j1}}) < \cdots < \cdots < x_j(\ P^{*j}_{\lambda'_{j\sigma},\mu'_{j\sigma}}) < \cdots$$

Clearly the above order is independent of a point $P^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$.

Fix a stratum $S^{*\delta-1}_{\lambda_{\delta-1},\mu_{\delta-1}}$, and consider the difference

$\pi^{*}_{\delta-1,\delta}(S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}) - \bigvee_{\lambda'_\delta\mu'_\delta} S^{*\delta}_{\lambda'_\delta\mu'_\delta}$, where $S^{*\delta}_{\lambda'_\delta\mu'_\delta}$ exhaust all the elements $S^{*\delta}_{\lambda'_\delta\mu'_\delta}$'s of first type/.

Define a connected manifold $S^{*\delta}_{\lambda''_\delta\mu''_{\delta,\sigma}}$ by

(2.1.16) $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda''_\delta\mu''_\delta}) = S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ and for a point $P^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}} \in S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$, the inverse $\pi^{-1}_{\delta-1,\delta}(S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}) \cap S^{*\delta}_{\lambda''_\delta\mu''_\delta}$ is given b

by

$\{ x_\delta : x_\delta(P^{*\delta}_{\lambda'_\delta\mu'_{\delta,\sigma}}) < x_\delta < x_\delta (P^{*\delta}_{\lambda'_\delta\mu'_{\delta,\sigma+1}})$

For t= 0 or t = t_0, the above inequality ( )

is replaced by one inequality in an obvious way.

Also if $\pi^{-1}_{\delta-1,\delta}(S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}) \cap V^{\delta} = \phi$ then $S^{*\delta}_{\lambda''_\delta\mu''_\delta}$ is defined to be the inverse image $\pi^{-1}_{\delta-1,\delta}(S^{*\delta-1}_{\lambda_{\delta-1},\mu_{\delta-1}})$. Now it is clear that $S^{*\delta}_{\lambda''_\delta\mu''_\delta}$'s exhaust all the connected c components of $\pi^{-1}_{\delta-1,\delta}(S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}) - \bigvee_{\lambda'_\delta,\mu'_\delta} S^{*\delta}_{\lambda'_\delta\mu'_\delta}$. An order : $\prec$ is given to the elements $\{S^{*\delta}_{\lambda''_\delta\mu''_\delta}\}$ (of second type) , so that $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda''_\delta\mu''_\delta}) = S^{*\delta-1}_{\lambda_{\delta-1}\mu_{\delta-1}}$ holds, by

(2.1.17) $S^{*\delta}_{\lambda''_{\delta,1}\mu''_{\delta,1}} \prec S^{*\delta}_{\lambda''_{\delta,2}\mu''_{\delta,2}}$ if and only if

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and $g(V^{+j}(S^{*j}_{\lambda_j\mu_j}))$, the comparison relations (2.1.1) are valid with these functions.

Propositions 2.1.~21 imply the following facts.

(2.1.18)$_1$ The collection $\mathcal{S}^{*j}$ is a pre-stratification of U^j.

(2.1.18)$_2$ Each closed set of U^j, which is expressed as a disjoint union of $S^{*j}_{\lambda_j\mu_j}$'s; $S^{*j}_{\lambda_j\mu_j}$ is in $\mathcal{S}^{*j}$, is (Q — D) — admissible data.

Given a series of directed set of C^∞ – thickings $\mathcal{T}(\mathcal{S}^{*j}) = \{\mathcal{T}^j_{\ell_j}(\mathcal{S}^{*j})\}_{\ell_j=1,2,\cdots}$ of $\mathcal{S}^{*j}$, the collection $\mathcal{T}(\mathcal{R}^*) = \{\mathcal{T}^j_{\ell_j}(\mathcal{S}^{*,j})\}_{\ell_j=1,2,\cdots,\, j=1,\cdots}$ will be called a directed set of C^∞-thickings of $\mathcal{R}^*$, if the following conditions are satisfied.

(2.1.19) If $\pi_{j-1,j}(S^{*j}_{\lambda_j\mu_j}) \neq S^{*j-1}_{\lambda_{j-1}\mu_{j-1}}$ holds, then $\pi_{j-1,j}(T^j_{\ell_j}(S^{*j}_{\lambda_j\mu_j})) = T^{j-1}_{\ell_j}(S^{*j-1}_{\lambda_{j-1},\mu_{j-1}})$

In the sequel of this section, when we speak of a directed set $\mathcal{T}(\mathcal{R}^*)$, we always assume that, for each directed set $\mathcal{T}^j(\mathcal{S}^*)$, the conditions (2.1.19) is valid.

In the aboves and in the sequel we denote C^{∞}-thickings used in this section by T(S)'s rather than by N(S B)'s. We formulate our problems in terms of C^{∞} thickings as follows

Proposition 2.1. 3. There exists a directed system $\mathcal{T}(\mathcal{R})$ in such a manner that the following conditions are satisfied.

(2.1.20)$_1$ There exists series of retract maps $\tau^{\delta}_{\lambda_{\delta_1}\cdots\mu_{\delta t}} : I \times T(S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}}, \ldots, S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}}) \longrightarrow T(S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}}, \ldots, S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}})_{\nabla}$ for any series $S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}} < \cdots < S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}}$ so that $\pi_{\delta-1,\delta} \circ \tau^{\delta}_{\lambda_{\delta_1}\cdots\mu_{\delta t}} = \tau^{\delta-1}_{\lambda_{(\delta-1)_1}\cdots\mu_{(\delta-1)_t}}$ hold, if $\pi_{\delta-1,\delta}(S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}}) = S^{\delta-1}_{\lambda_{(\delta-1)_1}\mu_{(\delta-1)_1}}, \ldots, \pi_{\delta-1,\delta}(S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}}) = S^{\delta-1}_{\lambda_{(\delta-1)_t}\mu_{(\delta-1)_t}}$ are true,

(2.1.20)$_{1,1}$ For each series $S^{\delta}_{\lambda_1\mu_1}, \ldots, S^{\delta}_{\lambda_t\mu_t}$ there exists a finite simple covering $\mathcal{U}^{\delta}_n(S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}} \cdots S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}})$ of $T^{\delta}_n(S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}}, \ldots, S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}})$ so that the following are valid.

(2.1.20)$_{1,2}$ $\mathcal{U}_n(S^{\delta}_{\lambda_{\delta_1}\mu_{\delta_1}} \cdots S^{\delta}_{\lambda_{\delta t}\mu_{\delta t}})$ has asymptotic behavior w.r.t. E

$(2.1.20)_{1.3}$ For each $\mathfrak{N}_n^{j}(S^{j}_{\lambda_1\mu_{0_1}},\dots,S^{j}_{\lambda_j\mu_j})$ there exist a partition of unity $\mathcal{U}_n^{j}(S^{j}_{\lambda_1\mu_1})$ subordinate to $\mathfrak{N}_n^{j}(S^{j}_{\lambda_1\mu_1}\dots S^{j}_{\lambda_{j_x}\mu_{j_x}})$ so that $\mathcal{U}_n^{j}(S^{j}_{\lambda_1\mu_1}\dots S^{j}_{\lambda_{j_x}\mu_{j_x}})$ has quantitative properties w.r.t. From ($S_{\lambda_{j_x}\mu_{j_x}}^{j}$) .

$(2.1.20)_{14}$ By a suitable choice of indices, the following relations are valid for each $S^{j}_{\lambda_1\mu_{j_1}} < \dots < S^{j}_{\lambda_{j_x}\mu_{j_x}}$

$(2.1.20)'_{1.4}$ For each $A^{j}_{\lambda_1\mu_{0_1}\dots\lambda_{j_x}\mu_{0_x}} \in \mathfrak{N}_n^{j}(S^{j}_{\lambda_1\mu_1}\dots S^{j}_{\lambda_j\mu_j})$, $\pi_{j,j}(A^{j}_{\lambda_1\mu_1\dots\mu_j})$ holds, and vice cersa.

$(2.1.20)''_{1.4}$ For each (n, n') and each series $S^{j}_{\lambda_1\mu_{0_1}} < \dots < S^{j}_{\lambda_{j_x}\mu_{0_x}}$ the set $\mathfrak{N}_n^{j}(S^{j}_{\lambda_1\mu_1}\dots S^{j}_{\lambda_j\mu_j})$ is the restriction of $\mathfrak{N}_{n'}^{j}(S^{j}_{\lambda_1\mu_1}\dots S^{j}_{\lambda_j\mu_j})$ to $\mathfrak{I}_n^{j}(S^{j}_{\lambda_j\mu_j})$

Now we summalize the above propositions in following

Lemma. 2.1. 1 . (C^{∞}- thickings and c^{∞} - forms with quantities) For a series $\mathcal{R}^*$ in proposit there exists a series of c^{∞} - thickings $\mathcal{T}(\mathcal{R}^*)$ so that the following condition is true.

$(2.1.21)$ For each j , and a pair $V_i^* \subsetneq V^*$ of closed sets (= disjoint union of strata in $\mathcal{S}^*_{\bar{\gamma}}$), the isomorphism

$$(2.1.21)' \quad H(V^*-V'^*:\mathbb{R}) \cong \frac{\mathfrak{F}_{asy}(V^*-V'^*:V'^*)}{d\Omega_{asy}(V^*-V'^*:V'^*)}$$

It is clear that the propositions 2.1.3 ~ 2.1. imply the above lemma 2.1.1 : From propositions 2.1.3, 2.1 we showed that $\mathcal{S}^*_{\bar{j}(\bar{j}=1,\cdots,N)}$ is (⇐ D) pre - stratified set . On the otherhand, propositions 2.1. show that conditions (2.1.2), (2.1.2) are satisfied by $\mathcal{T}(\mathbb{R}^*)$

Remark 1 . We note that the existence of a simple covering $\mathfrak{V}_n^{\bar{j}}(S^{*\bar{j}}_{\lambda\mu_{j_1}},\cdots,S^{*\bar{j}}_{\lambda\mu_{j_d}})$ of $\mathcal{T}_n(S^{*\bar{j}}_{\lambda\mu_{j_1}},\cdots,S^{*\bar{j}}_{\lambda\mu_{j_d}})$ contains more informations than required in lemma 3.1.1 . Actually It would be of interest to ask the existence of a finite simple covering $\mathfrak{U}(V^*-V'^*)$ of $V^*-V'^*$ with asymptotic behaviors w.r.t. V'^* . Actually the author's first intension was to show the existence of $\mathfrak{U}(V^*-V'^*)$ as above directly. when the author began the present

work in 1971 without enough knowledge of works of R.Thom, Lojasiewicz and H. Whitney.
It is found that to work with the notion of stratifications is quite basic.

Remark 2.. It would be of interest to persue our methods here from two reasos : (i) From a compytation of Betti groups of K - analytic varieties . (i i) From a point of view of a possible generalization of the notion of residue. (c.f. P.A. Griffith ( ) J.Leray ( ))

n. 3_1 Remaining parts of this section will be devoted to verifications of propositions and lemmas mentioned above. Our arguements will be done in the following divices.

(A) We show the implication

(B) We verify propositions 2. 1. 3 , 2. 1. 4 and 2. 1. 5 inductivaly on j. Arguement of this part will be divided into several steps. In each step, our arguement is of completely elementary nature. Arguements will be done by dividing the type of elements s^{*j} in $\mathcal{S}^{*j}$. Some cares should be taken further according to natures of elements s^{*j}s. We will argue ,in each step, the typical case and we indicate how to translate our arguement to other cases. Because the difference between the typical case and the other cases is quite small, there is no lack in our way of discussing problems.

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Because the steps in (II) involve considerable details we shall outline each step taken in the second step. In the first place (I), we settle a set of neighbourhood to each stratum $S^{*j}_{\lambda_j \mu_\gamma}$. This is done by dividing the strata into first and second type, and this part is of essential (but of quite simple and elementary) nature in the remaining parts of this section. Once we fix a set of neidhbourhoods to each stratum $S^{*j}_{\lambda_j \mu_\gamma}$, we investi-igate intersection relations of neighbourhoods for a series $S^{*j}_{\lambda_j \mu_{\gamma_1}} \sim S^{*j}_{\lambda_k \mu_{\gamma_2}}$. This part will be divided into three cases : (i) The case in which the series is composed of strata of first type only . (i i) The case where the series is composed of strata of second type only and finally (i i i) the mixed case where the series contains strata of both types If the intersections takes a simple form which is commodiate for our purpose, we say that the associated neighbourhoods are suitable. We ask quantitative conditions in order that the associated neighbourhoods are suitable. This will be done by dividing the cases into three cases (i), (ii), and (iii), and is also of quite elementary nature. In (III) the condition ( ) is quickly shown for the neighbourhoods which are suitable. Finally in (IV) we show the condition ( ). Our arguements in the last two problems will be done easily on basis of explict expressions of intersections of neighbourhoods.

Finally we note that, in the case of $\gamma = 1$ U^{γ} is a expressed as a disjoint finite union of points and one dimensional open segments. Therefore to verify all the desired conditions in propositions 1.5 is an obvious matter, and we do not enter into.

Remark . In discussing our problems in (I) ∼ (IV), we make the following remarks . These remarks are chiefly of notational nature . An element $S^{*i}_{\lambda,\mu} \in \mathcal{S}^{*i}$ will be called simply an element before we check conditions (2.1.1) ((2.1.1)′) After examining the conditions (2.1.1)((2.1.1)′) we call $S^{*i}_{\lambda\mu}$ stratum . (ii) To discuss the inductive steps , it is found to be commodiate to divide considerations into the horizontal and vertical directions . Especially neighbourhoods T′s of first type of first type will be designated by $T_{\bar{\sigma}^i,\bar{\delta}^i}(S^i_{\chi,\mu})$ rather than $T_{\sigma^i}(S^{*i}_{\chi,\mu})$. (iii) Thirdly we should make clear uses of distances whether to the analytical or the topological frontier . The above remarked points will be taken care in the arguements from now on .

n.32. From now on our task will be concentrated to do discussions mentioned in n° 3 .(I)$_0$ First we start with fixing neighbourhoods to each element $S^{\dot{\jmath}}_{\lambda_i,\mu_j}$. This will be done, as was mensioned previously, by dividing into two cases : (i) the element $S^{\dot{\jmath}}_{\lambda_i,\mu_j}$ is of first type, (i i,) The element $S^{\dot{\jmath}}_{\lambda''_j,\mu''_j}$ is of second type. The stratum of the first type will be written as $S^{\dot{\jmath}}_{\lambda'_j\mu'_j}$ while the stratum of second type will be written as $S^{\dot{\jmath}}_{\lambda''_j\mu''_j}$. The elements in $\mathcal{S}^{*\dot{\jmath}-1}$ will be written as $S^{\dot{\jmath}-1}_{\lambda_{j-1}\mu_{j-1}}$ .for a stratum $S^{j-1}_{\lambda_{j-1}\mu_{j-1}}$ we mean by $\mathcal{S}^{\dot{\jmath}}_1\{S^{\dot{\jmath}-1}_{\lambda_{j-1},\mu_{j-1}}\}$ the set of all the strata $S^{\dot{\jmath}}_{\lambda'_j\mu'_j}$ of first kind lying over $S^{j-1}_{\lambda_{j-1}\mu_{j-1}}$ while by $\mathcal{S}^{\dot{\jmath}}_2\{S^{\dot{\jmath}-1}_{\lambda_{j-1},\mu_{j-1}}\}$ we mean the set of all the elements of second type lting over $S^{j-1}_{\lambda_{j-1}\mu_{j-1}}$. Some notations introduced in n° will be used here without essesential changes : Neighbourhoods $N_\varepsilon($ $S^{*\dot{\jmath}}_{\lambda_i,\mu_j}$, Fron^{2n} $(\mathcal{S}^{\dot{\jmath}}_{\lambda_j})$ and $T_{\alpha,\sigma}($ $S^{*\dot{\jmath}}_{\lambda_j\mu_j}$, Fron^{2n} $(\mathcal{S}'^{\dot{\jmath}}_{\lambda_j})$) denote the unions $\bigcup_{P\in S^{*\dot{\jmath}}_{\lambda_j\mu_j}}$

$N_\delta($ P , $\mathrm{Fron}^{an}(S^{*i}_{\lambda_i}))$ and $\bigcup_{P\in S^{*j}_{\lambda_j\mu_j}} T_\sigma($ P, $\mathrm{Fron}^{an}(S^{*j}_{\lambda_j}))$

respectively. Note that , in view of the existence of the set of functions $\mathcal{F}^*$, we know the following eqiivalence relation

$$\{\mathcal{N}_\delta(S^{*i}_{\lambda_i\mu_i}\ \mathrm{Fron}(S^{*i}_{\lambda}))\}_\delta = \{T_\sigma^{i}(S^{*i}_{\lambda\mu_i}\ \mathrm{Fron}(S^{*i}_{\lambda}))\}_\delta$$

.

For a stratum of first type, $\bar{T}_{\bar\sigma}^{i}$-level set $\bar{T}_{\bar\sigma}^{i}(S^{*i}_{\lambda\mu_i} : \mathrm{Fron}(S^{*i}_{\lambda\mu}))$ was defined in the following way.

$$\bar{T}^{i}_{\bar\sigma}(P^{i}\ \mathrm{Fron}(S^{*i}_{\lambda,\mu})) = \{Q^{i} : x_\mu(Q^{i}) = x_\mu(P^{i})\ (\mu=1,\cdots,\mu_i),$$

$$\dim S^{*i}_{\lambda\mu},\ x_i(Q) = x_i(P^{i}),\ |x_{\mu+\tau}(Q^{i}) - x_{\mu+\tau}(P^{i})| \le \delta\, d(P, \mathrm{Fron}(S^{i}))\}$$

$(\tau=1,\cdots,i-1-\mu_i)$

Also, for $Q^{i}\in \hat{T}_{\hat\sigma}(P^{i}, \mathrm{Fron}^{an}(S^{i}_{\lambda}))$, $\hat{T}_{\hat\sigma^{i}}(Q^{i}, \mathrm{Fron}^{an}(S^{i}_{\lambda}))$

$$= \{R^{i} : x_\tau(Q^{i}) = x_\tau(R^{i})\ (\tau=1,\cdots,i-1),\ |x_i(R^{i}) - x_i(Q^{i})| < \hat\sigma^{i} d(Q^{i}, \mathrm{Fron}(S^{i}_{\lambda}))\}$$

On the otherhand $\bar{T}^{i}_{\bar\sigma^{i},\hat\sigma^{i}}(S^{i}_{\lambda\mu}, \mathrm{Fron}^{an}(S^{i}_{\lambda',\mu'}))$ is the union $\bigcup_{P^{i}} T^{i}_{\bar\sigma^{i}}(S^{i}_{\lambda,\mu}, \mathrm{Fron}^{an}(S^{i}_{\lambda'}))$ ($P\in S^{i}_{\lambda'}$),

$$T^{i}_{\bar\sigma^{i}\hat\sigma^{i}}(S^{i}_{\lambda',\mu'}, \mathrm{Fron}^{an}(S)) = \bigcup_{Q^{i}\in T^{i}_{\bar\sigma^{i}}} T^{i}_{\hat\sigma^{i}}(Q^{i}, \mathrm{Fron}(S^{i}_{\lambda'}))$$

It is easy to see that $\{T_{\sigma^{i}}(S^{i}_{\lambda\mu}), \mathrm{Fron}^{an}(S^{i}_{\lambda\mu})\}_{\sigma^{i}} \sim \{T_{\bar\sigma^{i}\hat\sigma^{i}}(S^{i}_{\lambda',\mu'})$, $\mathrm{Fron}^{an}(S^{i}_{\lambda'})\}_{\bar\sigma^{i}\hat\sigma^{i}}$. Moreover, for a series of

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couples (c^{i}) , $U^{i}_{\bar{\sigma},c^{i}}$(P^{i}) is defined by (1.-)
and $U^{i}_{\bar{\sigma},c^{i}}(S^{*i}_{\lambda\mu})$ is defined to be the union $\bigcup_{P\in S^{*i}_{\lambda\mu}} U_{\bar{\sigma},c^{i}}(P^{i})$ ($S^{*i}_{\lambda\mu}$). We shall make quite simple observations used here.

In $T^{i}_{\bar{\sigma},\hat{\sigma}^{i}}$ (or $\bar{T}^{i}_{\bar{\sigma}^{i}}$), the projection $\pi_{\lambda\mu}$: $T^{i}_{\bar{\sigma}\hat{\sigma}}$ (or $\bar{T}^{i}_{\sigma^{i}}$) onto $S^{*i}_{\lambda\mu}$ is defined by $\pi^{i}_{\lambda\mu}$(R^{i}) = $P^{i}_{\lambda\mu}$ where P^{i} is the uniquely determined point on $S^{*i}_{\lambda\mu}$ so that $T_{\bar{\sigma}^{i}\hat{\sigma}^{i}}$($P^{i}_{\lambda,\mu}$) $\ni$ R^{i} holds.

Let $S^{*i-1}_{\lambda_{i-1}\mu_{i-1}}$ be a stratum of $\mathcal{S}^{*i-1}$. Then , from (2.1.),) ( ) we know the existence of positive numbers ($a^{*i-1}_{\lambda_{i-1}\mu_{i-1},1}$ $a^{*i-1}_{\lambda_{i-1}\mu_{i-1},2}$) in such a way that the following statement is valid.

(2.1.22) Let $P^{*i-1}_{\lambda_{i-1}\mu_{i-1}}$ be a point in $S^{*i-1}_{\lambda_{i-1}\mu_{i-1}}$, and let $P^{*i}_{\lambda'_{1}\mu'_{1}} < \cdots < P^{*i}_{\lambda'_{t_0}\mu'_{t_0}}$ be points in $\bigcup S^{*i}_{\lambda'_{i_s}\mu'_{i_s}}$ ($S^{*i}_{\lambda'_{i_s}\mu'_{i_s}} \in \mathcal{S}$($S^{*i-1}_{\lambda_{i-1}\mu_{i-1}}$) lying over $P^{*i-1}_{\lambda_{i-1}\mu_{i-1}}$. Then the inequalities

(2.1.22)′ d ($P^{*i}_{\lambda'_{i_s}\mu'_{i_s}}$, $P^{*i}_{\lambda'_{i_t},\mu'_{i_t}}$) $\geq$ $a^{*i-1}_{\lambda_{i-1}\mu_{i-1}} \cdot d$ ($P^{*i-1}_{\lambda_{i-1},\mu_{i-1}}$, Fron($S^{*i-1}_{\lambda_{i-1}\mu_{i-1}}$))

holds for each pair ($P^{*i}_{\lambda'_{i_s}\mu'_{i_s}}$, $P^{*i}_{\lambda'_{i_t}\mu'_{i_t}}$).

From the above statement , we know that , for small couples ($\bar{\sigma}^{i}_{\lambda'_{i_s}\mu'_{i_s}}$), ($\hat{\sigma}^{i}_{\lambda'_{i_s}\mu'_{i_s}}$) ($s=1,\ldots,t_0$) the following disjoint condition is valid.

(2.1.22)' $T^{\delta}_{\sigma}(S^{*\delta}_{\lambda'_s\mu'_s}, \mathrm{Fron}^{2n}(S^{*\delta}_{\lambda'_s}))$ $\cap$ $T_{\sigma}(S^{*\delta}_{\lambda'_t\mu'_t}, \mathrm{Fron}^{2n}(S^{*\delta}_{\lambda'_t})) = \phi$, if $s \neq t$.

Next take strata $S^{*\delta-1}_{\lambda_2\mu_2} \succ S^{*\delta-1}_{\lambda_1\mu_1}$ of $\mathcal{S}^{*\delta-1}$. Take a sufficiently large n and consider a neighbourhood $T_n(S^{*\delta-1}_{\lambda_1,\mu_1})$ of $S^{*\delta-1}_{\lambda_1\mu_1}$. ~~Then the conditions (2.1.)$_{1,2}$ are valid in view of the induction s hypothesis for $\delta-1$.~~ ~~Assume that~~ $\pi_{\delta-1,\delta}$: $S^{*\delta}_{\lambda'_2\mu'_2} \to S^{*\delta-1}_{\lambda_2\mu_2}$ is (real analyticlly) biholomorphic .

Denote by $S^{*\delta}_{\lambda'_2,\mu_2}(\lambda'_1\mu'_1)$ the inverse image of $S^{*\delta-1}_{\lambda_2,\mu_2} \cap T^{\delta-1}_n(S^{*\delta-1}_{\lambda_1\mu_1})$ in $S^{*\delta}_{\lambda'_2\mu'_2}$. The map $\pi_{\delta-1,\delta}$, restricted to $S^{*\delta}_{\lambda'_2\mu'_2}(\lambda'_1\mu'_1)$, $\pi_{\delta-1,\delta} : S^{*\delta}_{\lambda'_2\mu'_2}(\lambda'_1\mu'_1) \to S^{*\delta-1}_{\lambda_2\mu_2} \cap T_n(S^{*\delta}_{\lambda_1\mu_1})$ is biholomorphic. Take a point $Q^{*\delta-1}_{\lambda_2\mu_2}$ in $T^{\delta-1}_n(S^{*\delta-1}_{\lambda_1\mu_1}) \cap S^{*\delta-1}_{\lambda_2\mu_2}$, and let $P^{*\delta-1}_{\lambda_1\mu_1}$ be the point on $S^{*\delta-1}_{\lambda_1\mu_1}$ so that $\pi^{\delta-1}_{\lambda_1\mu_1}(Q^{*\delta-1}_{\lambda_2\mu_2}) = P^{*\delta-1}_{\lambda_1\mu_1}$ holds. Join these two points $Q^{*\delta-1}_{\lambda_2\mu_2}$ and $P^{*\delta-1}_{\lambda_1\mu_1}$ by an arc $\gamma^{\delta-1}_{1,2}$ in

$T^{\delta-1}_{n}(S^{*\delta-1}_{\lambda_1\mu_1}) \cap T^{\delta-1}_{\sigma^{\delta-1}_{\lambda_2\mu_2}}(P^{*\delta-1}_{\lambda\mu_1}$, Fron $(S^{*\delta-1}_{\lambda\mu_1}))$. The point $Q^{*\delta}_{\lambda_2\mu_2}$ stands for the point on $S^{*\delta}_{\lambda_2\mu_2}$ so that $\pi_{\delta-1,\delta}(Q^{*\delta}_{\lambda_2\mu_2}) = Q^{*\delta-1}_{\lambda_2\mu_2}$ is valid. The arc $\gamma^{*\delta-1}_{12} = \gamma^{\delta-1}_{1,2} - \{P^{*\delta-1}_{\lambda\mu_1}\}$ is lifted to the arc $\tilde{\gamma}^{*\delta}_{12}$ and the arc $\tilde{\gamma}^{*\delta}_{12}$ has an uniquely determined point $P^{*\delta}_{\lambda_1\mu_1}$ as its boundary $(\neq Q^{*\delta}_{\lambda_1\mu_1})$. It is clear that $P^{*\delta}_{\lambda_1'\mu_1'}$ is in V^{δ} and $\pi_{\delta-1,\delta}(P^{*\delta}_{\lambda_1'\mu_1'}) = P^{*\delta-1}_{\lambda_1\mu_1}$ holds. Let $S^{*\delta}_{\lambda_1\mu_1}$ be the uniquely determined element of $\mathcal{S}^{*\delta}$ of first type which contains the point $P^{*\delta}_{\lambda_1\mu_1}$. In a little while our arguement will be done for a fixed point $Q^{*\delta-1}_{\lambda_1\mu_1}$ and for an arc $\gamma^{*\delta-1}_{1,2}$. We shall use the following terminology for the relation between $S^{*\delta}_{\lambda_2'\mu_2'}$ of first type and $S^{*\delta}_{\lambda_1'\mu_1'}$ of first type.

(2. 1 23) The element $S^{*\delta}_{\lambda_2'\mu_2'}$ converges to $S^{*\delta}_{\lambda_1\mu_1}$ along $\gamma^{*\delta-1}_{1,2}$ $(\ni Q^{*\delta-1}_{\lambda_1\mu_1})$.

We shall simply express the above fact by the following manner,

$$(2.1.23)' \qquad S^{*\delta}_{\lambda_2\mu_2} \xrightarrow[\gamma^{*\delta-1}]{} S^{*\delta}_{\lambda_1\mu_1}$$

We shall continue simple observations : It is clear that the following inclusion relation is valid in view of (1.).

$$(2.1.24) \qquad S^{*\delta}_{\lambda_2'\mu_2'}(\lambda_1',\mu_1') \subset \bigcup T^{\delta}_{\lambda_1'\mu_1'}(S^{*\delta}_{\chi_1\mu_1'}, \text{Fron}^{2n}(S^{*\delta}_{\chi_1}))$$

, where $S^{*\delta}_{\lambda_1'\mu_1'}$ exhaust all the elements in $\{\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda_1\mu_1})\}'$.

We remark that the righthand of (2.1.24) is disjoint union . Because $S^{*\delta}_{\lambda_2'\mu_2'}(\lambda_1'\mu_1')$ is connected we know that the set $S^{*\delta}_{\lambda_2'\mu_2'}(\lambda_1'\mu_1')$ is contained in a neighbourhood $T_{\lambda_\mu\lambda_\mu}(S^{*\delta}_{\chi_1\mu_1'}, \text{Fron}^{2n}(S^{*\delta}_{\lambda_1'}))$ for an uniquely determined element $S^{*\delta}_{\lambda_1'\mu_1'}$. It is clear that , for such an uniquely determined element $S^{*\delta}_{\chi_1\mu_1'}$, the converging relation (2.1.23) is valid for any point $Q^{*\delta-1}_{\lambda_2\mu_2} \in T_n(S^{*\delta-1}_{\lambda_1\mu_1})$ and for an arc $\gamma^{*\delta-1}_{12}$ joining $Q^{*\delta}_{\lambda_2\mu_2}$ and $P^{*\delta}_{\lambda_2\mu_2} = \pi^{\delta-1}_{\lambda_1\mu_2}(Q^{*\delta}_{\lambda_2\mu_2})$. The preparations done hitherto will be used in the sequel .

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(I)$_1$ After the above preparations , we shall fix neighbourhoods of certain types for $S^{*\delta}_{\lambda',\mu'}$ of first type. These neighbourhoods will be written as $T^{\delta}_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$, and we consider $T_{n,\sigma^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ parametrized by $(n, \hat{\sigma}^{\delta}_{\lambda',\mu'})$. In the course of our arguements to fix $T^{\delta}_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$, the condition (2.1.) will be shown. In this part (I)$_1$, we assume that n is chosen large enough so that the inclusion relation

$$\pi_{\delta-1,\delta}(T_{\sigma^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}\ \mathrm{Fron}^{2n}(S^{*\delta}_{\lambda',\mu'})) \supset T_n(S^{*\delta-1}_{\lambda',\mu'})$$

holds, where $S^{*\delta-1}_{\lambda,\mu} = \pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'})$.

For such an integer n and for a series of couples $(\hat{\sigma}^{\delta}_{\lambda',\mu'})$, define a neighbourhood $T_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ by

$$(2.1.95)\quad T_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}) = \bigcup_{Q^{*\delta}_{\lambda',\mu'}} T_{\sigma^{\delta}_{\lambda',\mu'}}(Q^{*\delta}_{\lambda',\mu'}, \mathrm{Fron}^{2n}(S^{*\delta}_{\lambda}));$$

$$Q^{*\delta}_{\lambda',\mu'} \in \pi^{-1}_{\delta-1,\delta}(T_n(S^{*\delta}_{\lambda',\mu'})) \cap T_{\sigma^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}, \mathrm{Fron}^{2n}(S^{*\delta}_{\lambda})) ;$$

From now on , we always assume the following condition for neighbourhoods $T_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$.

$(2.1.25)_1$ For each point $Q^{*i}_{\lambda',\mu'}$ appearing in $(2.1.25)$ the inclusion relation : $U^{\delta}_{c^{i\delta}_{\lambda',\mu'}}(Q^{*i}_{\lambda',\mu'}) \subset T_{\tilde{\sigma}_{\lambda',\mu'}}(Q^{*i}_{\lambda',\mu'}, \mathrm{Fron}^{2n}(S^{*i}_{\lambda',\mu'}))$

$(2.1.25)_2$ $(\sigma^{\prime i}_{\lambda',\mu'}) < (\sigma^{\delta,e}_{\lambda',\mu'})$

In a little while we consider neighbourhoods $T^{\delta}_{n,\hat{\sigma}^{i}_{\lambda',\mu'}}(S^{i}_{\lambda',\mu'}, \mathrm{Fron}^{2n}(S^{i}_{\lambda',\mu'}))$ parametrized by a pair $(n, \sigma^{i}_{\lambda',\mu'})$. From the induction hypothesis we have the following relation

$$\left\{T_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}}(S^{*i}_{\lambda',\mu'}\ \mathrm{Fron}^{2n}(S^{i}_{\lambda'})\right\}_{n,\hat{\sigma}^{\delta}_{\lambda',\mu'}} \sim \left\{\mathcal{N}_{\delta_{\lambda',\mu'}}(S^{*i}_{\lambda',\mu'}, \mathrm{Fron}^{2n}(S^{i\delta}_{\lambda'})\right\}$$

Thus, to show $(2.1.)$, it is enough to show the following

$$(2.1)' \quad T_{n,\hat{\sigma}^{i}_{\lambda',\mu'_1}}(S^{*i}_{\lambda',\mu'_1}, \mathrm{Fron}^{2n}(S^{\prime i}_{\lambda'_1})) \cap S^{*\delta}_{\lambda'_2\mu'_2} = \phi$$

, if $\bar{S}^{*i}_{\lambda'_2\mu'_2} \cap S^{*i}_{\lambda',\mu'_1} = \phi$, for a suitable pair $(n, \hat{\sigma}^{i\delta}_{\lambda',\mu'_1})$.

$(\mathrm{I})_{1.1}$ We show the conditions $(2.1.)$ (or $(2.1.)'$).

First consider the case where the both elements $S^{*\delta}_{\lambda',\mu'_1}$ and $S^{*i}_{\lambda'_2\mu'_2}$ are of first type so that

(i) $S^{*i}_{\lambda_2\mu_2} \neq S^{*i}_{\lambda_1\mu_1}$ and (ii) $\bar{S}^{*i}_{\lambda_1\mu_1} \cap S^{*\delta}_{\lambda_2\mu_2}$ hold .

Denote by $S^{*\delta-1}_{\lambda\mu_\lambda}$ projections $\pi_{\delta,\delta}(S^{*\delta}_{\lambda',\mu'_\lambda})$.

Divide cases into the following three cases.

(i)$_1$ $\overline{S}^{*\delta-1}_{\lambda_2'\mu_2'} \cap S^{*\delta-1}_{\lambda_1'\mu_1'}=\phi$ (ii) $\overline{S}^{*\delta-1}_{\lambda_2'\mu_2'} \not\supset S^{*\delta-1}_{\lambda_1'\mu_1'}$ (iii) $S^{*\delta-1}_{\lambda_2'\mu_2'} \neq S^{*\delta-1}_{\lambda_1'\mu_1'}$ $\pi_{\sigma_{\lambda\delta}}(S^{*\delta}_{\lambda_2'\mu_2'}) \supset S^{*\delta-1}_{\lambda_1'\mu_1'}$. Consider the first case. The inequality (2.1.22)$'$ shows that $\overline{S}^{*\delta}_{\lambda'\mu'} \cap S^{*\delta}_{\lambda,\mu} = \phi$ holds. Also, for $T^{\delta}_{n,\sigma^{\delta}_{\lambda,\mu'}}(S^{*\delta}_{\lambda',\mu'})$, the inequality(2.1.22)$'$ shows that the condition (2.1.) is valid. Next consider the sexond case: From the condition $\overline{S}^{*\delta-1}_{\lambda_2'\mu_2'} \cap S^{*\delta-1}_{\lambda,\mu_1}= \phi$ and from the induction hypothesis, (2.1.) holds for $T^{\delta}_{n,\sigma^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$. Our consideration of the third case is divided into further two cases: (iii)$_1$ $\overline{S}^{*\delta}_{\lambda'\mu_2'} \cap S^{*\delta}_{\lambda_1'\mu_1'} \neq \phi$ and (iii)$_2$ $\overline{S}^{*\delta}_{\lambda_2'\mu_2'} \cap S^{*\delta}_{\lambda_1'\mu_1'} = \phi$. First consider the case (iii)$_2$. Then we know that (2.1.23), (2.1.24) and (2.1.22)$'$ imply the condition (2.1.) for $T^{\delta}_{n,\sigma^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$. On the otherhand, in the case of (iii)$_1$ (2.1.23)$_1$ shows that $\overline{S}^{*\delta}_{\lambda_2'\mu_2'} \supset S^{*\delta}_{\lambda_1'\mu_1'}$ holds. Also from (2.1.24) it follows easily that the condition (2.1.) and (2.1.) hold for $T^{\delta}_{n,\sigma^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ (I)$_{1,2}$ Let us consider the case where both elements $S^{*\delta}_{\lambda_1'\mu_1'}$ $S^{*\delta}_{\lambda_2'\mu_2'}$ in question are of first type. Then the above observation leads easily to (2.1.)

In this connection, we remark the following : The variety V^i is expressed as a disjoint union $V^i = \bigcup_{\lambda',\mu'} S^{*i}_{\lambda'\mu'}$, where $S^{*i}_{\lambda'\mu'}$ exhaust all the elements of first type. From $(2.1.)_{1,1}$ we know that the conditions (2.1.) are valid for elements $S^{*i}_{\lambda',\mu'}$ of first type, and so the conditions (2.1) are valid for $V^i = \bigcup_{\lambda',\mu'} S^{*i}_{\lambda'\mu'}$ with sets of functions $\mathfrak{F}'(S^{*i}_{\lambda'})$, $\mathfrak{G}(S^{*i}_{\lambda'})$ and $\mathfrak{h}(S^{*i}_{\lambda'})$.

$(I)_2$ Here in, $(I)_2$, we consider the condition (2.1.) for the case where one of elements $S^{*i}_{\lambda'_1\mu'_1}$ is of first type and the other element $S^{*i}_{\lambda''_2\mu''_2}$ is of second type. Remark that the relation $S^{*i}_{\lambda'\mu'_1} \not> S^{*i}_{\lambda'_2\mu_2'}$ does never happens. What we show is the conditions (2.1.) and (2.1.), and arguements will be divided into three cases.

(I)$_3$ Let $S^{*\delta}_{\lambda'',\mu''}$ be an element of second type. Our problem here is to associate neighbourhoods of certain types with $S^{*\delta}_{\lambda'',\mu''}$. By $S^{*\delta-1}_{\lambda,\mu}$ we mean the projection $\pi_{\delta-1,\delta}(S^{*\delta-1}_{\lambda'',\mu''})$ and by $S^{\pm*\delta}_{\lambda'',\mu''}$ we mean walls of $S^{*\delta}_{\lambda'',\mu''}$. Our arguements will be done for the case where the element $S^{*\delta}_{\lambda'',\mu''}$ has two walls. Other cases will be discussed by modifying arguements for $S^{*\delta}_{\lambda'',\mu''}$ with two walls easily. Before we fix neighbourhoods of $S^{\pm\delta}_{\lambda'',\mu''}$, we remark the following: For a point $P^{\pm*\delta}_{\lambda'',\mu''} \in S^{\pm*\delta}_{\lambda'',\mu''}$, the comparison relation $d(P^{\pm*\delta}_{\lambda'',\mu''}, \mathrm{Fron}^{an}(S^{\pm*\delta}_{\lambda',\mu'})) \sim d(P^{\pm*\delta}_{\lambda'',\mu''}, \mathrm{Fron}(S^{\pm*\delta}_{\lambda'',\mu''}))$ holds, where we let $S^{\pm*\delta}_{\lambda'}$ be strata of $\tilde{\mathcal{S}}^{*\delta}$ containing $S^{\pm*\delta}_{\lambda'}$ respectively. We work with the distance measured to $\mathrm{Fron}(S^{\pm*\delta}_{\lambda'',\mu''})$ rather than the distance to $\mathrm{Fron}^{an}(S^{*\delta}_{\lambda',\mu'})$. In arguements of this part in (I I)$_3$ we shall fix couples $(\vec{\sigma}^{\pm*\delta,0}_{\lambda'',\mu''})$, $(\hat{\sigma}^{\pm*\delta,0}_{\lambda',\mu})$ to elements $S^{*\delta}_{\lambda'',\mu}$ of first kind in such a manner that the conditions (1,) (2,1.25)$_1$ and (2.1,25)$_2$ are true. Note that

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$\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda'',\mu''}) = \pi_{\delta-1,\delta}(S^{\pm*\delta}_{\lambda',\mu'})$ holds. Take an integer n so large that $\pi_{\delta-1,\delta}(\mathcal{T}_{\sigma^{\pm\delta,0}_{\lambda',\mu'}}(S^{\pm*\delta}_{\lambda',\mu'}, \text{Fron}(S^{*\delta}_{\lambda',\mu'}))) \supset \mathcal{T}^{\delta-1}_{n}(S^{*\delta-1}_{\lambda,\mu})$ holds. By $\mathcal{T}^{\delta}_{n}(S^{\pm*\delta}_{\lambda',\mu'}, S^{\pm*\delta-1}_{\lambda,\mu})$ the inverse image of $\mathcal{T}^{\delta-1}_{n}(S^{*\delta-1}_{\lambda,\mu})$ by $\pi_{\delta-1,\delta}$ in $\mathcal{T}^{\delta}_{\sigma^{\pm\delta,0}_{\lambda',\mu'}}(S^{\pm*\delta}_{\lambda',\mu'}, \text{Fron}(S^{*\delta}_{\lambda',\mu'}))$ is meant. Because of the biholomorphic property of $\pi_{\delta-1,\delta}$ on $\mathcal{T}^{\delta}_{\sigma^{\prime\delta\pm}_{\lambda',\mu'}}(S^{*\pm\delta}_{\lambda',\mu'}, \text{Fron}(S^{\pm*\delta}_{\lambda',\mu'}))$ it follows that the map $\pi_{\delta-1,\delta} : \mathcal{T}^{\delta}_{n}(S^{\pm*\delta}_{\lambda',\mu'}, S^{*\delta-1}_{\lambda,\mu}) \to \mathcal{T}^{\delta-1}_{n}(S^{*\delta-1}_{\lambda,\mu})$ is biholomorphic (and surjective). For any point $Q^{*\delta-1}_{\lambda,\mu} \in \mathcal{T}^{\delta-1}_{n}(S^{*\delta-1}_{\lambda,\mu})$ we mean by $Q^{\pm*\delta}_{\lambda',\mu'}$ the (uniquely determined) points on $\mathcal{T}^{\pm\delta}_{n}(S^{\pm*\delta}_{\lambda',\mu'}, S^{*\delta-1}_{\lambda,\mu})$ so that $\pi_{\delta-1,\delta}(Q^{\pm*\delta}_{\lambda',\mu'}) = Q^{*\delta-1}_{\lambda,\mu}$ holds. Let $(c^{\prime\delta}_{\lambda',\mu'})$ be a couple of positive numbers so that the inclusion condition $(2.1.25)_2$ is true between $(c^{\pm*\delta'}_{\lambda',\mu'})$, $(\overline{\sigma}^{\pm\delta,0}_{\lambda'',\mu''})$ and $(\hat{\sigma}^{\pm\delta,0}_{\lambda',\mu'})$. Also we assume that $(c^{\delta'}_{\lambda',\mu'}) < (c^{\delta,0}_{\lambda',\mu'})$ is true. From now on we always assume that the above two conditions are valid with fixed $(c^{\pm\delta,0}_{\lambda',\mu'})$, $(\sigma^{\pm\delta,0}_{\lambda'',\mu''})$ and $(\sigma^{\prime\delta,0}_{\lambda',\mu'})$ throughout this section. For a point $Q^{*\delta-1}_{\lambda,\mu} \in \mathcal{T}^{\delta-1}_{n}(S^{*\delta-1}_{\lambda,\mu})$ define $\hat{\mathcal{T}}^{\delta}_{c^{\pm*\delta-1}_{\lambda',\mu'}}(Q^{*\delta-1}_{\lambda,\mu})$, which is an open segment, as follows

$$(2.1.26)' \quad \hat{T}^{\delta}_{o^{\pm *\delta-1}_{\lambda',\mu'}}(Q^{*\delta-1}_{\lambda,\mu}) = \{ Q'^{\delta}_{\lambda',\mu'} \in U^{\delta} : \pi_{\delta-1,\delta}(Q'^{\delta}_{\lambda',\mu'})$$

$$= Q^{\delta-1}_{\lambda,\mu}, \quad x_{n_\delta}(Q^{*\delta}_{\lambda',\mu'}) - c_{\lambda,\mu,1} \cdot d(Q^{*\delta}_{\lambda',\mu'}, \pi^{\delta'}_{\lambda',\mu'}(Q'^{\delta}_{\lambda',\mu'}))$$

$$< x_{\xi}(Q^{\pm *\delta}_{\lambda',\mu'}) \leq x(\hat{Q}^{\pm *\delta}_{\lambda',\mu'}) - c^{\xi}_{\lambda',\mu'} \circ d(Q^{\pm *\delta}_{\lambda',\mu'}, \pi_{\lambda',\mu'}(Q^{\pm *\delta}_{\lambda',\mu'}) \}$$

A neighbourhood $\hat{T}^{\delta}_{n,c^{\pm\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda'',\mu''})$ of $S^{*\delta}_{\lambda',\mu'}$ is defined by

$$(2.1.26) \quad T^{\delta}_{n,c^{\pm\delta}_{\lambda',\mu'}}(S^{*\delta'}_{\lambda',\mu'}) = \bigcup_{Q^{*\delta}_{\lambda',\mu'}} \hat{T}^{\delta}_{c^{\pm\delta}_{\lambda',\mu'}}(Q^{*\delta}_{\lambda,\mu}) : Q^{*\delta}_{\lambda,\mu} \in T^{\delta-1}(S^{*\delta-1}_{\lambda,\mu})$$

Neighbourhoods considered here are of the above types $\hat{T}^{\delta}_{n,c^{\pm\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$. In a little while we consider $T^{\delta}_{n,c^{\pm\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ as parametrized by pairs $(n, (c^{\pm *\delta}_{\lambda',\mu'}))$.

Examine simple properties of such neighbourhoods $T^{\delta}_{n,c^{\pm\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$.

First we show the following relation

$$(2.1.26)'' \quad \{ T_{n,c^{\pm\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}) \}_{n,c^{\pm\delta}_{\lambda',\mu'}} \sim \{ T_{\sigma\delta}(S^{*\delta}_{\lambda',\mu'}) \}_{\sigma\delta},$$

where $(n, c^{\pm\delta}_{\lambda',\mu'})$ exhaust all the pair so that $(2.1.25)_{1,2}$ are valid, while $(\sigma \; \delta')$'s are chosen arbitrarily.

The above equivalence relation is shown by dividing our arguements as follows : (i) Outside of $T^{\delta}_{\sigma^{\pm\delta}_{\lambda',\mu'},\sigma'^{\delta}_{\lambda',\mu'}}(S^{\pm *\delta}_{\lambda',\mu'}$, From $(S^{*\delta}_{\lambda',\mu'})$) and (ii) In $T^{\delta}_{\sigma^{\pm\delta}_{\lambda',\mu'},\sigma'^{\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}$, From $(S^{*\delta}_{\lambda',\mu'})$.

First consider the case (i). Let $Q^{*i}_{\lambda'',\mu''}$ be a point in $\hat{T}^{\delta}_{n,c^{*i}_{\lambda'',\mu''}}(S^{*i}_{\lambda'',\mu''}) - \bigcup T_{\sigma^{*i,0}_{\lambda',\mu'},\sigma^{*i,0}_{\lambda',\mu'}}(S^{*i}_{\lambda',\mu'}, \mathrm{Fron}(S^{*i}_{\lambda',\mu'}))$. Then, for any element $S^{*i}_{\lambda',\mu'}$ of first type satisfying the condition : $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'}) = \pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'})$, the relation

$$(2.1.27) \qquad Q^{*\delta}_{\lambda'',\mu''} \notin T_{\sigma^{*\delta,0}_{\lambda',\mu'},\sigma^{*\delta,0}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}, \mathrm{Fron}(S^{*\delta}_{\lambda',\mu'}))$$

holds in view of (2.1.22)'.

Combining the above relation with the induction hypothesis we know the following fact.

$$(2.1.27)' \qquad Q^{*\delta}_{\lambda',\mu'} \notin \bigcup_{\lambda',\mu'} T_{\sigma^{*\delta,0}_{\lambda',\mu'},\sigma^{*\delta,0}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}, \mathrm{Fron}(S^{*\delta}_{\lambda',\mu'})),$$

where $S^{*\delta}_{\lambda',\mu'}$'s exhaust all the elements in $\mathcal{S}^{*\delta}$ of first kind : $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'}) \subset \tilde{S}^{*\delta-1}_{\lambda}$.

Recall that the analytical frontier $\mathrm{Fron}^{an}(S^{*\delta}_{\lambda'',\mu''})$ is expressed as a disjoint union of elements in the following way.

$$(2.1.27)'' \quad \mathrm{Fron}^{an}(S^{*\delta}_{\lambda'',\mu''}) = \pi^{-1}_{\delta-1,\delta}(\mathrm{Fron}^{an}(\tilde{S}^{*\delta}_{\lambda})) \cup \{\bigcup_{\lambda',\mu'} S^{*\delta}_{\lambda',\mu'}\}.$$

where $S^{*\delta}_{\lambda',\mu'}$ are the elements of first type satisfying the condition

$$\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'}) \subset \hat{S}^{*\delta-1}_{\lambda}.$$

Because the frontier of $S^{*\delta}_{\lambda',\mu'}$ (both in topological and analytical senses) are contained in $\mathrm{Fron}^{2n}(S^{*\delta}_{\lambda})$ the relations (2.1.27), (2.1.27)$'$ lead to the following comparison relation

$$(2.1.27)'' \quad \underline{d(\ Q^{*\delta}_{\lambda'',\mu''},\ \mathrm{Fron}^{2n}(\ S^{*\delta}_{\lambda'',\mu''})\ ) \sim d(\ Q^{*\delta}_{\lambda',\mu'},}$$
$$\underline{\pi^{-1}_{\delta-1,\delta}\left(\mathrm{Fron}^{2n}(\ S^{*\delta-1}_{\lambda'}\ )\ \right)},$$

so far as $Q^{*\delta}_{\lambda'',\mu''}$'s are in $T^{\delta}_{n,c^{\pm}_{\lambda',\mu'}}(\ S^{*\delta}_{\lambda'',\mu''})$ but are not in $\bigcup_{\lambda',\mu'} T^{\delta}_{\sigma^{\delta}_{\lambda',\mu'},\sigma^{*\delta}_{\lambda,\mu}}(\ S^{*\delta}_{\lambda',\mu'}\ )$.

Combining the above relation (2.1.27)$''$ with the obvious relation $d(\ Q^{*\delta}_{\lambda',\mu'},\ \pi^{-1}_{\delta-1,\delta}(\ \mathrm{Fron}^{2n}(\ S^{*\delta-1}_{\lambda})\ ) = d(\ \pi_{\delta-1,\delta}(\ Q^{*\delta}_{\lambda',\mu'})\ ,\ \mathrm{Fron}^{2n}(\ S^{*\delta-1}_{\lambda})\ )$, we know easily the follo- -wing facts.

(2.1.28)$_1$ For an arbitrary given (n, ($c^{\pm*\delta}_{\lambda',\mu'}$)), choosing $\sigma^{\delta}_{\lambda',\mu'}$ small enough leads to the inclusion relation

$$T_{\sigma^{\delta}_{\lambda',\mu'}}(\ S^{\delta}_{\lambda',\mu'}\ ,\ \mathrm{Fron}^{2n}(\ S^{*\delta}_{\lambda',\mu'}\ )\ ) - T_{\sigma^{*\delta}_{\lambda,\mu},\sigma^{\delta}_{\lambda',\mu'}}(\ S^{\pm\delta}_{\lambda',\mu'},$$
$$\mathrm{Fron}\ (\ S^{\pm*\delta}_{\lambda',\mu'}\ )\ ) \subset T_{n,c^{\pm*\delta}_{\lambda',\mu'}}(\ S^{*\delta}_{\lambda',\mu'},\ \mathrm{Fro}^{2n}(S^{*\delta}_{\lambda',\mu'})$$
$$-\bigcup T\ (\ S^{\pm*\delta}_{\lambda',\mu'},\ \mathrm{Fron}\ (\ S^{\pm*\delta}_{\lambda',\mu'}\ )\ )\ ,$$

and conversely ,

(2.1.28)$_2$ for an arbitrary given ($\sigma^{\delta}_{\lambda'',\mu''}$), choosing n sufficiently large and choosing ($c^{\pm*\delta}_{\lambda',\mu'}$, so that (2.1) holds, lead to the inclusion relation

$$T^{\dot{\imath}}_{n, C^{\pm *\dot{\imath}}_{\lambda', \mu'}}(S^{*\dot{\imath}}_{\lambda'', \mu''} \text{ Fron}^{2n} (S^{*\dot{\imath}}_{\lambda, \mu''})) - \vee T^{\dot{\imath}}_{\hat{\sigma}^{\pm}_{\lambda', \mu'} \hat{\sigma}^{\pm}_{\lambda, \mu}}(S^{\pm *\dot{\imath}}_{\lambda, \mu}, \text{ Fron}(S^{\pm *\dot{\imath}}_{\lambda, \mu}))$$

$$\subset T^{\dot{\imath}}_{\sigma^{\dot{\imath}}_{\lambda, \mu'}}(S^{\dot{\imath}}_{\lambda'', \mu'}, \text{ Fron}^{2n} (S^{*\dot{\imath}}_{\lambda', \mu''})) - T^{\dot{\imath}}_{\hat{\sigma}_{\lambda, \mu} \hat{\sigma}^{\dot{\imath}}_{\lambda, \mu}}(S^{\pm *\dot{\imath}}_{\lambda, \mu}, \text{ Fron}(S^{\pm *\dot{\imath}}_{\lambda, \mu}))$$

Next we shall make our arguements in $T^{\dot{\imath}}_{\sigma^{\pm}_{\lambda, \mu}, \sigma^{\pm}_{\lambda, \mu}}(S^{\pm *\dot{\imath}}_{\lambda, \mu}$, Fron $S^{\pm *\dot{\imath}}_{\lambda, \mu}))$. Note that , in view of (2.1.27)", we can assume that the following relation is valid.

(2.1.28)$_{2,1}$ $d(Q^{*\dot{\imath}}_{\lambda', \mu'}, \text{ Fron}^{2n}(S^{\dot{\imath}}_{\lambda'', \mu''})) = d (Q^{\pm *\dot{\imath}}_{\lambda, \mu}, S^{\pm *\dot{\imath}}_{\lambda, \mu})$,

so far as $Q^{\pm\dot{\imath}}_{\lambda', \mu'}$ is in $T_{\sigma^{\pm\dot{\imath}}_{\lambda, \mu} \sigma^{\pm\dot{\imath}}_{\lambda', \mu'}}(S^{\pm *\dot{\imath}}_{\lambda, \mu}$ Fron ($S^{\pm\dot{\imath}}_{\lambda, \mu}$)).

First fix a pair (n, ($c^{\pm *\dot{\imath}}_{\lambda', \mu'}$)). By the inductive condition (2.1.), we can choose $\sigma^{\prime\dot{\imath}-1}_{\lambda, \mu}$(n) in such a way that the inclusion relation

$$T^{\dot{\imath}-1}_{n}(S^{\lambda\dot{\imath}-1}) \subset T^{\dot{\imath}-1}_{\sigma^{\prime\dot{\imath}}_{\lambda, \mu}(n)}(S^{*\dot{\imath}-1}_{\lambda, \mu}, \text{ Fron} (S^{*\dot{\imath}-1}_{\lambda, \mu}))$$

is valid.

After fixing ($\sigma^{\prime\dot{\imath}-1}_{\lambda, \mu}$) so that the above inclusion relation is valid, choose a series of couples ($\sigma^{*\dot{\imath}}_{\lambda'', \mu}$), associated with the element $S^{*\dot{\imath}}_{\lambda'', \mu''}$, small enough. More precisely, from (2.1.28)$_2$, we know the existence of maps $\alpha^{1}_{\lambda'', \mu''}$, $\alpha^{2}_{\lambda'', \mu''}$ and $\alpha^{3}_{\lambda'', \mu''}$ of (e-$\mathcal{L}$)

types depending on $S^{*\partial}_{\lambda'',\mu''}$ only in such a manner that the following statement is valid.

(2.1.29) Inequalities of the following forms

$$\max (\sigma^{*\partial}_{\lambda'',\mu''_1} , \sigma^{*\partial}_{\lambda'',\mu''_2}) < \min \mathcal{L}^1_{\lambda'',\mu''}((c^{\pm\partial}_{\lambda',\mu',1})^{-1}, (c^{\pm\partial}_{\lambda',\mu',2})^{-1})$$

$$\max \mathcal{L}^2_{\lambda'',\mu''} (\sigma^{*\partial}_{\lambda'',\mu''}, \lambda^{*\partial}_{\lambda'',\mu''}) < \min \mathcal{L}^3_{\lambda'',\mu''} (\sigma^{\prime\prime\partial -1}(n))$$

imply the inclusion relation

$$T^{\partial}_{\lambda'',\mu''}(S^{*\partial}_{\lambda'',\mu''}, \operatorname{Fron}^{\partial n}(S^{*\partial}_{\lambda'',\mu''})) \cap T^{\partial}_{\sigma^{\pm\partial}_{\lambda',\mu'}, \sigma^{*\partial}_{\lambda',\mu'}}(S^{\pm\partial}_{\lambda',\mu'})$$

$$\subset T^{\partial}_{n, c^{\pm\partial}_{\lambda',\mu'}}(S^{*\partial}_{\lambda'',\mu''}) \cap T^{\partial}_{\sigma^{\pm\partial\, 0}_{\lambda',\mu'}, \sigma^{*\partial 0}_{\lambda',\mu'}} .$$

(Remark) Maps $\mathcal{L}^i_{\lambda'',\mu''}$ (i= 1,2,3) are, for example, given explicitly as follows.

$$\mathcal{L}^1_{\lambda'',\mu''}(c_1^{-1}, c_2^{-1}) = ((c^{\pm}_{\lambda',\mu',1})^{-1/c_2}, (c^{\pm}_{\lambda',\mu',2})^{-1}), \quad \mathcal{L}^2_{\lambda'',\mu''}(\sigma^{*\partial}_{\lambda'',\mu''_1}, \sigma^{*\partial}_{\lambda'',\mu''_2}) =$$

$$(\sigma^{*\partial}_{\lambda'',\mu'',1} \cdot (\sigma^{\pm\partial}_{\lambda',\mu',1})^{\sigma^{*\partial}_{\lambda'',\mu'',2}}, \sigma^{*\partial}_{\lambda'',\mu'',2} \cdot \sigma^{\pm\partial}_{\lambda',\mu',2}) \quad \mathcal{L}^3_{\lambda'',\mu''}(\sigma_1^{\prime\prime\partial -1}(n), \sigma_2^{\prime\prime\partial -1}(n)) =$$

$$(\sigma_1^{\prime\prime\partial -1}(n) \cdot (b^{\pm\partial\partial}_{\lambda',\mu'})^{\sigma_2^{\prime\prime\partial -1}(n)}, b^{\pm\partial\partial}_{\lambda',\mu',2} \cdot \sigma_2^{\prime\prime\partial -1}(n)).$$

Here ($b^{\pm\partial\partial}_{\lambda',\mu',1}$, $b^{\pm\partial\partial}_{\lambda',\mu',2}$) are appearing from the following situation : For any point $P^{\pm\partial}_{\lambda',\mu'} \in S^{\pm*\partial}_{\lambda',\mu'}$, d ($P^{\pm*}_{\lambda',\mu'}$, $\pi_{\partial}^{-1}(\operatorname{Fron}^{\partial n}(S_\lambda))$) $> b^{\pm\partial\partial}_{\lambda',\mu',1} \cdot d$ ($P^{\pm\partial}_{\lambda',\mu'}$, Fron ($S^{\pm\partial}_{\lambda',\mu'}$)) .

Conversely we start with a given ($\sigma^{*\partial}_{\lambda',\mu'}$). Choose ($\sigma^{\prime\prime\partial -1}_{\lambda,\mu}$) and ($c^{\pm\partial}_{\lambda',\mu'}$) so that the inequalities

$$\min (\sigma^{*\partial}_{\lambda'',\mu''_1}, \sigma^{*\partial}_{\lambda'',\mu''_2}) > \max \mathcal{L}^1_{\lambda'',\mu''} ((c^{\pm\partial}_{\lambda',\mu',1})^{-1} , (c^{\pm\partial}_{\lambda',\mu',2})^{-1})$$

$$\min \mathcal{L}^2_{\lambda'',\mu''} (\sigma^{*\partial}_{\lambda''_1,\mu''_1}, \sigma^{*\partial}_{\lambda''_2,\mu''_2}) > \max \mathcal{L}^3_{\lambda'',\mu''} (\sigma^{\prime\prime\partial -1}_{\lambda,\mu}).$$

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and the inclusion relation

$$T_n^{j-1}(S^{*j-1}_{\lambda,\mu}) \subset T^{j-1}_{\sigma^{*j-1}_{\lambda,\mu}}(S^{*j-1}_{\lambda,\mu},\ \mathrm{Fron}^{*n}(S^{*j-1}_{\lambda,\mu}))$$

imply the following inclusion relation

$$T^{j}_{n,C^{*j}_{\lambda',\mu'}}(S^{*j}_{\lambda'',\mu''}) \subset T^{j}_{\sigma^{*j}_{\lambda'',\mu''}}(S^{*j}_{\lambda'',\mu''}) \ .$$

Thus the equivalence relation (2.1.26)'' is assured.

(II)$'_3$ Once the above equivalence relation (2.1.26)'' is shown, it is quite easy to verify desired properties of $T^{j}_{n,C^{*j}_{\lambda',\mu'}}(S^{*j}_{\lambda'',\mu''})$: Here, concerning relations between (n, $\{C^{*j}_{\lambda',\mu'}\}$), the inclusion relation (2.1) only is assumed. Let ($S^{*j}_{\lambda'',\mu''}$, $S^{*j}_{\lambda',\mu'}$) be a pair of elements $S^{*j}_{\lambda'',\mu''}$ of second tipe and $S^{*j}_{\lambda',\mu'}$ of first tipe. From the definition of $T^{j}_{n,C^{*j}_{\lambda',\mu'}}(S^{*j}_{\lambda'',\mu''})$ itself, the following relation

$$(2.1.30)_1 \qquad T^{j}_{n,C^{*j}_{\lambda',\mu'}}(S^{*j}_{\lambda'',\mu''}) \cap S^{*j}_{\lambda',\mu'} = \phi$$

follows immeadiately.

Next let $S^{*j}_{\lambda'',\mu''}$ ($\neq$) $S^{*j}_{\lambda''_1,\mu''_1}$ be two elements of second type. As in (II)$_2$, divide our considerations into three cases : (i) $\Pi_{j+1,j}(S^{*j}_{\lambda'',\mu''}) = \Pi_{j+1,j}(S^{*j}_{\lambda''_1,\mu''_1})$,

(ii) $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'}) \not\subset \pi_{\delta-1,\delta}(\bar{S}^{*\delta}_{\lambda'',\mu''})$ (iii) $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'}) \neq \pi_{\delta-1,\delta}(S^{*\delta}_{\lambda'',\mu''})$ $S^{*\delta}_{\lambda'',\mu''} \subset \overline{\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'})}$. In the cases of (i) and (ii), the definition of $\Psi^{\delta}_{n_{\delta}, \lambda',\mu'}(S^{*\delta}_{\lambda',\mu'})$ and the induction hypothesis imply the relation

$$(2.1.30)_2 \quad T^{\delta}_{n,c^{*\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}) \cap S^{*\delta}_{\lambda'_1,\mu'_1} = \phi \quad .$$

consider the third case : Strata $S^{*\delta-1}_{\lambda,\mu}$, $S^{*\delta-1}_{\lambda_1,\mu_1}$ denote the projections $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda'',\mu''})$ and $\pi_{\delta-1,\delta}(S^{*\delta}_{\lambda',\mu'})$ respectively .

From (2.1,22)$'$, $\bar{S}^{*\delta}_{\lambda''_1,\mu''_1} \cap S^{*\delta}_{\lambda'',\mu''} \neq \phi$ is valid if and only if

$$(2.1.30)'_3 \quad S^{*\delta}_{\lambda'_1\mu'_1} \geq S^{+*\delta}_{\lambda_1\mu_1} \succ S^{+*\delta}_{\lambda'\mu'} \geq S^{*\delta}_{\lambda'\mu'}$$

holds. Here $S^{\pm*\delta}_{\lambda''_1,\mu''_1}$'s are those elements lying over $S^{*\delta-1}_{\lambda,\mu}$ so that $S^{*\delta}_{\lambda'_1\mu'_1} \to S^{\pm*\delta}_{\lambda''\mu''}(S^{*\delta-1}_{\lambda_1,\mu_1} \to S^{*\delta-1}_{\lambda,\mu})$ holds

(in this place we discuss the case where both elements $S^{*\delta}_{\lambda',\mu'}$, $S^{*\delta}_{\lambda''_1,\mu''_1}$ have two walls. Other cases are discussed similarly and we shall omit tedious arguements.)

From the definition of $T_{n,c^{*\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ and (2.1.), we know the following.

$$(2.1.30)_{3,1} \quad \text{If} \quad S^{*\delta}_{\lambda'',\mu''} \cap \bar{S}^{*\delta}_{\lambda''_1,\mu''_1} = \phi \quad \text{then} \quad T^{\delta}_{n,c^{*\delta}_{\lambda'',\mu''}}(S^{*\delta}_{\lambda'',\mu''}) \cap S^{*\delta}_{\lambda''_1,\mu''_1} = \phi .$$

On the other hand, if $S^{*\delta}_{\lambda',\mu'} \cap \bar{S}^{*\delta}_{\lambda''_1\mu''_1}$ is non-empty, then from $(2.1.30)'_3$ we know that

$(2.1.30)_{3.2}$ $T^{\delta}_{n,C^{*\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'}) \cap S^{*\delta}_{\lambda''_1\mu''_1} = \bigcup_{Q^{*\delta-1}_{\lambda''_1\mu''_1}} \hat{T}_{C^{*\delta}_{\lambda'',\mu'}}(Q^{*\delta-1}_{\lambda'',\mu'})$ holds,

where points $Q^{*\delta}_{\lambda''_1\mu''_1}$'s are in $T^{\delta-1}_{n}(S^{*\delta-1}_{\lambda,\mu}) \cap \bar{S}^{*\delta-1}_{\lambda_1\mu_1}$.

From the above expression, each 'fiber' $T_{C^{*\delta}_{\lambda',\mu'}}(Q^{*\delta}_{\lambda'_1\mu'_1})$ is an open segment. Thus the validity of the conditions (2.1.) are obvious. Now we summarize the above arguements in $(I)_{1,2}$. Then we know easily that the proposition 2.1. is true. Also we know that the conditions ( ) are valid for neighbourhoods $T^{\delta}_{n,C^{*\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ and $T^{\delta}_{n,C^{*\delta}_{\lambda'',\mu'}}(S^{*\delta}_{\lambda'',\mu'})$

((I I)$_{1.2}$ Let $S^{*\delta}_{\lambda_1''\mu_1''} < S^{*\delta}_{\lambda_2''\mu_2''}$ be two strata in $\mathcal{S}^{*\delta}$ of second types. For neighbourhoods $T^{\delta}_{n, c^{*\delta}_{\lambda_i''\mu_i''}}(S^{*\delta}_{\lambda_i''\mu_i''}) (i= 1,2)$ we ask the intersection relation : $T^{\delta}_{n,c^{*\delta}_{\lambda_1''\mu_1''}}(S^{*\delta}_{\lambda_1''\mu_1''}) \cap T^{\delta}_{n,c^{*\delta}_{\lambda_2''\mu_2''}}(S^{*\delta}_{\lambda_2''\mu_2''})$.

Strata $S^{\pm*\delta}_{\lambda_i''\mu_i''}(i =1,2)$ mean walls of $S^{*\delta}_{\lambda_i''\mu_i''}$ (i= 1, 2)

Here we will be concerned with the case where $S^{*\delta}_{\lambda_i''\mu_i''} (i=1,2)$ have two walls . Other cases are dealt in similar ways, and will be touched in the last part of (I I)$_{1.2}$. Strata $S^{*\delta-1}_{\lambda_i,\mu_i}$ (i = 1,2) denote projections of $S^{*\delta}_{\lambda_i''\mu_i''}$(i= 1, 2) while $S^{\pm*\delta}_{\lambda_i'',\mu_i''}$ denotes the stratum lying over $S^{*\delta-1}_{\lambda_i\mu_i}$ so that $S^{\pm*\delta}_{\lambda_1''\mu_1''} \to S^{\pm*\delta}_{\lambda_2''\mu_2''}$ ($S^{*\delta-1}_{\lambda_2\mu_2} \to S^{*\delta-1}_{\lambda_1\mu_1}$) holds. If $T^{\delta}_{n,c^{\pm*\delta}_{\lambda\mu}}(S^{\pm\delta}_{\lambda''\mu''}) \cap T_{n,c^{*\delta}_{\lambda_2''\mu_2''}}(S^{*\delta}_{\lambda_2''\mu_2''}) \neq \phi$, then the relations $S^{*\delta}_{\lambda_1''\mu_1''} \to S^{*\delta}_{\lambda_2''\mu_2''} (S^{*\delta-1}_{\lambda_2\mu_2} \to S^{*\delta-1}_{\lambda_1\mu_1})$ holds. Because we can assume the relation: $d(Q^{*\delta-1}_{12}, P^{*\delta-1}_{\lambda_2\mu_2}) < d (Q^{*\delta-1}_{12}, P^{*\delta-1}_{\lambda_1\mu_1})$, we know from (2.1.) the existence of maps $\mathcal{M}^{\delta}_{\lambda_1''\mu_1''\lambda_2''\mu_2''}$. of (e-$\ell$) - type depending on $S^{*\delta}_{\lambda_1''\mu_1''}$, $S^{*\delta}_{\lambda_2''\mu_2''}$ only, with which the following are valid.

(2. 1 .33) If couples ($c^{\pm*\delta}_{\lambda_i''\mu_i''}$) (i = 1,2) satisfy the condition : ($c^{\pm*\delta}_{\lambda_2''\mu_2''}$) $< \mathcal{M}^{\delta}_{\lambda_1''\mu_1''\lambda_2''\mu_2''}$($c^{\pm*\delta}_{\lambda_1''\mu_1''}$), then the following intersection relation holds.

$$(2.1.33)_{11} \quad T^{\delta}_{n,c^{*\delta}_{\lambda_1''\mu_1''}}(S^{*\delta}_{\lambda_1''\mu_1''}) \cap T^{\delta}_{n,c^{*\delta}_{\lambda_2''\mu_2''}}(S^{*\delta}_{\lambda_2''\mu_2''}) = \bigcup_{Q^{*\delta-1}_{12}} \hat{T}_{n,c^{*\delta}_{\lambda_2''\mu_2''}}(Q^{*\delta-1}_{12}),$$

where points $Q^{\lambda i-1}_{1,2}$'s are in $T^{i-1}_{n}(S^{\lambda i-1}_{\lambda_1'\mu_1'}, S^{\lambda i-1}_{\lambda_1''\mu_1''})$.

We say , in a similar way to (2.1.31) , that $T_{n,S_{\lambda\mu}}(S^{\lambda i}_{\lambda'\mu'})(i=1,2)$ ($i = 1,2$) are suitable if the intersection of them is expressed in the form of (2.1.33). . The conditions (2.1.33) give a sufficient condition in order that $T_{n,S^{i}_{\lambda'\mu'}}$ are suitable.

(Remark) The above arguements were done for the case where both strata have two walls and walls of $S^{\lambda i}_{\lambda_2''\mu_2''}$ converge to walls of $S^{\lambda i}_{\lambda''\mu_1''}$. In other cases situations are as follows : (i) If no walls of $S^{\lambda i}_{\lambda_2\mu_2}$ converge to walls of $S^{\lambda i}_{\lambda''\mu_1''}$, we do not require other conditions than $(2.1.35)_{1,2}$. In this case the intersection of neighbourhoods of $S^{\lambda i}_{\lambda''\mu''}$ (i = 1, 2) are given by $(2.1.33)_{1,1}$

(i i) if one wall of $S^{\lambda i}_{\lambda_2'\mu_2'}$ only converges to a wall of $S^{\lambda i}_{\lambda_1'\mu_1''}$ (without loss of generality , we assume $S^{\lambda i}_{\lambda_2''\mu_2''} \to S^{\lambda i}_{\lambda''\mu''}$ ($S^{i-1}_{\lambda_2\mu_2} \to S^{i-1}_{\lambda_1\mu_1}$)) we require $(2.1.33)_1$ for $S^{i}_{\lambda''\mu_2''}$ besides (2.1.). Then the intersection of neighbourhoods are given by (2.1.33). In both cases of (i), (ii) if the intersection of $T_{n,S^{i}_{\lambda\mu}}(S^{\lambda i}_{\lambda''\mu_1''})$ is given by (2.1.) we say that such neighbourhoods are suitable.

$(\mathrm{III})_{1,3}$ Thirdly we consider the case in which one of the strata is of first type while the other is of second type : Take a series $S^{*i}_{\lambda_1'\mu_1'} \prec S^{*i}_{\lambda_2''\mu_2''}$. There are tw possibilities : $\pi_{i-1,i}(S^{*i}_{\lambda_2''\mu_2'}) \neq \pi_{i-1,i}(S^{*i}_{\lambda_1'\mu_1'})$, $\pi_{i-1,i}(S^{*i}_{\lambda_2''\mu_2''}) = \pi_{i-1,i}(S^{*i}_{\lambda_1'\mu_1'})$. Here we conside the first case only . The second case is discussed in an analogous way. In the first case, for $S^{*i-1}_{\lambda_1\mu_1} = \pi_{i-1,i}(S^{*i-1}_{\lambda_1\mu_1})$ $S^{*i-1}_{\lambda_2\mu_2} = \pi_{i-1,i}(S^{*i}_{\lambda_2''\mu_2''})$ and walls $S^{+*i}_{\lambda_2''\mu_2'}$, the following relation

$$(2.1.34) \quad \hat{S}^{+*i}_{\lambda_2'\mu_2'} \geq S^{*i}_{\lambda_1\mu_1} \geq \hat{S}^{+*i}_{\lambda_2'\mu_2'}$$

is valid, where $S^{+*i}_{\lambda_2\mu_2} \rightarrow \hat{S}^{+*i}_{\lambda_2'\mu_2'}$ ($S^{*i-1}_{\lambda_2\mu_2} \longrightarrow S^{*i-1}_{\lambda_1\mu_1}$) .

We assume that $S^{+*i}_{\lambda_2'\mu_2'} = S^{*i}_{\lambda_1\mu_1}$ holds . Other cases are mensioned after discussions of this case. As in $(\mathrm{III})_{1,2}$, take a point Q^{*i-1}_{12} in $T^{i-1}_{n}(S^{*i-1}_{\lambda_1\mu_1}, S^{*i-1}_{\lambda_2\mu_2})$. Then points $P^{*i-1}_{\lambda_i\mu_i}$ (i= 1,2) $P^{*i}_{\lambda_1'\mu_1'}$, $P^{*i}_{\lambda_2''\mu_2''}$ are defined in the same way as in $(\mathrm{III})_{1,2}$. Also $Q^{+*i}_{12}(\lambda_2'', \mu_2'')$ is defined as in $(\mathrm{III})_2$ while $Q^{*i}_{12}(\lambda_1''\mu_1'')$ is in $T_{\sigma^{0i}_{\lambda_0\mu_0}}(S^{*i}_{\lambda_1'\mu_1'})$ so that the relation $\pi_{i-1,i}(Q^{*i}_{12}(\lambda_1'',\mu_1'')) = Q^{*i-1}_{1,2}$ is valid. The inequality

$$(2.1.34)_1 \quad d(Q^{*i}_{12}(\lambda_1'',\mu_1''), Q^{+*i}_{\lambda,\mu_2'}) < c^{*i}_{\lambda\mu} d(Q^{*i-1}_{12}, P^{*i}_{\lambda_1\mu_1})$$

holds.

2 – C?

In a completely similar manner to arguements in $(1\ I)_{1,2}$, we know the following:

(2.1.35) Inequalities : $c^{\pm *}_{\lambda''_1\mu''_1,1} \cdot (\bar{\delta}_1)^{c^{\pm\lambda\delta}_{\lambda''\mu''_2}} < \delta^{*\delta}_{\lambda'\mu'_1}$, $c^{\pm *}_{\lambda''\mu''_2} \cdot \bar{\delta}_2 > \delta^{\lambda\delta}_{\lambda'\mu'_2}$.

and the inclusion relation : $T^{\delta-1}_n(S^{*\delta-1}_{\lambda_i\mu_i}) \subset \Pi_{H,\delta} T^{\delta}_{o^{*\delta}_{\lambda_i\mu_i}}(S^{*\delta}_{\lambda_i\mu_i}$ from $(S^{*\delta}_{\lambda\mu}))$

imply the following inclusion relation.

$$(2.1.35)' \quad T^{\delta}_{n, o^{*\delta}_{\lambda'\mu'_1}}(S^{*\delta-1}_{\lambda_i\mu_i}) \cap T^{\delta}_{H,\delta, c^{*\delta}_{\lambda_i\mu_i}}(S^{*\delta}_{\lambda''_2\mu''_2}) = \bigcup_{Q^{*\delta-1}_{12}} \hat{T}^{\delta}_{c^{\pm *\delta}_{\lambda_i\mu_i}}(Q^{*\delta-1}_{12}),$$

where $Q^{*\delta-1}_{12}$'s are in $T^{\delta-1}_n(S^{*\delta-1}_{\lambda'_1\mu'_1}, S^{\lambda\delta-1}_{\lambda_2\mu_2})$.

In an accordance with (2.1.), (2.1.) if the above inter-section relation (2.1) is true, then neighbourhoods $T_{n,o^{*\delta}_{\lambda'\mu'_1}}(S^{*\delta}_{\lambda'_1\mu'_1})$, $T_{n,c_{\lambda,\mu}}(S^{*\delta}_{\lambda''\mu''_2})$ are suitable.

(Remark) The above arguements are done for the case where two walls $S^{*\delta}_{\lambda''\mu_\lambda}$ converge to $S^{*\delta}_{\lambda'_1\mu'_1}$.
In other cases, situations are as flllows : If one wall, for example $S^{\pm\lambda\delta}_{\lambda''_2\mu''_2}$ of $S^{*\delta}_{\lambda''\mu'_2}$, converbes to $S^{*\delta}_{\lambda'_1\mu'_1}$ then we replace the inequalities in (2.1.35) by one inequality only. If no walls of $S^{*\delta}_{\lambda''_2\mu''_2}$ converge to $S^{*\delta}_{\lambda'_1\mu'_1}$ then the intersection relations $(2.1.25)_1$, $(2.1.25)_2$ are sufficient. In both cases the intersection relations are expressed in the following manner.

$(2.1.35)_1$ $T^{\gamma}_{n,\sigma^{*\delta}_{\lambda''\mu''}}(S^{*\delta}_{\lambda'\mu'}) \cap T^{\delta}_{n,C^{*\delta}_{\lambda'\mu'}}(S^{*i}_{\lambda_2'\mu_2'}) = \bigcup_{Q^{*i}_{12}} T_{\sigma_{\lambda''\mu''}, C^{*\delta}_{\lambda'\mu'}}(Q^{*\delta-1}_{12})$,

where Q^{*i-1}_{12}'s are in $T^{i-1}_{n}(S^{*i-1}_{\lambda,\mu_1}, S^{*i-1}_{\lambda_2\mu_2})$ while the

fiber $\hat{T}^{\delta}_{\sigma^{*\delta}_{\lambda''\mu'', C^{*}}}(Q^{*\delta-1}_{12})$ is expressed in the following way

$(2.1.35)'_1$ $\hat{T}^{\delta}_{\sigma^{*\delta}_{\lambda''\mu''}, C^{*\delta}_{\lambda'\mu'}}(Q^{*\delta-1}_{12}) = \{ x_\delta : x_\delta(Q^{*i}_{12}) - \sigma^{*\delta}_{\lambda'\mu_1} d(\hat{Q}^{*i}_{12}$

From $(S^{*i}_{\lambda_1\mu_1}))\leq$ $x_\delta \leq x_\delta(Q^{\delta}_{12}) - c^{*\delta}_{\lambda_2\mu_2'} \circ d(Q^{*\delta}_{1,2}, P^{*\delta}_{1,2})$

In the above $Q^{*\delta}_{1,2}$, $Q^{*\delta}_{P,2}$ are points on $\bar{T}_{\sigma^{*\delta}_{\lambda'\mu_1}}(S^{*\delta}_{\lambda'\mu'})$ or

$\bar{T}_{\sigma^{*\delta}_{\lambda_2\mu_2}}(S^{*\delta}_{\lambda_2'\mu_2'})$ so that $\Pi_{\delta-1,\delta}(Q^{*\delta}_{12,i}) = Q^{*\delta-1}_{12}$ $(i = 1.2)$ holds.

If no walls of $S^{*\delta}_{\lambda''\mu''_2}$ converges to $S^{*\delta}_{\lambda'\mu'_1}$, then the

intersection is of the following form is suitable.

$(2.1.35)_2$ $T^{\gamma}_{n,\sigma^{*\delta}_{\lambda\mu}}(S^{*\delta}_{\lambda''_1\mu''_1}) \cap T^{\delta}_{n,C^{*i}_{\lambda'\mu'}}(S^{*\delta}_{\lambda_2'\mu_2'}) = \bigcup_{Q^{*\delta-1}_{1,2}} \hat{T}^{\delta}_{n,\sigma^{*\delta}_{\lambda\mu}}(Q^{*i}_{\lambda'\mu'_2})$

If $\Pi_{\delta-1,\delta}(S^{*\delta}_{\lambda'\mu'}) = \Pi_{\delta-1,\delta}(S^{*\delta}_{\lambda''})$ is true, then the

following

$(2.1.35)''$ $C^{\pm*\delta}_{\lambda''\mu'',1} \cdot (\bar{\delta}_{\lambda\mu_1}) < \hat{\delta}^{*\delta}_{\lambda\mu',1}$, $C^{\pm*\delta}_{\lambda''\mu'',2} \cdot$, $C^{\pm*}_{\lambda'\mu'_2} \bar{\delta}^{\delta}_{\lambda\mu_2} > \hat{\delta}^{i-1}_{\lambda\mu',2}$, $T^{i-1}_{n}(S^{*i}_{\lambda,\mu}) \triangleleft \Pi(T(S^{*i}_{\lambda\mu}))$

is sufficient in order that the intersection $T^{\delta}_{n,\sigma^{*\delta}_{\lambda\mu}}(S^{*i}_{\lambda\mu'})$

$\cap\ T^{\delta}_{n,C^{*\delta}_{\lambda\mu}}(S^{*\delta}_{\lambda_2'\mu_2'})$ is expressed as in $(2.1.35.1)$.

If the intersection is expressed as in $(2.1.35)_1$

then we say that $T^{\delta}_{n,\sigma^{*\delta}_{\lambda'\mu'}}(S^{*\delta}_{\lambda'\mu'})$ and $T^{\delta}_{n,C^{*}_{\lambda''\mu''_1}}(S^{*i}_{\lambda''\mu''_1})$

are suitable. Henceforth, we write the fiber $\hat{T}(Q^{*i}_{\lambda'\mu'})$ of

$T^{\delta}_{n,\sigma^{*\delta}_{\lambda\mu}}(S^{*\delta}_{\lambda'_1\mu'_1}) \cap T^{i}_{n,C^{*i}_{\lambda'\mu'}}(S^{*i}_{\lambda'_2\mu'_2})$ as $\hat{T}_{\sigma, C}(Q^{*\delta}_{12})$ if $T(S)$'s are suitable.

$(II)_2$ Now we combine the above results in the following manner. Assume that neighbourhoods $\{T^{\delta}_{n,\sigma^{*\delta}_{\lambda'\mu'}}(S^{*\delta}_{\lambda'\mu'})\}_{\lambda'\mu'}$ and $\{T^{\delta}_{n,c^{*\delta}_{\lambda''\mu''}}(S^{*\delta}_{\lambda''\mu''})\}_{\lambda'',\mu''}$ are associated with strata $S^{*\delta}_{\lambda',\mu'}$ of first type and $S^{*\delta}_{\lambda'',\mu''}$ of second type. We start with a situation in which conditions $(2.1.25)_1$ and $(2.1.25)_2$ for $(\hat{\sigma}^{*\delta}_{\lambda'\mu'})$, $(c^{*\delta}_{\lambda''\mu''})$ are valid. For neighbour-hoods $\{T^{\delta}_{n,\hat{\sigma}^{*\delta}_{\lambda'\mu'}}(S^{*\delta}_{\lambda'\mu'})\}_{\lambda',\mu'}$ $\{T^{\delta}_{n,c^{\delta}_{\lambda''\mu''}}(S^{*\delta}_{\lambda''\mu''})\}_{\lambda'',\mu''}$, we generalize the notion of the suitablity introduced in $(II)_{1,2,3}$.

Divide our consideration into the following three cases according to the nature of series $: S^{*\delta}_{\lambda_1\mu_1} < \cdots < S^{*\delta}_{\lambda_t\mu_t}$.

$(2.1.36)_1$ strata in series are composed of strata of first type only : $S^{*\delta}_{\lambda'_1\mu'_1} < \cdots\cdots < S^{*\delta}_{\lambda'_t\mu'_t}$.

In this case, let us consider situations in which the intersection of neighbourhoods of strata are expressed in the following way.

$$(2.1.36)'_1 \quad T^{\delta}_{n,\hat{\sigma}^{*\delta}_{\lambda_1\mu_1}}(S^{*\delta}_{\lambda_1\mu_1}) \cap \cdots \cap T^{\delta}_{n,\hat{\sigma}^{*\delta}_{\lambda_t\mu_t}}(S^{*\delta}_{\lambda_t\mu_t}) = \bigcup_{Q^{\eta*\delta}_{\lambda_1\mu_1\cdots\lambda_t\mu_t}} \hat{T}^{\delta}_{\hat{\sigma}^{\eta*\delta}_{\lambda_t\mu_t}}(Q^{\eta*\delta}_{\lambda_1\mu_1\cdots\lambda_t\mu_t})$$

where $Q^{\eta*\delta}_{\lambda\mu_1\cdots\lambda_t\mu_t}$ exhaust all the points in the set $\pi^{\eta-1}_{\hat{\sigma}\delta}(S^{\eta*\delta}_{\lambda_t\mu_t})(T^{\delta-1}_{n}(S^{*\delta}_{\lambda_1\mu_1},\cdots\cdots, S^{*\delta}_{\lambda_t\mu_t}))$.

(2.1.36)$_2$ Strata in series are composed of strata of second type only : $S^{*\delta}_{\lambda_1\mu_1'} < \dots < S^{*\delta}_{\lambda''_t\mu''_t}$: In this case the situation considered here is the case in which the intersection is expressed in the following way

$$(2.1.36)'_2 \quad T^{\delta}_{n,\sigma^{*\delta}_{\lambda_1\mu_1}}(S^{*\delta}_{\lambda_1\mu_1}) \cap \dots \cap T^{\delta}_{n,\sigma^{*\delta}_{\lambda_t\mu_t}}(S^{*\delta}_{\lambda_t\mu_t}) = \bigcup_{Q^{\delta-1}_{\lambda_1\mu_1\dots\lambda_t\mu_t}} T_{\sigma^{*\delta}_{\lambda_1\mu_1}}(Q^{*\delta-1}_{\lambda_1\mu_1\dots\lambda_t\mu_t}),$$

where $Q^{\delta-1}_{\lambda_1\mu_1\dots\lambda_t\mu_t}$'s are points in $T^{\delta-1}_n(S^{*\delta-1}_{\lambda_1\mu_1}, \dots, S^{*\delta-1}_{\lambda_t\mu_t})$.

(2.1.36)$_3$ Mixed cases : $S^{*\delta}_{\lambda'_1\mu'_1} < S^{*\delta}_{\lambda'_2\mu'_2} < \dots < S^{*\delta}_{\lambda''_1\mu''_1} < \dots < S^{*\delta}_{\lambda''_t\mu''_t}$. In this case, situations are as follows : The intersection

$\{T^{\delta}_{n,\sigma^{*\delta}_{\lambda_1\mu_1}}(S^{*\delta}_{\lambda\mu}) \cap \dots \cap T^{\delta}_{n,\sigma^{*\delta}_{\lambda_t\mu_t}}(S^{*\delta}_{\lambda_t\mu_t})\} \cap \{T^{\delta}_{n,c^{*\delta}_{\lambda_1\mu_1}}(S^{*\delta}_{\lambda_1\mu_1}) \cap \dots \cap T^{\delta}_{n,c^{*\delta}_{\lambda_t\mu_t}}(S^{*\delta}_{\lambda_t\mu_t})\}$ is expressed in the following fashion.

(2.1.36)$_{3.1}$ If two walls $S^{*\delta}_{\lambda''\mu''}$ of $S^{*\delta}_{\lambda''\mu''}$ converge to $S^{*\delta}_{\lambda''_1\mu''_1}$, then

$\{T^{\delta}_{n\sigma^{*\delta}_{\lambda''\mu''}}(S^{*\delta}_{\lambda''\mu''}) \cap \dots \cap T^{\delta}_{n,\sigma^{*\delta}_{\lambda\mu}}(S^{*\delta}_{\lambda\mu})\} \cap \{T(S^{*\delta}_{\lambda''\mu''}) \cap \dots \cap T_{n,c^{*\delta}_{\lambda\mu}}(S^{*\delta}_{\lambda''_t\mu''_t})\} = \bigcup_{Q^{*\delta}_{\lambda\mu_1\dots\lambda_t\mu_t}} \hat{T}_{c^{*\delta}_{\lambda\mu}}(Q^{*\delta}_{\lambda\mu_1\dots\lambda_t\mu_t})$

$(Q^{*\delta-1}_{\lambda_1\mu_1,\dots\lambda_t\mu_t} \in T^{-1}_n(S^{*\delta-1}_{\lambda\mu_1}, \dots, S^{*\delta-1}_{\lambda_{t_1+t_2}\mu_{t_1+t_2}}))$

(2.1.36)$_{3.2}$ If one wall $S^{*\delta}_{\lambda''\mu_1}$ only converges to $S^{*\delta}_{\lambda'_1\mu_1}$, then

$\{T^{\delta}_{n,\sigma^{*\delta}_{\lambda'\mu_1}}(S^{*\delta}_{\lambda\mu}) \cap \dots \cap T^{\delta}_{n,\sigma^{*\delta}_{\lambda\mu_t}}(S^{*\delta}_{\lambda\mu})\} \cap \{T(S^{*\delta}_{\lambda\mu}) \cap \dots \cap T_{n,c^{*\delta}_{\lambda\mu_2}}(S^{*\delta}_{\lambda''_2\mu''_2})\}$ T ( ) =

$$= \bigcup_{Q^{*\delta}_{\lambda\mu_1\dots\lambda_t\mu_t}} T_{\sigma^{*\delta}_{\lambda_1\mu_1},c^{*\delta}_{\lambda\mu}}(Q^{*\delta-1}_{\lambda\mu_1\dots\lambda_{t_2}\mu_{t_2}}) \quad (Q^{*\delta-1}_{\lambda\mu_1,\dots\lambda_t\mu_{t_2}} \in T^{\delta-1}_n(S^{*\delta}_{\lambda\mu_1}, \dots, S^{*\delta}_{\lambda_{t_1+t_2}\mu_{t_1+t_2}})).$$

(2.1.36)$_{3.3}$ If no wall of $S^{*\delta}_{\lambda''_2\mu''_2}$ converge to $S^{*\delta}_{\lambda'_1\mu'_1}$, then the intersection

$$T^{\delta}_{n,\sigma^{*\delta}_{\lambda'_1\mu_1}}(S^{*\delta}_{\lambda'_1\mu'_1}) \cap \dots \cap T^{\delta}_{n,\sigma^{*\delta}_{\lambda_t\mu_t}}(S^{*\delta}_{\lambda_t\mu_t}) = \bigcup_{Q^{*\delta}_{\lambda\mu_1\dots\lambda_t\mu_t}} T_{\sigma^{*\delta}_{\lambda_1\mu_1}}(Q^{*\delta}_{\lambda\dots\mu_t}) \quad (Q^{*\delta}_{\lambda\dots\mu_t} \in T^{\delta-1}_n(S^{*\delta-1}_{\lambda\mu_1}, \dots, S^{*\delta-1}_{\lambda_t\mu_t}))$$

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To be sure we see this fact quickly.

142 Note the following obvious facts: For series $S^{*\delta}_{\lambda'_1\mu'_1} \prec \cdots \prec S^{*\delta}_{\lambda'_t\mu'_t}$ composed of first tipe only or series $S^{*\delta}_{\lambda''_1\mu''_1} \prec \cdots \prec S^{*\delta}_{\lambda''_t\mu''_t}$ composed of second type only, the intersection relations $\bigcap_{\Delta=1}^{t'} T_{n,C^{*\delta}_{\lambda'\mu'}}(S^{*\delta}_{\lambda',\mu'}) = \bigcap_{\Delta=1}^{t-1} \{ T^{\delta}_{n,\sigma}(S^{*\delta}_{\lambda'_\Delta\mu'_\Delta}) \cap T^{\delta}_{n,\sigma}(S^{*\delta}_{\lambda'_t\mu'_t}) \}$ and $\bigcap_{\Delta=1}^{t'} T_{n,C^{*\delta}_{\lambda''\mu''}}(S^{*\delta}_{\lambda'',\mu''}) = \bigcap_{\Delta=2}^{t'} T^{\delta}_{n,c}(S^{*\delta}_{\lambda''\mu''}) \cap T_{n,c}(S^{*\delta}_{\lambda''_1\mu''_1}) \}$ are valid. Then the exist-
-ence of series $\mathcal{L}'^{\delta}(\mathcal{S}'^{\delta}), \mathcal{M}''^{\delta}(\mathcal{S}^{\delta})$ couples $\{\sigma'^{\delta}_0\}\{\sigma''^{\delta}_0\}$ with which the following statement is valid, follows from (2.1.32), (2.1.33) immeadiately.

(2.1.88)' For assignments $\hat{\sigma}'^{\delta}, \tilde{C}''^{\delta}$ of type $\{\mathcal{L}'^{\delta}, \mathcal{M}''^{\delta}\}$ satis-
-fying conditions $\hat{\sigma}'^{\delta}_1 < \hat{\sigma}'^{\delta}_0$, $C''^{\delta}_1 > C''^{\delta}_0$, there exist an integer $n' = n'(\tilde{\sigma}'^{\delta}_1, \tilde{C}''^{\delta}_1, \tilde{\sigma}'^{\delta}_0, \tilde{C}''^{\delta}_0)$ so that intersection of neighbourhoods of series, composed of strata of first (or of second) type only, are suitable.

Assume that assignments $\tilde{\sigma}'^{\delta}, \tilde{C}''^{\delta}$ of type $\{\mathcal{L}'^{\delta}, \mathcal{M}''^{\delta}\}$ are given so that the conditions $\hat{\sigma}'^{\delta}_1 < \hat{\sigma}'^{\delta}_0$, $C''^{\delta}_1 > C''^{\delta}_0$ hold. Fix an integer n in such a manner that the above condition (2.1.88)' is true. Next, for the assign-
-ments, $\tilde{\sigma}'^{\delta}, \tilde{C}''^{\delta}$, choose n'' sufficiently large

Let $l'^{\delta}_{t_0}$ and $l''^{\delta}_{t_0}$ be integers : $l'^{\delta}_{t_0}$ = $\max_{S^{\delta}_{\lambda',\mu'} \in V} l(S^{\delta}_{\lambda',\mu'}) (\cap V)$, $l''^{\delta}_{t_0} = \max l(S^{\delta}_{\lambda',\mu'})$

Given series $\mathcal{L}'^{\delta} = \{\mathcal{L}'^{\delta}_1, \dots, \mathcal{L}'^{\delta}_{l_0'^{\delta}-1}\}$ and $\mathcal{M}'^{\delta} = \{\mathcal{M}'^{\delta}_1, \dots, \mathcal{M}'^{\delta}_{l_0''}\}$ of maps of (e- $\mathcal{L}$) - types . , assignments $\hat{\sigma}'^{\delta} : \{1, \dots\} \ni t' \to \sigma'^{\delta}_{t'}$ and $\tilde{c}''^{\delta} : \{1, \dots, l''^{\delta}_{t_0}\} \ni t'' \to c''^{\delta}_{t''}$ are said to be of type $\mathcal{L}'^{\delta}$ and $\mathcal{M}'^{\delta}$ respectively if the following conditions are valid.

$$(2.1.37)_1 \quad \hat{\sigma}'^{\delta}_1 > \mathcal{L}'^{\delta}_1(\hat{\sigma}'^{\delta}_2), \dots\dots\dots, \hat{\sigma}'^{\delta}_{t-1} > \mathcal{L}(\hat{\sigma}'^{\delta}_t)$$

$$(2.1.37)_2 \quad (c''^{\delta}_1) < \mathcal{M}'^{\delta}_1(c''^{\delta}_2), \dots, c^{\delta}_{t-1} < \mathcal{M}(c''^{\delta}_t) .$$

For a given assignments $\hat{\sigma}'^{\delta}$, $\tilde{c}''^{\delta}$, define $\tilde{\sigma}'^{\delta}_{\lambda',\mu'}(S^{\delta}_{\lambda',\mu'})$, $c''^{\delta}(S^{\delta}_{\lambda'',\mu''})$ by $\hat{\sigma}'^{\delta}_{\lambda',\mu'}(S^{\delta}_{\lambda',\mu'}) = \hat{\sigma}'^{\delta}_{l(S^{\delta}_{\lambda',\mu'})}$ and $c''^{\delta}_{l}(S^{\delta}_{\lambda'',\mu''}) = c''^{\delta}_{l(S^{\delta}_{\lambda'',\mu''})}$. Now we show quickly the existence of series $\mathcal{L}'^{\delta}(\mathcal{S}'^{\delta}) = \{\mathcal{L}'^{\delta}_1, \dots, \mathcal{L}'^{\delta}\}$ and $\mathcal{M}'^{\delta}(\mathcal{S}'^{\delta}) = \{\mathcal{M}'^{\delta}_1, \dots, \mathcal{M}'^{\delta}_{l_0'-1}\}$ depending on $\mathcal{S}'$ only and also the existence of cou ($\hat{\sigma}'^{\delta}_a$), (c''^{δ}_0) so that the following statements are true.

(2.1.38) For any assignments $\hat{\sigma}'^{\delta}$ and $\tilde{c}''^{\delta}$ of types $\mathcal{L}$, $\mathcal{M}'^{\delta}(\mathcal{S}'^{\delta})$ so that $\hat{\sigma}'^{\delta}_1 < \hat{\sigma}^{\delta}_0$, $c''^{\delta}_1 > c''^{\delta}_0$ hold, there exists an integer $n = n(\hat{\sigma}'^{\delta}, \tilde{c}''^{\delta}, \hat{\sigma}^{\delta}_0, c''^{\delta}_0)$ with which neighbourhoods $\{T_{n,\sigma'^{\delta}_{\lambda',\mu'}}\}_{\lambda',\mu'}$ and $\{T_{n,c^{\delta}_{\lambda''}}\}_{\lambda'',\mu''}$ are suitable.

The above statement follows easily from (I I I)$_{1,2,3}$:

so that the condition (2. 1. 35)'s are true for any pair $S^{*\delta}_{\lambda',\mu'} \prec S^{*\delta}_{\lambda'',\mu''}$ (Remark that (2. 1. 35)'s shows that the mentioned fact is possible for $S^{*\delta}_{\lambda',\mu'} \prec S^{*\delta}_{\lambda'',\mu''}$ by choosing n'' large enough)

It is clear that, for such data, the assertion (2. 1. 38) is assured.

The above arguements finish this part of arguement to associate neibourhoods to S.

(III) Assume that neighbourhoods $T^{\delta}_{n,S^{*\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ $T^{\delta}_{n,S^{*\delta}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$ are fixed so that these neighbourhoods are suitable.

(Recall that the assertion (2. 1. 38) assures that we can ahoose suitable neighbourhoods arbitrarily small.)

For each series $S^{*\partial}_{\lambda'_1\mu'_1} < \quad < S^{*\partial}_{\lambda'_t\mu'_t} < S^{*\partial}_{\lambda''_1\mu''_1} \quad < S^{*\partial}_{\lambda''_{t_2}\mu''_{t_2}}$ the intersection of neighbourhoods $\bigcap_{\delta=1}^{t_1} \Psi^{\partial}_{n,\sigma^{*\partial}_{\lambda_\rho\mu'_\rho}}(S^{*\partial}_{\lambda'_\rho\mu'_\rho}) \bigcap_{\delta=1}^{t_2} \Psi^{\partial} (S^{*\partial}_{\lambda''\mu''_\rho}$ is written as $\Psi^{\partial}_{n,\sigma^*,c^*}(S^{*\partial}_{\lambda'\mu'_1}, \ldots\ldots\ldots, S^{*\partial}_{\lambda''_{t_2}\mu''_{t_2}})$.

The intersection of $\Psi^{\partial}_{n\sigma^*,c^*}(S^{*\partial}_{\lambda'\mu'_1}, \ldots\ldots, S^{*\partial}_{\lambda''_{t_2}\mu''_{t_2}})$ with V is written as $\Psi^{\partial}_{n,\sigma^*_\partial,c^*_\partial}(S^{*\partial}_{\lambda'\mu'_1}, \ldots\ldots, S^{*\partial}_{\lambda''_{t_2}\mu''_{t_2}})_V$. We show the existence of maps $\tau^{\partial}_{n,\sigma'',c^*} (\lambda'\mu'_1, \ldots\ldots\ldots\ldots, \lambda''_{t_2}\mu''_{t_2}) : \Psi_{n,\sigma,c^*}(S^{*\partial}_{\lambda'\mu'} \ldots S)$ $\rightarrow \Psi^{\partial}_{n,\sigma^\partial,c^\partial}(S^{*\partial}_{\lambda'\mu'_1}, \ldots\ldots, S^{*\partial}_{\lambda''_t\mu''_t})_{V^\partial}$ so that $\pi_{\partial,\partial+1} \cdot \tau^{\partial}_{n,\sigma'',c^*} = \cdot\, \pi_{\partial+1,\partial}$ is valid. Arguements are divided into two cases : (III)$_1$ The case where series $S^{*\partial}_{\lambda'\mu'_1} < \cdot\cdot \quad < S^{*\partial}_{\lambda'_t\mu'_t}$ is composed of strata of first type only. First define a retraction map $\tau'^{\partial}_{\lambda'_1\cdots\mu'_t}(n,\sigma^{*\partial}_{\lambda\mu'}\cdots\sigma^{*\partial}_{\lambda\mu'}) : \Psi^{\partial}_{n,\sigma^{*\partial}_{\lambda\mu'}}(S^{*\partial}_{\lambda'\mu'_1}, \cdots S^{*\partial}_{\lambda'_t\mu'_t}) \longrightarrow \Psi^{\partial}_{\sim} (S^{*\partial-1}_{\lambda\mu'_1}, \ldots\ldots, S^{*\partial}_{\lambda_t\mu_t})$ so that

[illegible] that

$\pi_{\partial,\partial}\, \tau'^{*\partial}_{\lambda'_1\cdots\mu'_t} = \pi_{\partial-1,\partial}$ $_{\lambda_1\cdots\mu_t}$ holds. After that define a retraction map $\tau''^{\partial}_{\lambda'_1\cdots\mu'_t}(n,\sigma^{*\partial}_{\lambda\mu'_1}\cdots\sigma^{*\partial}_{\lambda\mu'_t}) : \Psi''_n(\lambda_t' , \mu_t') (S^{*\partial-1}_{\lambda_1\mu_1}, \ldots\ldots, S^{*\partial}_{\lambda\mu}$ $\longrightarrow \Psi''_n(\lambda_t, \mu_t) (S^{*\partial-1}_{\lambda_1\mu_1}, \ldots, S^{*\partial-1}_{\lambda_t\mu_t})$ by the following equation

$$\Pi_{\delta-1,\delta} \cdot \mathcal{T}^{*\delta}_{\lambda'_1 \cdots \mu'_t} = \mathcal{T}^{*\delta-1}_{\lambda_1 \cdots \mu_t} .$$

Define a map $\tau^{\delta}_{\lambda'_1 \cdots \mu'_t}(n, \sigma^{*\delta}_{\lambda'_1\mu'_1}, \cdots \sigma^{*\delta}_{\lambda'_t\mu'_t})$ by $\tau^{\delta}_{\lambda'_1 \cdots \mu'_t}(n \quad) \circ = \tau'^{\delta}_{\lambda'_1 \cdots \mu'_t}(n, \sigma^{*\delta}_{\lambda'_1\mu'_1} \cdots \sigma^{*\delta}_{\lambda'_t\mu'_t}) \circ \tau''^{\delta}_{\lambda' \cdots \mu'_t}(n, \sigma^{*\delta}_{\lambda'_1\mu'_1} \cdots \sigma^{*\delta}_{\lambda'_t\mu'_t})$. Then the commutativity

$$\Pi_{\delta-1,\delta} \circ \tau^{\delta}_{\lambda'_1 \cdots \mu'_t}(n, \sigma^{*\delta}_{\lambda'_1\mu'_1 \cdots}) = \tau^{\delta-1}_{\lambda_1 \cdots \mu_t}(n) \circ \Pi_{\delta-1,\delta}$$

holds.

From the relation (2. 1.), the map $\tau_{\lambda' \cdots \mu'_t}(n, \sigma^{*}_{(n)})$ is a retraction of $T^{\delta}_{n,\sigma,t}(S^{*\delta}_{\lambda'_1\mu'_1}, \cdots\cdots, S'^{\delta}_{\lambda'_t\mu'_t})$ onto $T^{\delta}_{n,\sigma,t}(S'^{\delta}_{\lambda'_1\mu'_1}, \cdots\cdots, S'^{\delta}_{\lambda'_t\mu'_t}) \cap \bar{S}'^{\delta}_{\lambda'_t\mu'_t}$. Thus , in this case, our assertion (2 .1.) is shown.

(III)$_2$ Next we consider our assertion in which all the strata $S^{*\delta}_{\lambda''_1\mu''_1} \prec \cdots\cdots \prec S^{*\delta}_{\lambda''_t\mu''_t}$ in series are of second type. We consider the case where the stratum $S^{*\delta}_{\lambda''_1\mu''_1}$ has two walls.

Other cases are treatable in a similar way.

Denote by $S^{*\delta-1}_{\lambda_i\mu_i}$ the projection $\Pi_{\delta-1,\delta}(S^{*\delta}_{\lambda''_t\mu''_t})$.

Obviously the relation $S^{*\delta-1}_{\lambda_1\mu_1} \prec \quad \prec S^{*\delta-1}_{\lambda_t\mu_t}$ holds.

For a point $Q^{*\delta-1}_{1\cdots t} \in T_n(S^{*\delta-1}_{\lambda_1\mu_1}, \cdots, S^{*\delta-1}_{\lambda_t\mu_t})$, $x^{\pm}_{\delta}(Q^{*\delta-1}_{1\cdots t})$, denotes x_{δ} - coefficients of points $Q^{*\delta-1}_{1\cdots t}$ on $S^{\pm*\delta}_{\lambda'_1\mu'_1}$

so that $\Pi_{j-1,j}(\hat{Q}^{*j}_{1\cdots t}) = Q^{*j-1}_{1\cdots t}$ holds. Coordinates $x^{*j}_{j}(Q^{*j}_{1\cdots t})$ is holomorphic in the variable $Q^{*j-1}_{1\cdots t}$. Define a map $\tau_{\lambda_1''\cdots\mu_t''}(n, \sigma^j(n))$ $I \times T_{n,C^{*j}_{\lambda\mu}}(S^{*j-1}_{\lambda_1''\mu_1''}, \ldots, S^{*}_{\lambda_t\mu_t})$ $\to T_{n,C^{*j}_{\lambda\mu}}(S^{*j-1}_{\lambda_1'\mu_1'}, \ldots, S^{*j-1}_{\lambda_t''\mu_t''})$ by the following requirements

(2.1.39) For a point $R^{*j-1}_{1\cdots t} \in T_{n,C^{*j}_{\lambda}}(S^{*j-1}_{\lambda_1''\mu_1''}, \ldots, S^{*j-1})$,

$(2.1.39)_1$ $R^{*j}_{1\cdots t,\rho} = \tau_{\lambda_1\cdots\mu_t'}(n, \sigma^j(n))(\rho, R^{*j}_{1\cdots t})$ is defined by the following conditions.

$(2.1.39)_{1.1}$ $\Pi_{j-1,j}(R^{*j}_{1\cdots t,\rho}) = Q^{*j-1}_{1\cdots t,\rho}$

$(2.1.39)_{1.2}$ $\{x_{n_j}(R^{*j}_{1\cdots t,\rho}) - x_{n_j}(Q^{*j}_{1\cdots t,\rho})\} : \{x_j(\hat{Q}^{j}_{1\cdots t,\rho}) - x_j(Q^{*j}_{1\cdots t,\rho})\}$

$$= \{x_j(R^{*j}_{1\cdots t}) - x_j(Q^{*j}_{1\cdots t})\}$$

$$: \{x_j(R_{1\cdots t}) - x_j(Q^{*j}_{1\cdots t})\},$$

where $Q^{*j}_{1\cdots t,\rho} = \tau_{\lambda_1\cdots\mu_t}(\rho, \Pi_{j-1,j}(R^{*j}_{1\cdots t}))$.

It is clear that the commutativity condition (2.1) for $\tau_{\lambda_1''\cdots\mu_t''}$ is valid. Also from (2.1), the retraction condition (2.1) is valid.

(IV) Thirdly consider the series of the mixed type : $S^{*j}_{\lambda_1'\mu_1'} \prec \cdots \prec S^{*j}_{\lambda_{k_1}'\mu_{k_1}'} \prec S^{*j}_{\lambda''\mu''} \cdots \prec S^{*j}_{\lambda_{t_2}''\mu_{t_2}''}$.

Divide this case into further following

cases : (i) $\pi_{j-1,j}(S^{*j}_{\lambda'_k \mu'_k}) \neq \pi_{j-1,j}(S^{*j}_{\lambda''_1, \mu''_1})$ and (ii) $\pi_{j-1,j}(S^{*j}_{\lambda'_k \mu'_{k_1}}) = \pi_{j-1,j}(S^{*j}_{\lambda''_1 \mu''_1})$. In the case (i) define a map $\tau^j_{n,\sigma,c} = \tau^j_{n,\sigma,c}(\lambda'_1, \mu'_1, \ldots\ldots, \lambda''_{k_2} \mu''_{k_2}) : T^j_{n,\sigma',c'} = T^j_n (S^{*j}_{\lambda'_1 \mu'_1}, \ldots, S^{*j}_{\lambda''_{k_2} \mu''_{k_2}}) \longrightarrow T^j_{n,\sigma'',c''}(S^{*j}_{\lambda'_1 \mu'_1}, \ldots, S^{*j}_{\lambda''_{k_2} \mu''_{k_2}})$ by the following requirements

$$(2.1.39)_{2.1} \quad \pi_{j-1,j} \cdot \tau^j_{n,\sigma,c} = \tau^{j-1}_{n,} \cdot \pi_{j-1,j}$$

$(2.1.39)_{2.2}$ The coordinates $x_j(R^j_{\lambda'_1 \mu'_{k_2} \rho})$: $R^j_{\lambda'_1 \mu'_{k_1+k_2} \rho} = \tau^j_{n,\sigma,c}(\rho, R^j_{\lambda'_1 \mu'_{k_1 k_2}})$ and $R^j_{\lambda'_1 \mu'_{k_1 k_2}, \rho} = R^j_{\lambda'_1 \mu'_1 \cdots \lambda''_{k_2} \mu''_{k_2}}$ is defined by

$$(2.1.39)_{2.3} \quad \{x_j(\bar{R}^j_{\lambda'_1 \mu'_1 \rho}) - x_j(\bar{R}^{-j}_{\rho})\} : \{x_j(\bar{R}^{+j}_{\rho}) - x_j(\bar{R}^{-j}_{\rho})\} = \{x_j(R^j) - x_j(R^{-j})\} : \{x_j(R^{+j}) - x_j(R^{-j})\}.$$

Here $R^{\pm j}$ is defined by the conditions (i) $\pi_{j-1,j}(R^{\pm j}) = \pi_{j-1,j}(R^j) = R$ (ii) $\bar{R}^{\pm j}$ are in the boundary of the 'fiber' $T^j_{n,\sigma',c''}(R^{j-1})$ and (iii) $x_j(R^{+j}) > x_j(R^{-j})$ hold. Explicit expressions of $\bar{R}^{\pm j}$ are easily given according to the cases divided in ( ). Then it is clear that, for the value $\rho = 0$, $\tau^j_{n,\sigma,c}(0, T^{+j}_{n,\sigma'',c''}) = \pi^{-1}_{0,j}(T^{j-1}_{0}, T^{+j-1}_{n,\sigma}) \cap (\cap T_{n,\sigma,c}(S^{*j}_{\lambda'_1 \mu'_1}, \cdots S^{*j}_{\lambda'_k \mu'_k}))$.

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Because of the relation $(2.1.36)_3$, the condition ( ) is assured . In the second case the situation is completely similar : Namely let

$\mathcal{T}_n^{j-1}(\overset{*}{S}{}^{j-1}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j-1}_{\lambda_t\mu_t})$ be the retraction map $\mathcal{T}_n^{j-1}(\overset{*}{S}{}^{j-1}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j-1}_{\lambda_t\mu_t}) : T_n^{j-1}(\overset{*}{S}{}^{j-1}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j-1}_{\lambda_t\mu_t}) \longrightarrow T_n^{j-1}(\overset{*}{S}{}^{j-1}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j-1}_{\lambda_t\mu_t})$. Then, from $(2.1.36)_3$, the map is liftable to a map $\mathcal{T}_n^{j}(\overset{*}{S}{}^{j}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j}_{\lambda_{t+k}\mu_{t+k}})$ satisfying ( ) . (Observe that the fiber is an open segment).
Therefore the condition (2.1.) is assured .

The above assertion leads to the condition 2.1 .

(IV) Finally we show the existence of simple coverings $\mathcal{U}_n^{j}(\overset{*}{S}{}^{j}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j}_{\lambda_t\mu_t})$ and sets of C^∞ functions $u^{j,n}_{\lambda_1\ldots\mu_t}$ subordinate to $\mathcal{U}_n^{j}(\overset{*}{S}{}^{j}_{\lambda_1\mu_1},\ldots,\overset{*}{S}{}^{j}_{\lambda_t\mu_t})$. We start with suitable neighbourhoods $T^{j}_{\lambda,c_{\lambda\mu}}(\overset{*}{S}{}^{j}_{\lambda,\mu})$ ' s as well as $T^{j}_{\lambda,c_{\lambda\mu}}(S^{j}_{\lambda,\mu})$.
Take a series $S^{j}_{\lambda_1\mu_1} < \cdots < S^{j}_{\lambda_t\mu_t}$. The series $S^{j-1}_{\lambda_1\mu_1} < \cdots < S^{j-1}_{\lambda_t\mu_t}$ means the series of projections of $S^{j}_{\lambda_1\mu_1},\ldots,S^{j}_{\lambda_t\mu_t}$.
Recall that $T^{j}_{\lambda,c}(S^{j}_{\lambda_1\mu_1},\ldots,S^{j}_{\lambda_t\mu_t})$ is a fiber space with base $T_n^{j-1}(S^{j-1}_{\lambda_1\mu_1},\ldots,S^{j-1}_{\lambda_t\mu_t})$ and fiber an open segment (C.f. explicit expressions $(2.1.36)_1$, $(2.1.36)_2$ $(2.1.36)_3$

For each element $A^{j-1}_{n,\lambda\mu_1\cdots\lambda\mu_t\sigma} \in \mathcal{U}(S^{j-1}_{\lambda\mu_1}\cdots, S^{j-1}_{\lambda_t\mu_t})$, define an open set $A^{j}_{n,\lambda_1\cdots\mu_t,\sigma}$ by $A^{j}_{n\lambda\cdots\mu_t\sigma} = \pi^{-1}_{j-1,j}(A^{j-1}_{n,\lambda\cdots\lambda\mu_t\sigma}) \cap {}^{2}F^{j}_{n\sigma}(S^{j}_{\lambda\mu_1}\cdots S^{j}_{\lambda_t\mu_t})$.

It is clear that $\mathcal{U}^{j}_{n,\lambda\cdots\mu_t} = \{A^{j}_{n,\lambda\cdots\lambda_t\sigma}\}_{\sigma}$ is a simple covering of $T^{j}_{n,\sigma,c}(S^{j}_{\lambda\mu_1}\cdots S^{j}_{\lambda_t\mu_t})$.

Foe each $A^{j}_{n,\lambda\cdots\lambda_t,\sigma}$ define a C^{∞} function $\mathcal{U}^{j}_{n,\lambda\cdots\lambda_t,\sigma}$ by $\pi^{*}_{j-1,j}(\mathcal{U}^{j-1}_{n,\lambda\cdots\lambda_t\sigma})$.

Then from the induction hypothesis, we know that the function $\mathcal{U}^{j}_{n,\lambda\cdots\lambda_t\sigma}$ has an asymptotic behavior w.r.t. $\pi^{-1}_{j-1,j}(\mathrm{Fron}(S^{j-1}_{\lambda_t\mu_t}))$. But from (2.1) and from that the function $\mathcal{U}^{j}_{n,\lambda\cdots\lambda_t\sigma}$ is considered in $T^{j}_{n,\sigma,c}(S^{j}_{\lambda\mu_1},\cdots,S^{j}_{\lambda_t\mu_t})$, the function $\mathcal{U}^{j}_{n,\lambda\cdots\lambda_t\sigma}$ has asymptotic behavior w.r.t. $\mathrm{Fron}(S^{j}_{\lambda_t\mu_t})$.

Thus the remained problem is to verify the condition (2.1). This will be done by dividing the cases: A common method in every cases is to integrate a given form in the direction of the fiber at first and reduce the problem to the case of j-1. (i) In the case in which the strata in series are composed of first type only. In this case, for a given

The case in which the series contains at least t two strata of second kind . $(\ i\ i\ )_2$ In the case in which the strata of second kind appear in the series . In the case of $(\ ii\ )_2$ the form $\varphi^{*\delta/}_{\delta}$ and $\varphi'_{\delta}=\varphi^{\delta}_{\delta}-d\varphi^{\delta/}_{\delta}$ have asymptotic behavior w.r.t. $\pi^{-1}_{\delta-1,\delta}(\mathrm{Fron}(S^{*\delta-1}_{\lambda_t\mu_t})$

Thus the problem is reduced to the case $(\delta-1)$.

In the second case, we divide our consideration in $: S^{*\delta}_{\lambda,\mu,\kappa}\cdots < S^{*\delta}_{\lambda_t\mu_t} < S^{*\delta}_{\lambda''\mu''}$, $T^{\delta}_{\tilde\sigma^{\delta\circ}_{\lambda'\mu'},\hat\sigma^{\delta\circ}_{\lambda''\mu''}}(\ S^{t*\delta}_{\lambda'',\mu''}\ )$ and outside of $T^{\delta}_{\tilde\sigma^{\delta\circ}_{\lambda'\mu'},\hat\sigma^{\delta\circ}_{\lambda''\mu''}}(\ S^{t*\delta}_{\lambda''\mu''})$. For a point $Q_{\lambda''\mu''}\notin T^{\delta}(\ S^{t*\delta}_{\lambda''\mu''}\ )$ the form $\varphi^{\delta}_{\delta}$ has asymptotic behavior w.r.t. $\pi^{-1}_{\delta-1,\delta}(\mathrm{Fron}(S^{\delta-1}_{\lambda_t\mu_t})) : S^{*\delta}_{\lambda_t\mu_t}=\pi(S^{*\delta}_{\lambda\mu})$. On the otherhand , in $T_{\tilde\sigma^{\delta\circ}_{\lambda'\mu'},\hat\sigma^{\delta\circ}_{\lambda''\mu''}}(\ S^{t*\delta}_{\lambda'',\mu''}\ )$ the matter is as follows : Recall that for a point $Q^{*\delta}_{\lambda''\mu''}\in T_{\tilde\sigma^{\delta\circ}_{\lambda'\mu'},\hat\sigma^{\delta\circ}_{\lambda''\mu''}}(\ S^{t*\delta}_{\lambda'',\mu''})$, $d\ (\ Q^{*\delta}_{\lambda'',\mu''},\mathrm{Fron}(S^{t*\delta}_{\lambda''\mu''}))$ $= d\ (\ Q^{*\delta}_{\lambda'',\mu''}\ ,\ \overline{S}^{*\delta}_{\lambda'',\mu''}\ )$ holds . Moreover for a point $Q^{*\delta}_{\lambda'',\mu''}$ in $T\ (\ S\ )$ $T_{n,C^{*\delta}_{\lambda'\mu'}}(\ S\ )$, there exists a unique points $Q^{t*\delta}_{\lambda'',\mu''}$, $\bar Q^{*\delta}_{\lambda'',\mu''}$ so that the relations

$$(2.1.40)_1 \qquad \pi_{\delta-1,\delta}\,(Q^{*\delta}_{\lambda'',\mu''}) = \pi_{\delta-1,\delta}\,(\overset{+}{Q}{}^{*\delta}_{\lambda'',\mu''}) = \pi_{\delta-1,\delta}\,(\bar Q^{*\delta}_{\lambda',\mu'}) .$$

$$(2.1.40)_2 \qquad Q^{*\delta}_{\lambda,\mu''} \in \overline{T}^{\delta}_{n,C^{*\delta}_{\lambda\mu}}(S^{*\delta}_{\lambda'',\mu''}) - T_{n,C^{*\delta}_{\lambda,\mu}}(S^{*\delta}_{\lambda'',\mu''})$$

hold .

Note that the comparison relation : $d(Q^{*\delta}_{\lambda'\mu''}, \bar{S}^{\pm*\delta}_{\lambda'\mu''}) \sim d(Q^{*\delta}_{\lambda'\mu'}, Q^{\pm*\delta}_{\lambda'\mu'})$ is valid . Put $\mathfrak{X}'^{\pm}_{\delta} = x_{\delta}(Q^{*\delta}_{\lambda',\mu'}) - x_{\delta}(Q^{\pm*\delta}_{\lambda',\mu'})$. After these preparatory considerations consider the integration $\int_{\mathfrak{X}_\delta} \varphi^{g}_{\delta}$. In this case we integrate the form explicitly as follows .

$$(2.1.41)\quad \varphi^{g-1}_{\delta} = \int_{\mathfrak{X}_\delta = \mathfrak{X}_\delta(Q^{\delta-1}_{\lambda'\mu'})} \varphi^{g}_{\delta} \quad : \quad \mathfrak{X}_\delta(Q^{\delta-1}_{\lambda'\mu'}) = \mathfrak{X}_\delta(Q^{\delta}_{\lambda'\mu'}) + \tfrac{1}{2}\{\mathfrak{X}_\delta(Q^{+\delta}_{\lambda'\mu'}) - \mathfrak{X}_\delta(\sigma^{\delta}_{\lambda'\mu'})\},$$

then it is clear that the form φ^{g-1}_{δ} has asymptotic behavior w.r.t. $\pi^{-1}_{\delta,\lambda}(\mathrm{Fron}(S^{*\delta}_{\lambda_\delta\mu_\delta}))$ outside $T_{\bar{\sigma}^{\delta,0}_{\lambda'\mu'}}(S^{*\delta}_{\lambda'\mu'})$. On the other hand , in the set $T^{\tilde{r}}_{\bar{\sigma}^{\delta,0}_{\lambda',\mu'}}(S^{*\delta}_{\lambda',\mu'})$, the integration φ^{g-1}_{δ} has asymptotic behavior w.r.t. $\bar{S}^{\pm*\delta}_{\lambda',\mu'}$ (Here we speak about the case where $S^{*\delta}_{\lambda'\mu'}$ has two walls . Other cases are treatable in the same man in the whole the form φ^{g-1}_{δ} and the difference $\varphi'^{g}_{\delta} = \varphi^{g}_{\delta} - d\varphi^{g-1}_{\delta}$ has asymptotic behavior w.r.t. $\mathrm{Fron}(S^{*\delta}_{\lambda',\mu'})$. By the same reason as in ( ) , the problem is reduced to $(\delta-1)$. ( ) Now the arguements in ( ) ~ ( ) leads to the desired fact : Indeed the conditions (2.1.36) for the suitablity shows that we can choose a directed set $\mathcal{T}(\mathcal{R}^*)$ of $\mathcal{R}^*$ so

§ 2.2. Retractions with quantity

§ 2.2 n.1. Our position here is to give precise meanings of propetties of retraction maps which we call simply ' quantitative properties of retraction maps.' Let (U ,V , P be a triple composed of a neighbourhood U , a variety V in U and an 'origin' P of V . In this and later arguements, when we speak of a retraction map τ of a pair (U, V) to a point P we are always concerned with not only with the couple (U, V) but also with the couple(U,V) endowed with a pre-stratification $\mathcal{S}^*$ of V (and so a natural stratification of U induced from $\mathcal{S}^*_0$) Our problem will be formulated in terms of a given prestratification of $\mathcal{S}^*_0$ rather than the couple (U, V) itself.

Let I =[0,1] be an interval in $\mathbb{R}$. Variables in I will be denoted by ρ . A continious map τ :

$\tau: I \times (U, V) \longrightarrow (U, V)$ is said to be a **retraction of $\mathcal{S}^*_0$** if the following conditions are satisfied.

$(2.2.1)_1$ $\tau(0, U) = P$, $\tau(1, Q) = Q$, for any $Q \in U$.

$(2.2.1)_2$ For each stratum $S^* \in \mathcal{S}^*$, an inclusion relation

$$\tau : (I - 0) \times S^* \hookrightarrow S^*$$

is true.

Given a subclosed set W of V and triples $(\beta), (\beta')$ of positive numbers, a retraction map τ of $\mathcal{S}^*$ will be called to satisfy $(\beta) - (\beta')$ - distance preserving property (w.r.t. W) if the following inequality is valid.

$(2.2.2)_1$ $\beta_1 d(Q, W)^{\beta_2} \cdot \varepsilon^{\beta_3} \leqq d(Q_\varepsilon, W)$ $\leqq \beta_1' d(Q, W)^{\beta_2'} \cdot \varepsilon^{\beta_3'}$, for any point $Q \in U$ and $\varepsilon \in (0, 1]$.

In the above $Q_\varepsilon = \tau(\varepsilon, Q)$. In the sequel the following abbriviation of the inequality(2) will be used.

$$(2.2.2)'_1 \qquad (\beta)\cdot d(\mathcal{Q}, W) \leq d(Q_\ell, W) \leq (\beta')\cdot d(Q, W).$$

Given a pair (W_1, W_2) of closed sets of V so that $W_1 \supsetneq W_2$ holds. we define a notion of <u>inclusion property</u> in the following manner:

Let (σ), (σ^1) be couples. A retraction map τ of $\mathcal{S}^*$ has <u>$(\sigma)-(\sigma^1)$-inclusion property</u> <u>(w.r.t. (W_1, W_2))</u> (<u>$(\sigma)-(\sigma^1)$-i.p. w.r.t. (W_1, W_2)</u>) if the following condition is valid.

$$(2.2.2)_2 \quad \tau : [0,1] \times N_\sigma(W_1, W_2') \hookrightarrow N_{\sigma^1}(W_1', W_2),$$

On the otherhand, for a given pair (σ'), (σ'^2) of couples, a map τ is said to have <u>$(\sigma')-(\sigma'^2)$-exclusion property w.r.t. (W_1, W_2) ($(\sigma')-(\sigma'^2)$-ex.p. w.r.t. (W_1, W_2))</u>, if the following condition is valid.

$$(2.2.2)'_2 \quad \tau : [0,1] \times \{U - N_\sigma(W_1, W_2)\} \to U - N_\sigma(W_1, W_2)$$

Examine certain immeadiate consequence of the above definitions : Let W_s ($s = 1, \ldots, s_0$) be a finite set of sub closed sets of V . We assume the following d.p. properties

$$(\beta^s) \circ d(Q, W_s) \leqq d(Q_\rho, W_s) \leqq (\beta'^s) \cdot d(Q, W_s) \quad (s = 1, \cdots s_0),$$

, for each subset W_s with suitable couples .

Then a simple observation leads to the following

$$\tilde{\beta}_1 \cdot d(Q, \cup_s W_s)^{\tilde{\beta}_2} \rho^{\tilde{\beta}_3} \leqq d(Q_\rho, \cup_s W_s) \leqq \tilde{\beta}'_1 d(Q, \cup_s W_s)^{\tilde{\beta}'_2} Q^{\tilde{\beta}'_3},$$

where $(\tilde{\beta})$, $(\tilde{\beta}')$ are triples defined to be

$$(\tilde{\beta}) = (\min_s \beta_1^s, \max_s \beta_2^s, \max_s \beta_3^s) \quad (\tilde{\beta}') = (\mathrm{Max}_s \tilde{\beta}_1'^s, \min_s \tilde{\beta}_2'^s, \min_s \tilde{\beta}_3'^s)$$

Next we examine elementary relations between the notitions of d.p. and (i.p., ex.p.) . We speak relations in terms of neighbourhoods $M_{\tau s}'$. Let τ be a retraction map of $\mathcal{S}_c^*$, and let (W_1 , W_2) be a pair of subvarieties of V . We assume that τ has (σ) - (σ^1) (resp. (σ') - (σ'^2)) d.p w.r.t. W_2 (resp. W_2) . Take a point $Q \in U$ which satisfies the inequality

$$(2.2.3)' \qquad d(Q, W_1) \leqq h_1 d(Q, W_2)^{h_2}$$

Moreover, we assume that the map τ has (β)-(β') - d.p. w.r.t. W_1 (and ($\hat{\beta}_1$) - ($\hat{\beta}_2$) - d.p. w.r.t. W_2). Then a simple computation leads to the following

$$(2.2.3)''_1 \quad d(Q_e, W_1) \leqq \beta'_1 \cdot (\hat{\beta}_1)^{-(\beta'_2/\hat{\beta}_2) h_2} \cdot h_1^{\beta'_2} \cdot d(Q_e, W')^{(\beta'_2/\hat{\beta}_2) h_2} \times \rho^{(\beta'_3 - (\beta'_2/\hat{\beta}_2)\hat{\beta}_3 \cdot h_2)}$$

Thus we know the following fact .

(2.2.3)$_1$ If the inequalities

$$\beta'_3/\beta'_2 > (\hat{\beta}_3/\hat{\beta}_2) \cdot \delta_2 , \quad \delta_1^{\beta'_2} \cdot e^{\beta'_2} \cdot \beta'_1 \cdot (\hat{\beta}_1)^{-(\beta'_2/\hat{\beta}_2)\delta_2} \leqq 1/2$$

is valid, then the inclusion relation

$$\tau : [0, 1] \times N_\delta(W_1, W_2) \hookrightarrow N_{\delta'}(W_1, W_2)$$

with $\delta' = (\delta'_1, \delta'_2)$: $\delta'_1 = \delta_1^{\beta'_2} \cdot e^{(1+\beta'_2)\delta_2}$, $\delta'_2 = (\beta'_2/\hat{\beta}_2) \cdot \delta_2$ holds .

On the other hand, assume the following

$$(2.2.3)'_2 \quad d(Q, W_2) \leqq h'_1 \cdot d(Q, W_1)^{h'_2}$$

Then we know the following

$$(2.2.3)''_2 \quad d(Q_e, W_2) \leqq \hat{\beta}'_1 \cdot (\beta_1)^{-(\hat{\beta}'_2/\hat{\beta}_2) \cdot h'_2} \cdot h_1'^{\hat{\beta}'_2} \cdot d(Q_e, W)^{(\hat{\beta}'_2/\hat{\beta}_2) h'_2} \cdot \rho^{\hat{\beta}'_3 - (\hat{\beta}'_2/\hat{\beta}_2)\beta_3 h_2}$$

The above inequality

Then combining (2.2.) and (1.) we know

(2 . 2. 3)′ Inequalities $\delta_2'(\hat{\beta}_3'/\hat{\beta}_2') \geqq \beta_3/\beta_2$, $\hat{\beta}_1^{-\hat{\beta}_2'/\beta_2 \cdot \delta_2'} \cdot \beta_1' \cdot \delta_1'^{\beta_2'} \leqq (1/2)$

implies the following exclusion relation

$$(2.2.3)' \quad \tau : I \times {}^2\mathcal{T}_{\sigma'}(\mathbb{W}_1, \mathbb{W}_2) \hookrightarrow {}^2\mathcal{T}_{\sigma'^2}(\mathbb{W}_1, \mathbb{W}_2)$$

with $(\sigma'^2) = (\sigma_1'^2, \sigma_2'^2)$ $\sigma_1'^2 = (\hat{\beta}_1')^{-\hat{\beta}_2'/\beta_2 \cdot \delta_2} \cdot \beta_1 \cdot \delta_1 \cdot c^{\beta_2 - (\beta_2+1)\delta_2}$, $\sigma_2'^2 = (\beta_2/\beta_2') \cdot \delta_2$ . .

In connection with the above observations, we shall remark the following. Concerning the exclusion property, (2. 2)′ tells us that , by taking (σ') sufficiently small, the exclusion relation is assured (c.f. 2.2.). In this sense we say that the d.p. implies the ex. p. On the otherhand, if we consider the i.p. , the situation becomes a little subtle : The relation (2.2.) shows that , if we take (σ) sufficiently large then the inequality relation is formally valid. But we point out that the couple (σ) should be suitably small in order that the set $\mathcal{N}_g(\mathcal{G})$ reflects the property of $\mathcal{G}$. In this sense we regard that i.p. is not an immeadiate consequence of d.p. Further relations among the above notio will be discussed in later .

Now we shall be concerned with our formulation of our problems : Let U be a domain in $\mathbb{R}^N$, and let V be an analytic variety in U . Moreover, let P be a point P in V . Moreover, let $\mathcal{R}$ be a regular series and let $\mathcal{F}$ be a normalizing datum of $\mathcal{R}$. Of course we assume that ($\mathcal{R}$, $\mathcal{F}$) is attached to V. We assume the condition (R. C) . Our problem will be formulated in terms of the series $\mathcal{R}^*$ (c.f. §2.1) We note that we formulate the i.p. and ex. p . in terms of the T – neighbourhoods rather than N – neighbourhoods . A series $\mathcal{T}(\mathcal{R}^*) = \{\mathcal{T}^\delta\}_{\delta=1,\cdots N}$ of maps will be called a c^2- retraction of the series $\mathcal{R}^*$ if the following conditions are valid.

160 $(2.2.4)_0$ $\pi_{j+1,j} \cdot \tau^{j-1} = \tau^{j} \cdot \pi_{j,j-1}$.

$(2.2.4)_1$ Maps τ^{j} are c^{∞}- differentiable on each set $[0,1] \times Q^{j}$; $Q^{j} \in \mathfrak{U}^{j}$.

$(2.2.4)_2$ Maps τ^{j} are c^{∞}- differentiable in $[0,1] \times U - \mathfrak{B}^{j}$.

(Remark) Because of the existence of ramification loci, it is not possible to sharpen differentiablity condition $(2.2)_2$ concerning the ramifications in general .

Now our concerns about quantitative properties of retraction maps come from sources ; (i) d.p with respect to closures of strata S^* .

(i i) Sizes of neighbourhoods T^{j}_{j-j} s where the i.p. and the ex. p. are valid.

(i i i) Quantitative properties of c^{∞}- differentiable properties of τ^{j} . These problems will be formulated in the following manner.

(i) Let us assume that a series of assignments $([\beta^{j}])$, $([\beta'^{j}])$ are given : $[\beta^{j}]: S^{t_j}_{\lambda\mu} \to \beta^{j}_{\mu}(S^{t_j}_{\lambda\mu})$, $[\beta'^{j}]: S^{t_j}_{\lambda\mu} \to \beta'^{j}_{\mu}(S^{t_j}_{\lambda\mu})$, where (β^{j}), (β'^{j}) are triples of positive numbers . The collection $\{\beta^{j}\}_{j=1,\dots,N}$, $\{\beta'^{j}\}_{j=1,\dots,N}$ will be denoted by $[\beta]$, $[\beta']$. A retraction τ of $\mathcal{R}$ will be called <u>$([\beta])-([\beta'])$- distance preserving</u> if the map τ^{j} is $(\beta^{j})-(\beta'^{j})$- d.p. for each pre-stratification $\mathcal{S}^{*j}$ ($([\beta])-([\beta'])$- d.p.).

(i i) Given a series of couples $\{(\sigma^{j})\}_{j=1,\dots,N}$, $\{(\sigma'^{j})\}_{j=1,\dots,N}$, $(\sigma'^{j})_{j=1,\dots,N}$, $(\sigma'^{j,2})_{j=1,\dots,N}$, where (σ^{j}), $(\sigma^{j,1})$, (σ'^{j}) and (σ'^{j2}) are couples of positive numbers . the collections $\{(\sigma^{j})\}_{j=1,\dots,N}, \dots, \{(\sigma'^{j2})\}_{j=1,\dots,N}$ will be denoted by $[\sigma], \dots, [\sigma'^{2}]$.

A map τ is <u>$([\sigma])-([\sigma^{1}])$ i.p.</u> or <u>$([\sigma'])-([\sigma'^{2}])$- ex. p.</u> if the map τ has the properties : τ^{j} is $(\sigma)-(\sigma^{1})$ i.p. (or $(\sigma')-(\sigma'^{2})$ - ex. p.) for each pre stratification $\mathcal{S}^{*j}$. : (, ( )) .

(i i i) To state quantitative properties of C^{∞}- differentiablity, we introduce the following notations . Let (k, k^{j}) be a set of non negative integers $\in \mathbb{Z} \times \mathbb{Z}^{j} (j=1,\dots,N)$. Let τ be a map of $\mathfrak{R}^*$ and let ( ) be the collection.

will be denoted by $[\gamma]$. A map τ will be called to have $([\gamma])$ – **differentiable property** if the following is valid.

(2.2.5)$_1$ For each D_K^{j} and for a point $\mathfrak{a}^j$ ( ) the following estimation is valid.

(2.2.5)$_2$ $|D_K^j x_\mu(a)| \leq \gamma_{k,K,1}^{j} \cdot d(a, \mathfrak{D})^{-\gamma_{k,K,2}^{j}} \cdot e^{-\gamma_{k,3}^{j}}$,

Using the above notations, our problem will be formulated in the following fashion.

Lemma 2.2.1. (Retractions with quantities)

For a given series $\mathfrak{R}^*$ and normalizing functions $\mathfrak{F}$, there exists a set of assignments $[\beta]$, $[\beta']$, $[\sigma]$, $[\sigma^1]$, $[\sigma']$, $[\sigma']$, $[\gamma]$ as well as a series of maps $\mathfrak{P} = \{\mathfrak{P}_1, \cdots, \mathfrak{P}\}$ in such a manner that the following are true.

(2.2.6)$_1$ For any series of positive

numbers (Υ) of type (c.f §1.), there exists a retraction map $\tau = \tau(\mathcal{R}^*)$ of $\mathcal{R}^*_{( )}$, so that the map τ has the following properties.

$(2.2.6)_1$ τ IS of (β) - (β') d.p.

$(2.2.6)_2$ τ is of (σ)- (σ^1)- i.p. and (σ') - (σ'') – ex.p.

$(2.2.6)_3$ τ is of (γ) - ( )

Remark . our argoements will be done in such a manner that explicit computations of data $(\beta), \cdots, (\gamma)$ are computable in an inductive way , though we do not enter into tedious computations , Key points which enables us to make computations of $(\beta), \cdots$ are as follows :
The above lemma is spoken in terms of a fixed point P_0.
Next we will be concerned with the variance of the retraction with quantities :

Remakk . In the connection with the basic distance preserving propertis (c.f. (2.2.)), we shall add the following : Let $S^{*\delta}_{\lambda,\mu}$ be a stratum $\in \mathcal{S}^{*\delta}$, then , for a retraction map τ^{δ} with properties (2.2.) , it may happen that the d.p. is sharpened in a small neighbourhood of $S^{*\delta}_{\lambda,\mu}$: .

By this reason , we introduce one another notion of local distance preserving property as follows.

As in (2.2.) , a map τ^{δ} is said to have (β_{loc})-(β'_{loc})-local distance property around S^{*} (or more strictly in $T^{\delta}_{\sigma}(S^{*})$) if the inequalities

$$(\beta_{loc}(S^{*}))\cdot d(Q, \overline{S}^{*}) \leq d(Q_{t}, \overline{S}^{*}) \leq (\beta'_{loc}(S^{*}))\cdot d(Q, \overline{S}^{*})$$

are valid for a point P in $T^{\delta}_{\sigma}(S^{*\delta}, \mathrm{Fr}(S))$.

This notion will be used for a map τ defined locally in a neighbourhood of $S^{*\delta}$ only :

Our arguements will be done in turn by fixing quamtities ($\beta_{loc}(S^{*})$), ($\beta_{loc}(S'^{*})$) at first in local situations and after that we make use of the obtained results to yield a ' global ' result .

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is assured for j . Take a stratum S^* of the dimension $(i+1)$. If a point Q^* is in $N_\delta(S^*, \mathrm{Fron}(S^*))$, then we have the following

$$(2.2.7)_1 \quad (\beta_{loc}(S^*)) \circ d(Q, \overline{S}^*) \leq d(Q_e, \overline{S}^*) \leq (\beta'_{loc}(S^*)) \circ d(Q, \overline{S}^*).$$

Assume that a point Q is not in $N_\delta(S^*, \mathrm{Fron}(S^*))$. From the induction hypothesis we know the following

$$(2.2.7)_2 \quad \hat{\beta}(S^*) \circ d(Q, \mathrm{Fron}(S^*)) \leq d(Q_e, \mathrm{Fron}(S^*)) \leq \hat{\beta}'(S^*) \circ d(Q, \mathrm{Fron}(S^*))$$

with suitable triples $\hat{\beta}(S^*)$, $\hat{\beta}'(S^*)$.

Combining this inequality with (2.2.) we have the following

(2.2.7)

with = , = .

This finishes our verification of (1.9) for j

The avove observation enables us to concentrate our attension to local distance preserving property in each neighbourhood of S and exclusive property (as in ( ), ( ) .

In connection with the above remark we shall make the following observation : Let τ be a retraction of $\mathcal{S}$. Moreover, we assume the following proprrties of the map τ.

(i) For each stratum $S^* \in \mathcal{S}^*$, there exist a couple (δ), triples ($\beta_{loc}(S^*)$), ($\beta'_{loc}(S^*)$) in such a manner that the map τ has (β_{loc}) - (β'_{loc}) - d.p. property in $N_\delta(S^*, \tau_{loc}(S^*))$.

(ii) For each $S^* \in \mathcal{S}^*$, there exist couples (δ') (< (δ) and (δ'^1) in such a way that the map τ has the(δ')- (δ'^1) - ex.p .

Then we know that (d.p) the map τ has (β) - (β')- d.p.(in U) for each S with suitable (β),(β')'s.

This is shown quickly by the induction on the dimension of S^*. If dim (S) = 0, then the assumption(i) gives an answer for (d.p) . Assume that (d.p)

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Before we enter into details, we shall make preparations

What we will be concerned here are notational properties :
As neighbourhoods , we take those neighbourhoods $T_{\hat{\sigma}^{\dot{\delta}}_{\lambda,\mu}\hat{\sigma}^{\dot{\delta}}_{\lambda,\mu}}(S^{*\dot{\delta}}_{\lambda,\mu}$ Fron $(S^{*\dot{\delta}}_{\lambda,\mu}))$ rather than neighbourhoods $T_{\sigma^{\dot{\delta}}_{\lambda,\mu}}(S^{*\dot{\delta}}_{\lambda,\mu}$, Fron$(S^{*\dot{\delta}}_{\lambda,\mu}))$. This use of neighbour-
-hood $\hat{T}_{\hat{\sigma}\dot{\delta}}$ is for purpose of discussing inductive steps . The symbols $(\hat{\beta}^{\dot{\delta}}_{\lambda,\mu})$, $(\hat{\beta}'^{\dot{\delta}}_{\lambda,\mu})$ are used to express distance relations to Fron $(S^{*\dot{\delta}}_{\lambda,\mu})$. Then $(\hat{\beta}^{\dot{\delta}}_{\lambda,\mu})$'s and $(\hat{\beta}'^{\dot{\delta}}_{\lambda,\mu})$'s are compared in the following manner.

$$(\hat{\beta}) \qquad (\hat{\beta}^{\dot{\delta}}) = \left(\hat{\beta}^{\dot{\delta}-1}_1\cdot(b^{\dot{\delta}}_1\cdot t)^{-\hat{\beta}^{\dot{\delta}-1}_2/b^{\dot{\delta}}_2},\ \hat{\beta}^{\dot{\delta}-1}_2/b^{\dot{\delta}}_2,\ \hat{\beta}^{\dot{\delta}-1}_3/b^{\dot{\delta}}_2\right),$$

$$(\hat{\beta}'^{\dot{\delta}}) = \left(\hat{\beta}'^{\dot{\delta}-1}_1\cdot(b^{\dot{\delta}}_1\cdot t)^{\hat{\beta}^{\dot{\delta}-1}_2\cdot b^{\dot{\delta}}_2},\ \hat{\beta}'^{\dot{\delta}-1}_2\cdot b^{\dot{\delta}}_2,\ \hat{\beta}'^{\dot{\delta}-1}_3\cdot b^{\dot{\delta}}_3\right).$$

Also in a small neighbourhood of $S^{*\dot{\delta}}_{\lambda',\mu'}$, the distance to $\overline{S}^{*\dot{\delta}}_{\lambda',\mu'}$ will be replaced by $\hat{d}$, This is also for purpose of discussing inductive procedures . Finally to express natures of points in a neighbourhood of $S^{*\dot{\delta}}_{\lambda,\mu}$, rules ( ) will be preserved.

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n. 2, Remaining parts of the present section will be devoted to verifications of the lemmas stated in n°. 1 . Before we enter int details, we shall outline our arguements . Principally our arguement will be done by the induction on j .Namely we ask possiblities of lifting a given retraction map τ^j with a desired quantities (β),-- . . . . . .(γ) . Here we note that , if we ask quantitative conditions then a problem of a liftablity becomes subtle than a case where we do not consider quantitative conditions . It seems reasonable, to the author 's investigation , to impose certain additional quantitative properties for τ^j's in order that the given map τ^j is liftable to a map τ^{j+1} with suitable quantities . Therefore our arguement will be done in such a manner that cares are taken of about conditions on $\{(\beta^j) \cdots (\gamma^j)\}$ so that τ^j is liftable to a map τ^{j+1} with quantities .

Remaining parts of this section will be devoted to verifications of assertions in n. 1° . Our arguements will be done inductively on j :

n. 2.1 . We start with local situations around a stratum $S^{*j}_{\lambda,\mu}$. An underlying datum is a series of retraction maps $\tau^{j'}$ ($j'=1,\cdots,j-1$) with quantitative properties : $(\sigma^{j,j'})-(\sigma^{j',1})$- inclusion , $(\beta^{j'})-(\beta'^{j'})$- distance property , and $(\beta^{j'}_{loc})-(\beta'^{j'}_{loc})$- local distance property ($j'=1,\cdots,j-1$). We assume also $(\gamma^{j'})$ - differentiable property ($j'=1,\cdots,j-1$).

Let us consider a map τ^{*j} defined in a set $T_{\bar\sigma^j_{\lambda\mu},\hat\sigma^j_{\lambda\mu}}(S^{*j}_{\lambda\mu}, \mathcal{F}_{con}(S^{*j}_{\lambda\mu}))\cap\{\bigcup T_{\bar\sigma^j_{\tilde\lambda\tilde\mu},\hat\sigma^j_{\tilde\lambda\tilde\mu}}(S^{*j}_{\tilde\lambda\tilde\mu},\mathcal{F}_{con}(S^{*j}_{\tilde\lambda\tilde\mu})) ; S^{*j}_{\tilde\lambda\tilde\mu}\in\mathcal{S}^+(S^{*j}_{\lambda\mu})\}$. We consider following conditions for such a map τ^{*j} :

(2.2.)$_1$ $\pi_{j-1,j}\cdot\tau^{j} = \tau^{j-1}\cdot\pi_{j-1,j}$,

(2.2.)$_2$ For each $S^{*j}_{\tilde\lambda,\tilde\mu}$, τ^{j} has $(\bar\sigma^j_{\lambda\mu},\hat\sigma^j_{\lambda\mu})$ - $(\bar\sigma^j_{\lambda,\mu},\hat\sigma^{j,1}_{\lambda\mu})$- inclusion property .

(2.2.)$_3$ τ^{*j} has $(\beta^{j}_{loc,\tilde\lambda\tilde\mu})$ - $(\beta'^{j}_{loc,\tilde\lambda,\tilde\mu})$- local distance properties with quantities (β^{j}_{loc}) , (β'^{j}_{loc}) , in each $T_{\bar\sigma^j_{\tilde\lambda\tilde\mu},\hat\sigma^j_{\tilde\lambda\tilde\mu}}(S^{*j}_{\tilde\lambda\tilde\mu}, \mathcal{F}_{con}(S^{*j}_{\tilde\lambda\tilde\mu}))$.

(2.2.)$_4$ τ^{*j} is C^∞- differentiable outside $(0)\times\mathcal{D}^{j}$ and is estimated in ~~the~~ a manner

(2.2) $|Q_{k,K^j}(x^j)| \leq \gamma_{k,K^j,1} \cdot d(Q, \mathcal{D})^{-\gamma_{k,K^j,2}} \cdot \rho^{-\gamma_{k,K^j,3}}$

Moreover, we consider the following

(2.2) The x_δ - coordinate $x^\delta(\tau^j(\rho, Q))$ is C^∞ differntiable in the variable x_δ .

Furthermore, if a point Q is not in $\mathcal{D}$ then

(2.2) $\frac{d\tilde{x}^j(\rho,x)}{dx^\delta} \geq \gamma_1^\delta \cdot d(Q, \mathcal{D}^\delta)^{\gamma_2^\delta} \cdot \rho^{\gamma_3^\delta}$ if $Q \notin \mathcal{D}$

and if Q is in $\mathcal{D}$ then

(2.2) $\frac{d\tilde{x}^j(\rho,x)}{dx^\delta} \geq \gamma_1^\delta \cdot d(Q, \mathcal{D}^\delta)^{\gamma_2^\delta} \cdot \rho^{\gamma_3^\delta}$

where S is a (uniquely determined) stratum containing the point Q .

The above conditions correspond to ( )

Finally , as a technical condition, we consider the following

(2.2) For each $S^{*j}_{\lambda\mu} \in \mathcal{S}(S^{*}_{\lambda,\mu})$ there exists a couple ($\sigma^j_{\lambda\mu}$') so that , for a point $Q \notin T_{\sigma^j_{\lambda\mu}, \sigma^j_{\lambda,\mu}}$, the relation

$$\left| \frac{d\tilde{x}_j}{dx_\delta} \right| \leq K$$

is valid with a constant K .

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if a map , defined in $\bigcup_{\tilde{\lambda},\tilde{\mu}} T_{\sigma\hat{\sigma}}(S^{*i}_{\tilde{\lambda},\tilde{\mu}}, \mathfrak{F}_{\ell o n}(S^{*i}_{\tilde{\lambda}\tilde{\mu}}))$, satisfies the above cited conditions (2.2) ~ (2.2), we say that a map is adequate .

Let Σ be a subset of $\mathcal{T}_{\sigma\hat{\sigma}}(S^{*i}_{\lambda,\mu}, \mathfrak{F}_{\ell o n}(S^{*i}_{\lambda\mu}))$ A map defined τ in Σ , will be called an extesion of τ (in Σ) if coincides with τ in a set of the form $\{\bigcup_{\tilde{\lambda},\tilde{\mu}} \mathcal{T}_{\tilde{\sigma}\tilde{\sigma}}(S^{*i}_{\tilde{\lambda},\tilde{\mu}}, \mathfrak{F}_{\ell o n}(S^{*i}_{\tilde{\lambda},\tilde{\mu}}))\}\Sigma$. Here we note that $\mathcal{T}_{\tilde{\sigma},\hat{\sigma}} \subsetneq \mathcal{T}_{\tilde{\sigma},\hat{\sigma}}$ holds in general .

Our problem will be concerned with an extension of a given map τ . In doing our arguements , quantitative problems which we will be concerned a are of the following two features .

(2.2) Differential properties of τ :

(2.) Local distance preserving properties of $\mathcal{T}_{\lambda\mu}$ to $\bar{S}_{\lambda\mu}$

Our discussion will be divided into two parts : (1) arguements in a set of the form $U_{\bar{\sigma}^i_{\lambda\mu}, C^i_{\lambda\mu}}(S^{*i}_{\lambda,\mu})$ and (i i) arguements in a set of the form $T_{\bar{\sigma}^i_{\lambda\mu}\hat{\sigma}^i_{\lambda\mu}} - U_{\bar{\sigma}^i_{\lambda\mu}C^i_{\lambda\mu}}$.

Differentiable properties will be cared in the first situation while distance relations will be examined in the second part . Arguements in both cases are quite elementary . However discussions will be done to certain details .

n. 2. . First we show discuss a problem of extending the given map $\mathcal{T}$ to a set U_{σ, c^i}. Arguements in this part will be done as follows : Take a stratum $S^{*i}_{\bar{\lambda}\bar{\mu}} \in \mathcal{S}(S^{*i}_{\lambda\mu})$ and let $S^{*i-1}_{\bar{\lambda}\bar{\mu}}$ be its projection We first construct an extension of a map $\mathcal{T}$ in the set of the form $\pi^{-1}_{\sigma^{i}}(T_{\sigma^{i}_{\bar{\lambda}\bar{\mu}}}(S^{*i-1}_{\bar{\lambda},\bar{\mu}}. \mathrm{Fron}(S^{*i}_{\bar{\lambda}\bar{\mu}}))$.

First we fix certain notations used here .

Take a point $Q^{*j-1}_{\tilde{\lambda}\tilde{\mu}}$ in $T_{\sigma^{*j-1}_{\tilde{\lambda}\tilde{\mu}}}(S^{*j-1}_{\tilde{\lambda}\tilde{\mu}}, \mathrm{Fron}(S^{*j-1}_{\tilde{\lambda}\tilde{\mu}}))$.
Then points $Q^{*j}_{\tilde{\lambda}\tilde{\mu},i}$'s lying over $Q^{*j-1}_{\tilde{\lambda}\tilde{\mu}}$ (in $\bigcup T_{\sigma^{j}_{\tilde{\lambda}\tilde{\mu}_i}}(S^{*j}_{\tilde{\lambda}\tilde{\mu}_i}, \mathrm{Fron}(S^{*j}_{\tilde{\lambda}\tilde{\mu}_i}))$)
satisfies an inequality of the following form

$$(2.2)\qquad |x_j(Q^{*j-1}_{\tilde{\lambda}_i\tilde{\mu}_i}) - x_j(Q^{*j-1}_{\tilde{\lambda}_i\tilde{\mu}_i'})| \geqq (\tilde{a}^{*j-1}_{\tilde{\lambda}\tilde{\mu}})\cdot d(Q^{*j-1}_{\tilde{\lambda}\tilde{\mu}}, \mathrm{Fron}(S^{*j-1}_{\tilde{\lambda}\tilde{\mu}}))$$

with suitable positive numbers $(\tilde{a}^{*j-1}_{\tilde{\lambda}\tilde{\mu}}) = (\tilde{a}^{*j-1}_{\tilde{\lambda}\tilde{\mu}1}, \tilde{a}^{*j-1}_{\tilde{\lambda}\tilde{\mu}2})$.

From (2.2) we obtain an inequality of the form

$$(2.2)\qquad |x_j(Q^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i,e}) - x_j(Q^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i,e})| \geqq \tilde{a}^{*j-1}_{\tilde{\lambda}\tilde{\mu},2}\cdot(\rho^{*j-1}_{\tilde{\lambda}\tilde{\mu},1})^{a^{*j-1}_{\tilde{\lambda}\tilde{\mu},2}}\cdot d(Q^{*j-1}_{\tilde{\lambda}\tilde{\mu}}, \mathrm{Fron}(S^{*j-1}_{\tilde{\lambda}\tilde{\mu}}))^{\beta^{j-1}_{\tilde{\lambda}\tilde{\mu},2}\cdot \tilde{a}^{*j-1}_{\tilde{\lambda}\tilde{\mu}2}}$$

For a couple $(a^{*j-1}_{\tilde{\lambda}_i\tilde{\mu}_i})$, define a point $\overset{\pm}{Q}{}^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i}(a)$ by

$$\pi_{j-1,j}(\overset{\pm}{Q}{}^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i}(a)) = Q^{*j-1}_{\tilde{\lambda}_i\tilde{\mu}_i}, \qquad x_j(Q^{\pm*j}_{\tilde{\lambda}_i\tilde{\mu}_i}(a)) = x_j(Q^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i}) \pm (a)\cdot h(Q^{*j-1}_{\tilde{\lambda}\tilde{\mu}}).$$

Here $Q^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i}$'s are all the points (lying over $Q^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i}$) in $\bigcup T_{\sigma^{j}_{\tilde{\lambda}_i\tilde{\mu}_i}}(S^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i})$ ordered by : $x_j(Q^{*}_{\tilde{\lambda}_i\tilde{\mu}_i}) > x_j(Q^{*j}_{\tilde{\lambda}_i\tilde{\mu}_i'})$.

Let $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ be a triple of positive numbers.

define functions by

$$(2.2\quad) \qquad \chi(S^{\lambda j}_{\bar{\lambda},\bar{\mu}}, (a), (\hat{a})) = \chi_0\left(\frac{\pm x_j \mp \xi_j(\theta, Q^{\lambda j}_{\bar{\lambda},\bar{\mu}})}{\hat{a}_1 \cdot d(\theta^{j-1}_{\bar{\lambda},\bar{\mu}}, \ldots)^{\hat{a}_2} \rho^{\hat{a}_3}}\right).$$

This function $\chi((a),(\hat{a}))$ is c^∞-differentiable in a set $(0,1] \times \pi^{-1}_{j,j} \circ T_{\sigma^{j-1}_{\bar{\lambda},\bar{\mu}}}(S^{\lambda j}_{\bar{\lambda},\bar{\mu}}, \mathrm{Fron}(S^{\lambda j}_{\bar{\lambda},\bar{\mu}}))$. Note that we can replace the set $T^j(S^{\lambda j}_{\bar{\lambda},\bar{\mu}}, \mathrm{Fron}(S))$ by $T_{\sigma^j_{\bar{\lambda},\bar{\mu}}}(S, (a))$ and vice versa ; and we can assume that the estimation (2.) is valid in a set $T(S, (a'))$, $(a') < (a)$. Also note that the functions χ are estimated in a manner

$$(2.2\quad) \qquad |D_{k,K}\, \chi(S^{\lambda j}_{\bar{\lambda},\bar{\mu}}, (a), (\hat{a}))| \leq \gamma_{k,K,1}\, d(\theta, \mathrm{Fron}(S))^{-\gamma_{k,K,2}} \rho^{-\gamma_{k,K,3}}.$$

Now we first extend a map to a set of the form $\{(x) : (x_1, \ldots, x_{j-1}) \in T(S^{\lambda j}_{\bar{\lambda},\bar{\mu}}), \xi_j(\theta(a)) \leq x_j \leq \xi_j(\theta_{t_0}(a))\}$. For this purpose define functions V_t, independent of x_j by

$$(2.2\quad) \qquad V_t(\rho, x_1, \ldots, x_{j-1}) = \frac{x_j(\theta^j(a)_\rho) - x_j(\theta^j_t(a)_\rho)}{x_j(\theta^j_{t+1}(a)) - x_j(\theta^j_t(a))} \quad (t < t_0).$$

Then V_t is estmated as

$$(2.2\quad)_1 \qquad |V_t(\rho, x_1, \ldots, x_{j-1})| \geq \gamma_{t,1}\, d(\theta^{j-1}, \mathrm{Fron}(S^{j-1}_{\bar{\lambda},\bar{\mu}}))^{\gamma_{t,2}} \rho^{\gamma_{t,3}}$$

$$(2.2\quad)_2 \qquad |D_{k,K} V_t(\rho, x_1, \ldots, x_{j-1}, x_j)| \leq \gamma'_{t,1}\, d(\theta^{j-1}, \mathrm{Fron}(S^{j-1}_{\bar{\lambda},\bar{\mu}}))^{-\gamma'_{t,2}} \rho^{-\gamma'_{t,3}}$$

with suitable (γ)'s ; $(2.2)_1$ follows in view of (2. 2.), (2. 2.) while $(2.2)_2$ is obvious.

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Now we show the existence of functions $\hat{x}^i_t(\theta, x)$ $t=1,\cdots,t_0$), defined in a set Σ_t so that the conditions (2.2) are valid . Moreover we assume that $\hat{x}^j_t(\theta,x)$ coincides with the previously given x^j in the set $\bigcup_{t'=1}^{t} \tilde{\Sigma}_{t'}$. This is shown inductively on t . Assume that our assertion was shown for $t-1$ ($\geqq 1$), using functions define $\tilde{v}^t_j$'s by

$$(2. \quad) \qquad \tilde{v}^t = \frac{d\tilde{x}}{dt^j} \times \chi_{t-1}(a,(a)) + (1-\chi_{t-1}(a,(a)))v_t$$

$$\text{if} \quad x_j \leqq x_j(Q_{t-1}(a)) + \frac{\rho(Q_{t-1}(a)) - \chi_j(Q_{t-1}(a))}{2}$$

$$= \frac{d\tilde{x}^j}{dx^j}(1-\chi_t(a,(a))) + \chi_t(a,(a)) \cdot v_t$$

we regard that this function is defined in $\pi^{-1}_{j-1,j}(\Phi^{j-1}_{j-1}(S^{t_{j-1}}_{\pi,\pi}))$ $([0,1]\times$

2−20の続き

Then quantitative properties of , which we make use are as follows .

(2.2) $\tilde{v}^t$ is C^∞- ~~diffe~~ function of x_j

(2. 2) $\tilde{v}^t$ satisfies ~~is~~ estimation ( ), ( ) with suitable (γ) in $\pi^{-1}_{j-1,j}(\Phi^{j-1}_{j-1}(S^{t_{j-1}}_{\pi,\pi}))$

(2. 2) $\tilde{v}^t$ satisfies ( )

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$\tilde{v}$ is estimated in a manner as ( ) with suitable (γ), (γ) 's. Define a function $\hat{x}_t^i(\rho, x)$ by

$$(2.2)_1 \quad \hat{x}_t^i(\rho, x) = x_{t-1}^i \quad (x^i \leq \theta_j(a) + (a'))$$

$$(2.2)_2 \quad \hat{x}_t^i(\rho, x) = \int_{x_j(\theta_j(a))}^{x_j} \tilde{v}_t \, dx_j$$

Then the difference $\hat{x}_t^i - x_t^i$ is independent of x_j and is estimated as $|k_t^i| = |\hat{x}_t^i - x_t^i| \leq \int_{\theta(a)}^{\theta(a)+(a')(k,\rho)} |\tilde{v}_t - \frac{dx_t^i}{dt}| \, dx_j + \int_{\theta(a)-(a')(k,\rho)}^{\theta^i(a)} |v_t - \frac{dx_t^i}{dt}| \, dx_j$

Then taking (a') sufficiently small one can assume , in view of (2.) an inequality of the following form

$$(2.2) \quad |k_t^i(\rho, x)| \leq a_1 \cdot d(Q, F_{\rho m}(S))^{a_2} \cdot \rho^{a_3}$$

with arbitrarily given (a)

Define a function x_t^i by $x_t^i = \hat{x}_t^i + (1-\chi)k_t^i$. This function sarisfies conditions (2.2) because of (.) :

To be sure we check required conditions it is quite clear that coimcides with x_t^i in a set and so satisfies (2. 2). On the otherhand it is also quite clear that the derivatives are estimated from above in a form. Note that the above checked facts are valid for any choice of (a'). What is remainded is to see that is estimated from below as in (2.2); $2 - 2 - 2 /$

Because k_t^i is independent of x_j, we have

$$\frac{d\tilde{x}_j}{dx_j} = \frac{d\hat{x}_t^j}{dx} + \left(\frac{d(1-\delta_{t+1,t})}{dx}\right) k_t^j$$

Note that $\frac{d\hat{x}_t^j}{dx_j} = \tilde{v}_t$ is estimated as in ( ), so that quantities $(\tilde{\gamma})$'s are chosen independently from $(\hat{a}')$. On the otherhand $\delta_{t+1,t}$ is obviously independent of (a'). Because $|k_t^i|$ can be assumed arbitrarily small ((12.2.)) by taking $(\hat{a}')$ sufficiently small, the estimation (from below) for x is assured.

ext extend a map to the set $L^{-1}_{H,i}(T^{i}(S^{\pm i}_{\lambda\mu}))$ U U

this will be done in an entirely same way as above.

For a point $Q^{\pm j}_{0}$ define $Q^{\pm j}_{0,e}$ by $x_j(Q^{\pm j}_{0,e}) = c_1 \cdot \hat{d}(Q^{\pm j}_{e})^{\hat{c}_2}$.

Because the given map satisfies ( ) -'( )-

inclusion property , we know that inequalities

$$x_j(Q^{+j}_{0,e}) > x_j(Q^{+j(a)}_{e,t}) > \cdots \quad x_j(Q^{+j}_{l,e}(a)) > x_j(Q^{-j}_{0,e})$$

and estimations

$$|x_j(Q^{+j}_{0,e}) - x_j(Q^{+j}_{l,t}(a))|, \quad |x_j(Q^{+j}_{l,t}(a)) - x_j(Q^{-j}_{0,e})| > c_1 d(Q^{\pm j}_{\lambda\mu})^{\hat{c}_2} \cdot \rho^{c_3}$$

is valid . with suitabe

Cleary x(Q) 's are C^{∞} differentiable outside

and are estimated in a manner ( )

with suitable (γ)' s .

Define $v^{\pm 0}$ by

$$(2.2). \quad v^{\pm}_{0} = \frac{\pm x_j(Q^{\pm j}_{0,e}) \mp x_j(Q^{\pm j}_{0,e})}{\pm x_j(Q^{\pm j}_{0,e}) \mp x_j(Q_{t})}$$

Moreover define $\tilde{v}^{\pm}_{0}$ by $\tilde{v}^{\pm}_{0} = \chi_0(Q,(a),(d)) \cdot \frac{d\hat{x}_j}{dx_j} + (1 - \chi_0(Q,(a),(d))) \cdot v^{+}_{0}$

and $\chi_0(Q,(a),(d)) \frac{d\hat{x}_j}{dx_j} + (1 - \chi_0(Q,(a),(d))) \cdot v^{-}_{0}$.

Then define a function $x^{\pm}_{0}$ by [$x^{+}_{0} = \int \tilde{v}^{+}_{0} dx_j : x_j \geq x^{j}_{t}(Q(a))$, $x^{+}_{0} = \hat{x}^{j}_{t}$, $x^{-}_{0} = \int \tilde{v}^{-}_{0} dx_j : x_j < x^{j}_{t}(Q(a))$, $x^{-}_{0} = \hat{x}^{-}_{t}$]. This function

coincides with x^{j}_{0} in S_t and $x_j(Q) - x_j( )$ [$R \pm 0$]

is estimated as $\int_{Q}^{Q+(a)\cdot(R\rho)} |v^{\pm}_{0}| dx_j$. As in ( ) we obtain a func

tion x so that the conditions (, 2-2

Now it is easy to derive an extension of in a set U (S) so that conditions are satisfied

This is done inductively on the length of S :

Namely we show the following facts inductively on

(2.2.)$_\ell$ For a set $\pi^{-1}(\bigcup \mathcal{T}_{\tilde{\lambda}\tilde{\mu}}(S^{*\sigma}_{\tilde{\lambda}\tilde{\mu}}))\cap U_{\sigma,c}$ ($\ell(S^{*\sigma}_{\tilde{\lambda}\tilde{\mu}}) \leq \ell$) there exists an τ extension of τ^{σ} in $\pi^{-1}_{\sigma}(\bigcup \mathcal{T}_{\tilde{\lambda}\tilde{\mu}})\cap U_{\sigma,c}$ so that ( ) and ( ) are valid .

(2.2.) $\quad x^{\pm}_{\sigma}(Q^{*\sigma}_{\sigma,\ell}) = c_1 \hat{d}(\hat{Q}^{*\sigma}_{\sigma,\ell})^{c_2}$

We show the above inductively on : If $\ell = 1$, the assertion i simple : Let S be a stratum of length ℓ . Choose a function χ by $\chi = \chi\left(\frac{g(Q:S)}{h(Q:S)}\right)$. Define a function $\hat{x}_i$ as in the step ( ) . To see the induction step it is sufficient to put $x = \chi\cdot\hat{x} + (1-\chi)\cdot\hat{x}^{\sigma-1}$

Now we summarize the above ~~computations~~ observations :

(2.2.) For an arbitrarily given adequate map τ^{σ}, there exists an extension τ ~~in~~ $U_{\bar{\sigma},\hat{c}}(S^{*\sigma}_{\tilde{\lambda},\tilde{\mu}})$ so that (i) τ is adequate and, furthermore,

(2.) $\quad x_{\sigma}(Q_{\sigma,\ell}) = c_1 \cdot x_{\ell}^{c_2}$

No2 we shall continue arguements in a neighbourhood of S .

M our arguements here are also quite elementary : We start with the following situation : Given a series of

maps so that these series satisfy quantitative conditions

( ) ,....., ( ), assume that a map $\tau^{*\sigma}$ in $\mathbf{U}_{\bar\sigma,c}(S^{*\sigma}_{\lambda,\mu})$

is already given : Namely we assume the existence of a map $\tau^{*\sigma}$, defined in $U_{\bar\sigma,c}(S^{*\sigma}_{\lambda,\mu})$, so that $\tau^{*\sigma}$ is adequate

and moreover $\tau^{'*\sigma}$ satisfies the condition

$$(\quad) \qquad \mathcal{X}_\sigma(Q^{*\sigma}_{\varrho}) = c_1 \mathcal{X}_\sigma(Q^{*\sigma}_{\varrho})^{c_2}$$

Now we want to extend this map τ to τ' in a set $T_{\sigma,c}(S^{*\sigma}_{\lambda,\mu}, \ldots S^{*\sigma}_{\lambda\mu})$ so that '

' main part ' of τ' will be ϱ^{η} outside $U_{\bar\sigma,c}$.

Here we assume the inequality : $(\tilde{c}) > (2c_1, c_2)$.

For a given $\eta (>c)$ we define a function $\mathcal{V}(\eta)$ in $T_{\bar\sigma,\hat\sigma}(S^{*\sigma}$, Fron $(S^{*\sigma}))$ by

$$\mathcal{V}(\eta) = \varrho^{\eta}_{x} \mathcal{X}(Q^{*}) + \mathcal{V}_t \mathcal{X} \quad {}^{*}$$

This function $\mathcal{V}(\eta)$ has similar properties as were considersd in the previous numero : (i) is C^∞ function outside $\mathcal{D}$ and whose derivatives are estimated as (2.2) with

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* $$\mathcal{X} = \mathcal{X}_0\left(\frac{x_j - \mathcal{X}_j(Q^*)}{c_1 \hat{\mathcal{X}}(Q^*)^{c_2}}\right) \qquad 2-2-25$$

a suitable (η) (ii) τ is estimated (from below) as (with a suitable η). Moreover we note that $\nu(\eta)$ is estimated as : $|\nu(\eta)| \leq K$ in a set ($P_{\bar\sigma} - P_{\sigma_2}$) ~~of the form~~ (with a suitable $\hat\sigma^1$).

To investigate quite simple properties of such a map we shall make the following observations. Consider inequalities

$$(2.2.\ )\qquad t \leq \hat\sigma_1\, d(Q, F_{con}(S))^{\hat\sigma_2}$$

$$(2.2.\ )\qquad t \leq c_1'\, d(Q, F_{con}(S))^{c_2'}.$$

Then a simple calculation leads to the following

(2.2.) an inequality ~~implies an inclusion~~ $\hat\sigma_1^{\circ,1} \geq \hat\sigma_1^{\circ}\cdot(c\cdot\hat\beta_1^{\circ})^{-\hat\sigma_2^{\circ}/\hat\beta_2^{\circ}}$, $\hat\sigma_2^{\circ}/\hat\beta_2^{\circ} \geq \hat\sigma_2^{\circ,1}$, $\eta \geq \bar\sigma_2^{\circ}(\hat\beta_3^{\circ}/\hat\beta_2^{\circ})$ implies the relation

$$t_\rho = e^{\eta} t \leq \hat\sigma^1 . d(Q_\rho, F_{con}(S))^{\hat\sigma_2}$$

(2.2.) Also ~~implies~~ $\hat c_1^{\circ,1} \geq c_1^{\circ}(c\cdot\bar\beta_1^{\circ})^{-c_2^{\circ}/\bar\beta_2^{\circ}}$, $c_2^{\circ}/\bar\beta_2^{\circ} \leq c_2^{\circ}$, $\eta \geq (\bar\beta_3^{\circ}/\bar\beta_2^{\circ})\cdot c_2$

~~implies~~ 1行アケル Moreover a simple computation shows the following fact.

(2.2.) an inequality $\bar\sigma_1^{\circ} \geq c_1^{\circ}\cdot(\bar\sigma_1^{\circ})(\bar\beta_1^{\circ})^{-c_2/\bar\beta_2^{\circ}} \times (c)^{\bar\sigma_2^{\circ,1}\cdot c_2}$, $\hat\sigma_2^{\circ} \leq \bar\sigma_2^{\circ,1}\cdot c_2^{\circ,1}$ implies an inclusion relation

$$U_{\bar\sigma^1, c} \subset P_{\bar\sigma^1, \sigma^1}$$

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Examine simple propreties of a map $\tau(n)$.

(2.2.2) From the definitinn of τ and from ( )

we know that an inequality

$$(\quad) \quad \hat{\sigma}^{j,1} \geqq c_1^j \cdot (\bar{\sigma}_1^j) \cdot (\varepsilon)^{\bar{\sigma}_2^j c_2^j} \cdot (\varepsilon\, \hat{\beta}_1^j)^{-\frac{\bar{\sigma}_2^j \cdot c_2^{j\prime}}{\hat{\beta}_2^j}} , \quad \hat{\sigma}_2^{j,\prime} = \frac{c_2^{j\prime}}{\hat{\beta}_2^j} \times \bar{\sigma}_2^{j}$$

implies the following inclusion relation of $\tau(n)$.

$$(\ 2.2.\ ) \quad \tau(n) : \ \mathcal{P}_{\bar{\sigma},\hat{\sigma}}(S) \hookrightarrow T_{\bar{\sigma}',\hat{\sigma}'} ,$$

$$T_{\bar{\sigma}',\hat{\sigma}}(S) \supset U_{\bar{\sigma},\varepsilon}$$

$$\text{with} \quad \bar{\sigma}_1' = (c_1^j \cdot (\bar{\sigma}_1^j) \cdot (\varepsilon)^{\bar{\sigma}_2^j c_2^j} , \quad \bar{\sigma}_2^{j\prime} \cdot c_2^j \quad)$$

Assume the inequality (2.2.) and so the inclusion

relation (2.2) . Then we obtain the following distance

relations to S .

$(2.2.\ )_1$ [illegible] For $R \in \mathcal{P}_{\bar{\sigma}\hat{\sigma}}(S)$,

$$\chi_j(R_\rho) \leqq \chi_j(R)\, \rho^{n} + \overset{\infty}{c_1}\, \chi_\rho^{\ c_2}$$

$(2.2\ )_{2,1}$ For $R \notin \mathcal{P}_{\bar{\sigma}\hat{\sigma}}(S) - U_{\bar{\sigma},\varepsilon}(S)$,

$$\chi_j(R_\rho) \geqq \frac{1}{2} \cdot g \cdot \rho^{n} ,$$

2-2-2') $(2.2)_{2,1}$ For $R \in U_{\varepsilon}(S)$, we use the following obvious relation

$$\chi_j(\rho) + \chi_j(R_\rho) \geqq \text{[illegible]}\ \chi_j(R_\rho)$$

From the aboves we know easily the following distance preserving properties

$$(2.2 .) \quad \bar{\beta}_1 (\bar{c}_1^{-1} c_1^{-1})^{\bar{\beta}_2/b_2} \times \hat{d}^{\,\bar{\beta}_2/b_2} \cdot \rho^{\bar{\beta}_3/b_3} \leq \hat{d}(\rho_\rho, S) \leq 2\tilde{c}_1 \bar{\bar{\beta}}_1^{\tilde{c}_2} \cdot \hat{d}(\rho, S)^{\bar{\beta}_2 c_2} \cdot \rho^{m\bar{\beta}_3 c_2, \eta}$$

From the above elementary observations we remark that

($\bar{\sigma}_1^{\,i} \{ (c\hat{\beta}_1^{\,i})^{-1} \}^{\bar{\sigma}_2^{\,i}/\hat{\beta}_2^{\,i}}, \bar{\sigma}_2^{\,i\prime}/\hat{\beta}_2^{\,i}$) defined by is small enough only if (there exist certain relations between , so far as we use . Consider following expressions ;

Then substituting ( ), ( ) we obtain recursive relations

$$(\quad) \quad \bar{\sigma}_1^{\,i} \{ c(\hat{\beta}_1^{\,i})^{-1} \}^{\bar{\sigma}_2^{\,i}/\hat{\beta}_2^{\,i}} \leqq c_1^{\,i} \cdot (\bar{\sigma}_1^{\,i})^{2} \{ c(\hat{\beta}_1^{\,i})^{-2} \}^{\bar{\sigma}_2^{\,i}/\hat{\beta}_2^{\,i}} \times (b_1^{-1/b_2})^{\bar{\sigma}_2^{\,i\prime} c_2^{\,i} b_2^{\,i}}, \quad \bar{\sigma}_2^{\,i\prime}/\hat{\beta}_2^{\,i} = c_2^{\,i\prime} \hat{\beta}_2^{\,i} \times \bar{\sigma}_2^{\,i\prime}/(\hat{\beta}_2^{\,i})^2$$

Being suggested from the above relation , we say that is satisfies (m_1, m_2, m_3) - (ε_1), (m_3) - (ε_2) condition if the following additional relations are valid .

$$(\quad) \quad \bar{\sigma}_1^{\,i} \{ m_1 (\hat{\beta}_1^{\,i})^{-m_2} \}^{\bar{\sigma}_2^{\,i}/(\hat{\beta}_2^{\,i})^{m_3}} < \varepsilon_2 \quad , \quad \bar{\sigma}_2^{\,i} \geqq (\hat{\beta}_2^{\,i})^{m_3} \cdot \varepsilon_2^{-1} .$$

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SO fax as (c1, c2) > (0)

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From (2.), (2.) we derive statements below easily

( ) For an arbirarily given $(\hat{\sigma}^{\delta_1})$, $(m_1^{\delta}, m_2^{\delta}, m_3^{\delta})$ choosing

$(m_1, m_2, \varepsilon_1)$, (m_3, ε_2) sufficiently small enables us to assure the existence of a map τ^{δ} so that the inclusion relation (2.) hilds with suitable $(\sigma\ )$.

Moreover one can assume that satisfies $(n_1 - n_2)$ - (ε)-

(m_3) — (ε_2) - condition

n. From the above observations our assertions in n.1 follow quite easily : Actually we show the lemma .2.2.1 in the following manner (: We consider additional conditions fo r series in question from technical reasons .

( ) Aseries of maps is said to satisfy

( ) - ( )- conditions if inequalities ( )- ( ) are valid for any stratum of $\mathcal{S}^{*}$ besides conditions given in n°. 1.

We show the above assertion inductively on .

Because , for i =1 , no new conditions imposed by means of

( )- ( ) conditions we assume that ( ) was shown for

( ) . We first show the ex in a neighbourhood of .

This part will be sh argued in the following way . We show

the following

( ) For each l and a any ( ) , ( )

there exists a map , defined in a set , so

that is adequate and (ii) satisfies ( )- ( )

condition . Moreover one can assume a further condition

( )

This fis easily checked to be true : If ℓ =1, it suffices

to define a map by : $x_{i+} = \rho^{n} \cdot x^{i}$, where (σ_1, σ_2) is small

enough , and n is large enough On the otherhand , for = , statements (2.2)

(2.2) suffices for our purpose .

In order to extend a map in a neighbourhood of

to a whole space , there is no difficulty in arguements

if we follow procedure in n. 2-2-3 (inductive step) . this finishes our q e d

§ 3.1 Cohomology with algebraic growth condition.

3.1 n.1. Let $\mathbb{C}^N$ be an N-dimensional complex euclidean space with coordinates $(x_1, \ldots, x_N)$ and let U be a bounded domain in $\mathbb{C}^n$. coordinates of $\mathbb{C}^n$ will be denoted by $(w_1, \ldots, w_n)$. The product $\mathbb{C}^N \times U$ will be denoted by $\Sigma_{n,N}$. In the sequel we simplify, without essential loss of generality, U to be a product of rectangulars. $U = \prod_{i=1}^{n} \square_{a_i b_i} \times \prod_{j=1}^{n} \square_{\tilde{a}_j b_j}$. We assume that $\max_{i,j}(\tilde{a}_i, \tilde{b}_i, \tilde{a}_j, \tilde{b}_j)$ is smaller than 1. we will be at first concerned with the structure sheaf $\mathcal{O}$ over $\mathbb{C}^N \times \mathbb{C}^n$. We note that when we speak of a neighbourhood of a point Q in $\Sigma_{n,N}$, we are concerned with a set in $\mathbb{C}^N \times \mathbb{C}^n$ rather than its restriction to $\overline{\Sigma}_{n,N}$. For a point $Q \in \overline{\Sigma}_{n,N}$ we mean by $\mathfrak{N}^{\delta}(\overline{\Sigma}_{n,N})$ the open set $\{Q' : |x_j(Q') - x_j(Q)| < \delta_1 (|x(Q)|+1)^{-\delta_2}$, $|w_j(Q) - w_j(Q')| < \delta_1^{-1}(|x|+1)^{-\delta_2} (a_i, b_j)\}$. For a fixed couple $(\delta) = (\delta_1, \delta_2)$ we mean by $\hat{\mathfrak{N}}^{\delta}(\overline{\Sigma}_{n,N})$ the open covrring of $\overline{\Sigma}_{n,N}$ defined to be $\{\hat{\mathfrak{N}}^{\delta}(Q) : Q \in \overline{\Sigma}_{n,N}\}$. This open covering is not locally finite. But it is found to be commodiate to formulate our problem in terms of such covering for our application.

A q- cochain will be said to be of

algebraic growth $(\alpha)=(\alpha_1, \alpha_2)$ if the following estimation is valid .

$$(A.G) \quad |\varphi_{Q_0\cdots Q_p}(z,w)| \leq \alpha_1(|z|+1)^{\alpha_2}.$$

Our first concern will be to discuss a type of vanishing theorems for such cochains :

Namely we show the following lemma, which we call a vanishing theorem with algebraic growth condition (We simplify to call v.a.) .

Lemma 3.1.1. (V. A)

There exists a datum $(\mathcal{L}, \mathbb{P}_1, \mathbb{P}_2, \mathcal{L},)$

N, with which the following facts are valid .

(T. A. I)$_1$ The datum $(\delta, C, \mathbb{P}_1, \mathbb{P}_2, L)$ is depending on ($\mathcal{N}$ n, q) only .

(T. A. I)$_2$ For a given cocycle φ^q in $C^q(N(\mathfrak{M}^q(\Xi_{n,N})))$ there exists a (q-1) - cochain so that the equation

$$\Upsilon^* \varphi(\cap_{v=1}^{q} \square^{\delta'}_{I_v J_v K_v}) = \delta_{\text{Cech}} (\varphi^{q-1})(\cap_{v=1}^{q} \square^{\delta'}_{I_v J_v K_v})$$

is valid with a suitable refinement map Υ^* so far as $\square^{\delta'}_{I_v J_v K_v} \subset \mathbb{C}^N \times D$ $(v=1,\cdots,q+1)$.

(T. A. I)$_3$ Quantities (∂_1' , ∂_2') are given by

$$\partial_1' = \min_v(\tilde{a}_v, \tilde{b}_v)^{-c} \times \mathbb{P}_1(\partial_1, \delta_1) e^{\mathbb{P}_2(\partial_1, \delta_2)}, \quad \partial_2' = L(\partial_2 + \delta_2)$$

Remark . In the above statement , the equality (T. A. I)$_2$ is understood so that the right hand side is understood to be a restriction of to by taking a suitable refinement . We note that we can assume (δ') is smaller than (δ). In the sequel we understand the meaning of coboundary relations as is understood here .

Before we enter into details of prrof, we shall fix certain notations. Let $\mathbb{C}^{\ell}$ be a complex Eucklidean space. We say that a geometric figure $\square$ is of elementary type if $\square$ is a product of rectangulars : $\square = \Pi_{j=1}^{\ell} \square_{a_j b_j}$. Unless we do not mention otherwise, we assume that $\max(a_j, b_j)$ is smaller than 1. Consider a figure $\Sigma_{n,N} = \partial^{\nu}\nabla$, where ∇ is a figure of elementary type. We shall introduce certain notations used here : Henceforth we assume that (δ_1, δ_2) is always a couple of positive integers in this numero : n. 1. By P_I, we mean the point in $\mathbb{C}^N$ with coordinates $(i_1^1, i_1^2, \cdots, i_N^1, i_N^2)$. On the otherhand we mean by P_{IJ}^{δ} the point in $\mathbb{C}$ with coordinate $(x(P_I) + \frac{J}{\delta_1 (|I|+1)^{\delta_2}}, \cdots$ $\cdots) 0 \leq J \leq \delta_1(|I|+1)^{\delta_2}$. Furthermore we mean by P_{IJK}^{δ} the point in $\mathbb{C}^N \times \mathbb{C}^n$ with coordinates $(x(P_{IJ}^{\delta})) \times \{(\frac{\tilde{a}_{\nu} \times K_1^{\nu}}{\delta_1(|I|+1)^{\delta_2}}, \frac{\tilde{b}_{\nu} \times K_2^{\nu}}{\delta_1(|I|+1)^{\delta_2}})\}_{\nu=1,\cdots n} : 0 \leq K \leq \delta_1(|I|+1)^{\delta_2}$. By $\square^{\delta}(P_{IJK}^{\delta})$ we mean the neighbourhood of P_{IJK}^{δ} defined to be $\{ Q : |x_{\nu_i}(Q) - x_{\nu_i}(P_{IJK}^{\delta})| < \frac{1}{\delta_1(|I|+1)^{\delta_2}} (\nu=1,\cdots,N),\ |w_{\nu}^1(Q) - w_{\nu}^1(P_{IJK}^{\delta})| < \frac{\tilde{a}_{\nu}}{\delta_1(|I|+1)^{\delta_2}},\ |w_{\nu}^2(Q) - w_{\nu}^2(P_{IJK}^{\delta})| < \frac{\tilde{b}_{\nu}}{\delta_1(|I|+1)^{\delta_2}} \}_{\nu=1,\cdots n}$ Moreover we mean by $\tilde{\mathfrak{M}}^{\delta}(\Sigma_{n,N})$ the open covering of defined to be $\tilde{\mathfrak{M}}^{\delta}(\Sigma_{n,N}) = \{ \square^{\delta}(P_{IJK}^{\delta}),\ I \in \mathbb{Z}^{2N},\ 0 \leq J, K \leq \delta_1(|I|+1)^{\delta_2} \}$

Here x_{ν_i}'s are real and imaginary parts of x.

Our formulation of lemma 3.1.1. is found to be suitable for our later discussions ((c.f.) § 4 . Our formulation of lemma 3.1.1. is found to be quite suitable to discuss theories of differentiable forms) . However , in our practical arguement , it is more suitable to work with $\tilde{\mathfrak{N}}^{\delta}$ than $\hat{\mathfrak{N}}^{\delta}$.

A cochain $\tilde{\varphi}^{q} \in N(\tilde{\mathfrak{N}}^{\delta}(Z_{n,m}))$ is , in a similar way to (A . G) of algebraic growth (∂_1 , ∂_2) if $\tilde{A}^{q}$ is estimated in the manner (A . G) . Then we show the following lemma 3.1.1.' which is completely similar to lemma 3.1.1.

Lemma 3.1.1.' There exists a datum $\{d, \ell, \mathbb{P}_1, \mathbb{P}_2, \mathcal{L}\}$ depending on (N, n, q) only so that the following ~~conditions~~ are valid .

(V.A.I)' For a given cocycle of algebraic growth (∂_1 , ∂_2) , there exists a (q- 1) - cochain $\varphi^{q-1} \in (\mathbb{C}^N \times \mathfrak{V})$ so that the equation

$$(\quad) \qquad r^{*} \varphi\Big(\bigcap_{\nu=1}^{q+1} \square^{\delta'}_{I_\nu J_\nu K_\nu} \mathfrak{V}\Big) = \delta_{cech} (\varphi^{q-1}) \Big(\bigcap_{\nu=1}^{q+1} \square^{\delta'}_{I_\nu J_\nu K_\nu \, n} \mathfrak{V} \Big)$$

as well as relations ( ) $(\partial_1' , \partial_2')_{3-1-5} = \left(\min(\hat{a}_\nu, \hat{b}_\nu)^{-\ell} \cdot \mathbb{P}_1(\partial_1, \delta_1)\, e^{\mathbb{P}_2(\partial_2, \delta_2)} , \mathcal{L}(\partial_2 + \delta_2) \right)$

it is easy to see that lemma 3.1.1. implies lemma 3.1.1 .
Before we enter into detailed arguements of the above lemma , we shall give an outline of our proof.

Outline of proof. As in the case of the standerd vanishing theorem on Stein variety, our discussion, where quantitative properties enter into , will be done in two steps : (I) One dimensional case where examina-tions of Cousin integrals and a method of an approxi-mation play central rolls. (I I) General cases where inductive divices from lower dimensional cases to higher dimensional cases play key rolls. In both cases quantitative arguements are important ,though arguements themselves are completely elementary. We shall discuss both cases in details for the sake of completeness and also for the purpose of learning such a basic fact as the vanishi- theorem on Stein variety through our problem.

(I) Let $\mathfrak{D}$ be a domain in $\mathbb{C}^n$. We do not assume the boundedness of $\mathfrak{D}$. By $\mathfrak{M}^{\delta,\varepsilon}$, we mean the open covering of $\mathfrak{D} \times \mathbb{C}^N (N=1)$ composed of all the open sets of the form $\mathfrak{D} \times \square^{\varepsilon}_{i_1 i_2}$:

$$\square^{\varepsilon}_{i_1 i_2} = \{ x : \quad |x_1 - x_1(i_1)|, |x_2 - x_2(i_2)| < \tfrac{1}{2} + \varepsilon \} : (i_1, i_2) \in \mathbb{Z} \times \mathbb{Z} .$$

Given a 1- cocycle $\varphi^1 \in C^1(N(\mathfrak{U}), \mathcal{O})$ estimated as

(3.1.1) $$|\varphi^1_{i_1,i_2,i_1',i_2'}(z,w)| \leq \alpha_1 \cdot \{|z|^{\alpha_2} + |w|^{\alpha_2} + 1\}.$$

We show the existence of absolute independent constants*

$c_0, c_1, \cdots c_6$, with which the following statements are valid.

(3.1.2) There exists a p- 1 cochain $\{\varphi^0_{i_1 i_2}\} \in C^0(N(\mathfrak{U}), \mathcal{O})$ so that the equation

(3.1.2)$_1$ $$\hat{\delta}_{\check{c}ech}(\varphi^0_{i_1 i_2}) = \varphi^1$$

and an estimation of the following forms

(3.1.2)$_2$ $$|\varphi^0_{i_1 i_2}(w,z)| \leq \varepsilon^{-c_1} \cdot c_2^{c_3\alpha_2 + c_4} \cdot \alpha_1 \{|z|^{c_5\alpha_2 + c_6} + |w|^{c_5\alpha_2 + c_6} + 1\}$$

are valid.

Our proof is of a completely elementary nature and will be given in two steps:(i) First we show the existence of constants $c_6', \cdots, c_6'$ with which the following assertions , which is weaker than (3.1.2) , are valid:

(3.1.2)$'$ There exist absolutely independent constants $c_0', \cdots, c_5'$ in such a way that the following $(3.1.2)'_{1,2}$ holds.

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* Here we mwan by ' absolutely independent ' the fact that datum is depending on the dimension of C =]

$(3.1.2)^{\sim\prime}$ There exists a o- cochain $\psi^{0}_{i_1 i_2} \in C^{0}(N(\pi^{\delta\varepsilon}), \mathcal{O})$ with which the relation

$(3.1.2)^{\sim\prime}_1$ $\varphi^1 - \delta_{\text{čech}}(\varphi^0_{i_1 i_2})$ is zero on the intersections $\square'_{12}$ of the form $\square'_{12} = \square_{i_1+1, i_2} \cap \square_{i_1 i_2}$.

as well as the estimations of the following forms

$(3.1.2)^{\sim\prime}_2$ $|\varphi^0_{i_1 i_2}| \leq \varepsilon^{-c'_6} \cdot c_1'^{\,c'_2 d_2 + c'_3} \cdot d'_1 \cdot (|z|^{c'_4 d_2 + c'_5} + |w|^{c'_4 d_2 + c'_5} + 1)$

are valid.

Let P_0 be the point with coordinates $(0, i_2, w)$

We let $\Psi^2_{i_1 i_2}$ be the <u>Cousin integrals</u> defined by

$(3.1.3)$ $\Psi^1_{i_1 i_2} = (1/2\pi i)\int_{\gamma^1_{i_1 i_2}} \varphi^1_{i_1 i_2}(\zeta - z)^{-1} d\zeta, \quad \Psi^2_{i_1 i_2} = (1/2\pi i)\int_{\gamma^2_{i_1 i_2}} \varphi^1_{i_1 i_2}(\zeta - z)^{-1} d\zeta$

, where $\gamma^j_{i_1 i_2}$ (j=1,2) are characterized by the reqoirements $\bigcup_{j=1}^{2} \gamma^j_{i_1 i_2} = \partial(\square^{\varepsilon}_{i_1 i_2})$, and $z \in \gamma^1_{i_1 i_2}$ or $z \in \gamma^2_{i_1 i_2}$ according to whether $Re(z) \leq i_1 + 1/2$ or $Re(z) \geq i_1 + 1/2$. For the value $i \geqq 0$, we expand P in the following form : $\Psi^1_{i_1 i_2} = \sum_{j=0}^{\infty} \Psi^1_{i_1 i_2, j}(w) z^j$ where

$$\Psi^1_{i_1 i_2, j}(w) = (-1)^j (1/2\pi i) \cdot \int_{\gamma^1_{i_1 i_2}} \varphi_{i_1 i_2} \zeta^{-(j+1)} \cdot d\zeta$$

is holomorphic in w .

Define a polynomial $\Psi_{i_1 i_2}(\mathbb{P})$ in z by

$$(3.1.3)_1 \qquad \Psi_{i_1 i_2}(\mathbb{P}) = \sum_{j=0}^{d_2+2} \Psi_{i_1 i_2, j}(w) \cdot (z - i_2)^j .$$

ALso define a holomorphic function $\Psi_{i_1 i_2}(\infty)$ = infinite part of $\Psi_{i_1 i_2}$

by

$$(3.1.3)_2 \qquad \Psi_{i_1 i_2}(\infty) = \Psi_{i_1 i_2} - \Psi_{i_1 i_2}(\mathbb{P}).$$

Note that the above function $\Psi_{i_1 i_2}(\infty)$ is considered in the set

$$\mathcal{D}' \times \{ z : \quad Re(z) \leqq i_2 + \tfrac{1}{2} + \varepsilon ,\ i_2 - \tfrac{1}{2} - \varepsilon < I_m(z) < i_2 + \tfrac{1}{2} + \varepsilon \}$$

1 Also note that the function $\Psi_{i_1 i_2}(\infty)$ coincides with the

following power series

$$(3.1.3)_2' \qquad \sum_{j=d_2+3}^{\infty} \Psi_{i_1 i_2, j}(w) \cdot (z - i_2)^j$$

in the set $\mathcal{D}' \times \{ z : \quad Re(z) < i_2 , \quad i_2 - \tfrac{1}{2} - \varepsilon < I_m(z) < i_2 + \tfrac{1}{2} + \varepsilon \}$.

From the explict expressions of $\Psi_{i_1 i_2, j}(w)$, $\Psi_{i_1 i_2}(\mathbb{P})$, $\Psi_{i_1 i_2}$ and $\Psi_{i_1 i_2}(\infty)$

we know easily that following estimations are valid for functions

$\Psi_{i_1 i_2, j}(w)$, $\Psi_{i_1 i_2}$, $\Psi_{i_1 i_2}(\mathbb{P})$ and $\Psi_{i_1 i_2}(\infty)$.

$$(3.1.4)_1 \qquad | \Psi_{i_1 i_2, j}(w) | \leqq 4 \cdot 4^{d_2+1} \cdot d_1 \cdot \{ |w|^{d_2} + |i_1|^{d_2} + |i_2|^{d_2} + 1 \} \cdot (i_1 + \tfrac{1}{2})^{-j-1},$$

$$(i_1 \geqq 0),$$

$$(3.1.4)_2 \qquad | \Psi^{I}_{i_1 i_2}(w, z) | \leqq 4 \cdot 4^{d_2+1} \cdot \varepsilon^{-1} \cdot d_1 \cdot \{ |w|^{d_2} + |i_1|^{d_2} + |i_2|^{d_2} + 1 \} ,$$

In the second inequality (3.1.4.) , we consider the function $\Psi^{1}_{i_1 i_2}(w,z)$ in the set where z satisfies the condition : $\mathrm{Re}\, z \leq i_1 + \frac{1}{2} + \varepsilon$, $|\mathrm{Im}\, z - i_2| \leqq \frac{1}{2} + \varepsilon$.

Define functions $\sigma^{0}_{i_1 i_2}$ $(i_1 \geqq 0)$ by the following equations

$$(3.1.5) \qquad \sigma^{0}_{i_1 i_2} = \sum_{0 \leq i_1' \leq i_1 - 1} \Psi^{2}_{i_1' i_2} - \sum_{0 \leq i_1' \leq i_1 - 1} \Psi^{1}_{i_1' i_2}(\mathrm{P}) + \sum_{i_1 \leq i_1'} \Psi^{1}_{i_1' i_2}(\infty) \quad (i_1 \geqq 0).$$

We note that these functions are considered in a set where z satisfies the condition: $\{ z : |\mathrm{Re}\, z - i_1|, |\mathrm{Im}\, z - i_2| < \frac{1}{2} + \frac{\varepsilon}{2} \}$.

Examine preeperties of these functions $\sigma^{0}_{i_1 i_2}$'s .

At first it is clear that a ' coboundary condition '

$$(3.1.6)_1 \qquad \sigma^{0}_{i_1+1, i_2} - \sigma^{0}_{i_1, i_2} = \sigma^{1}_{(i_1 i_2),(i_1+1, i_2)}$$

holds .

On the otherhand , we obtain the following estimations of $\sigma^{0}_{i_1 i_2}$ ' s in view of $(3.1.4)_{1,2}$.

$$(3.1.6)_2 \qquad |\sigma^{0}_{i_1 i_2}| \leqq \varepsilon^{-c_0'} C_1^{c_2' d_2 + c_3'} \cdot d_2 \times \{ |w|^{c_4' d_2 + c_5'} + |i_2|^{c_4' d_2 + c_5'} + |i_1|^{c_4' d_2 + c_5'} + 1 \}$$

with absolutely independent constants $(c_1', \cdots, c_5')$.

Repeat an entirely same procedure for $i_1 \leqq 0$. Then we obtain functions $\sigma^{0,-}_{i_1 i_2}$'s for $i_1 \leqq 0$, in such manner that the equations

$$(3.1.6)'_1 \quad \sigma^{0,-}_{i_1,i_2} - \sigma^{0,-}_{i_1-1,i_2} = \sigma_{i_1 i_2, i_1-1, i_2} \qquad (i_1 \leqq 0)$$

as well as the estimations of the forms

$$(3.1.6)'_2 \quad |\sigma^{0,-}_{i_1 i_2}| \leqq C_1^{\prime\prime c_3 d_2 + c_3''} \cdot (\varepsilon)^{-c_6''} d_1 \cdot \{ |w|^{c_4'' d_2 + c_5''} + i_1^{c_4'' d_2 + c_5''} + i_2^{c_4'' d_2 + c_5''} + 1 \}$$

are valid. (4)

We let $\sigma^{0}_{i_2}$ be the function defined to be $\sigma^{+}_{0,i_2} - \sigma^{-}_{0,i_2}$ in $\mathcal{D} \times \{ z : -\frac{1}{2} - \frac{\varepsilon}{2} \leqq \operatorname{Re} z \leqq \frac{1}{2} + \frac{\varepsilon}{2}, \ i_2 - \frac{1}{2} - \frac{\varepsilon}{2} \leqq \operatorname{Im} z \leqq i_2 + \frac{1}{2} + \frac{\varepsilon}{2} \}$. Define functions $\sigma^{+}_{i_1,i_2}$ $(i_1 \geqq 0)$ and $\sigma^{-}_{i_1,i_2}$ $(i_1 \leqq 0)$ in $\mathcal{D} \times \{ z : -\frac{\varepsilon}{4} < \operatorname{Re} z < \frac{1}{2} + \frac{\varepsilon}{4} + i_1, \}$ and in $\mathcal{D} \times \{ z : -\frac{1}{2} - \frac{\varepsilon}{4} - i_1 < \operatorname{Re} z < \frac{1}{2} + \frac{\varepsilon}{4} + i_1, \}$ respectively by Cousin integrals $\int_{\gamma_0^+} \sigma^{0}_{i_2} \cdot (\zeta - z)^{-1} d\zeta$ and $\int_{\gamma_0^-} \sigma^{0}_{i_2} \cdot (\zeta - z)^{-1} d\zeta$: $\gamma_0^{\pm} = \partial(\square^{\varepsilon/2}_{0,i_2}) \cap \{ \operatorname{Re} z \gtrless 0 \}$.

Define finally functions $\sigma_{i_1 i_2}$ by $\sigma_{i_1 i_2} = \sigma^{\pm}_{i_1 i_2} + \sigma^{\pm 0}_{i_1 i_2}$ $(i_1 = 0, \pm 1, \cdots)$ in $\mathcal{D} \times \{ z : -\frac{1}{2} - \frac{\varepsilon}{4} - i_1 \leqq \operatorname{Re} z \leqq \frac{1}{2} + \frac{\varepsilon}{4} + i_1, \ -\frac{1}{2} - \frac{\varepsilon}{4} - i_2 \leqq \operatorname{Im} z \leqq \frac{1}{2} + \frac{\varepsilon}{4} + i_2 \}$

Then, from (3.1.5), $(3.1.6)_1$, $(3.1.6)_2$, $(3.1.6)'_1$, $(3.1.6)'_2$, and from an estimation of $\sigma^{\pm}_{0,i_2}$'s obtained from the above, our assertions $(3.1.2)_1$, $(3.1.2)_2$ are assured.

3-1-10

(ii) Next let us consider the following situation. We let $\square_{i_2,\varepsilon'} = \{z : i_2 - (1/2) - \varepsilon' \leq \operatorname{Im} z \leq i_2 + (1/2) + \varepsilon'\}$. Let $\widetilde{\mathfrak{N}}_{\varepsilon}$ be the covering of $D'_n \times \mathbf{C}$ composed of sets $\{D'_n \times \square_{i_2,\varepsilon'} : i_2 \in Z.\}$ Take a cocycle $\tilde{\varphi}^1 \in C^1(N(\widetilde{\mathfrak{N}}_\varepsilon), \mathcal{O})$ which is estimated as follows..

$(3.37)_1$ $|\tilde{\varphi}^1| \leq \tilde{\alpha}_1'' \cdot (|z|^{\tilde{\beta}''} + |w|^{\tilde{\beta}''} + 1)$.

Now we show, in an analogous way to $(\mathrm{I})_1$ the existence of $\tilde{\sigma}^0 \in C^0(N(\widetilde{\mathfrak{N}}_{\varepsilon}), \mathcal{O})$ satisfying the following conditions .

$(3.38)_1$ $\hat{\delta}\tilde{\sigma}^{0} = \tilde{\varphi}^1$ and $\varphi^{1,0}$ is defined in $\square_{i_2, \varepsilon'/2}$.

$(3.38)_2$ $|\tilde{\varphi}^{0}| \leq e_1^{-c_0''} c_1'' \, c_2'' \tilde{\alpha}_1'' \cdot (|z| + |w| + 1)^{c_3'' c_4''}$ with absolutely independent constants c_0'', c_1'', c_2'', c_3'', c_4'' and c_5''.

We verify the above assertions $(3.38)_{1,2}$ as follows: We let S_{pi_2} be the geometric figure defined by $S_{pi_2} = \{z : -1/2 - p - \varepsilon' \leq \operatorname{Re} z \leq 1/2 + p + \varepsilon', -\varepsilon' - 1/2 + i_2 \leq \operatorname{Im} z \leq \varepsilon' + (1/2) + i_2\}$, and let S'_{pi_2} be $D'_n \times S'_{pi_2}$. We define the figures $\Gamma^1_{pi_2}$ and $\Gamma^2_{pi_2}$ by the requirements

(i) $\bigcup_{i=1}^{2} \Gamma^{i}_{pi_2} = \partial(S'_{pi_2} \cap S'_{pi_2+1})$ and (ii) $\operatorname{Im} z \geq i_2 + (1/2)$ or $\operatorname{Im} z_2 \leq i_2 + (1/2)$ according to $z \in \Gamma^1_{pi_2}$ or $z \in \Gamma^2_{pi_2}$. We expand the Cousin integral $\Psi^1_{pi_2} = \int_{\Gamma^1_{pi_2}} \varphi_{i_2} \cdot (\zeta - z)^{-1} . d\zeta$ at the origin 0 in the form

$\Psi'_{p i_2} = \sum_{j=0}^{\infty} \Psi_{p i_2, j}$. In this case the following estimation for $\Psi'_{p i_2, j}$ is valid.

$$(3.1.9)_1 \quad |\Psi_{p i_2, j}| \leqq 4^{d_2} d_1 \cdot (p^{d_2} + |i_2|^{d_2} + |w|^{d_2}) \cdot (2p+2) \cdot (i_2 + 1/2)^{-j-1} \quad (i_2 \geqq 0)$$

It is clear that Cousin integral $\Psi'_{p i_2}$ itself is estimated in the manner

$$(3.1.9)_2 \quad |\Psi^{\tau}_{p i_2}| \leqq 4^{d_2} (\varepsilon')^{-1} \times d_1 \cdot (p^{d_2} + |i_2|^{d_2} + |w|^{d_2} + 1)(2p+2) \quad (i_2 \geqq 0)$$

$(\tau = 1, 2)$ in the set $\mathfrak{D} \times \{ z : Im(z) < i_2 + 1/2 + \varepsilon/2 , \ -p - 1/2 - \varepsilon/2 < Re(z) < p + 1/2 + \varepsilon/2 \}$

In an analogous way to ( ), define functions $\Psi^{+}_{p,s}(\mathbb{P})$ and $\Psi^{+}_{p,s}(\infty)$ by

$$(3.1.9)_3 \quad \Psi^{+}_{p,s}(\mathbb{P}) = \sum_{j=0}^{d_2+2} \Psi^{+}_{p,s,j} \cdot z^j , \quad \Psi^{+}_{p,s}(\infty) = \Psi_{p,s}(\infty) - \Psi^{+}_{p,s}(\mathbb{P})$$

Using the estimation $(3.1.9)_{1,2}$ the above functions $\Psi^{+}_{p,s}(\mathbb{P})$ and $\Psi^{+}_{p,s}(\infty)$ are estimated respectively in the following ways.

$$(3.1.10)_1 \quad |\Psi^{+}_{p,s}(\mathbb{P})| \leq 4^{d_2+2} d_1 \cdot (p^{d_2} + |i_2|^{d_2} + |w|^{d_2} + 1)(2p+2)\left(\sum_{j=0}^{d_2+2} |z|^j\right)$$

$$(3.1.10)_2 \quad |\Psi^{+}_{p,s}(\infty)| \leq 4^{d_2+2} d_1 \cdot (p^{d_2} + |i_2|^{d_2} + |w|^{d_2} + 1)(2p+2)|z|^{d_2+3} \cdot (i_2 + 1/2)^{-d_2-2}$$

in $\mathfrak{D} \times \{ z : -p - 1/2 - \varepsilon/2 \leq Re\, z \leq p + 1/2 + \varepsilon/2 , \quad Im\, z \leq i_2 \}$.

Next define a function σ_{p,i_2} in the set $\mathcal{D}\times\{z: |\mathrm{Re}\, z|<p+1/2+\varepsilon/2,$ ~~Im z~~ $<$ $\mathrm{Im}\, z < i_2+1/2+\varepsilon/2\}$ by

$$(3.1.11)\qquad \sigma^{0,+}_{p,i_2} = \sum_{i_2>t_2\geq 0} \Psi^{2}_{p,t_2} + \sum_{i_2>t_2\geq 0} \Psi^{+}_{p,t_2}(\mathbb{P}) - \sum_{t_2\geq i_2} \Psi^{+}_{p,t_2}(\infty)$$

Then, from the definition of $\sigma_{p i_2}$ itself, the relation

$$(3.1.11)_1\qquad \sigma^{0,+}_{p\,i_2+1} - \sigma^{0,+}_{p\,i_2} = \sigma_{i_2}$$

is valid. in $\mathcal{D}\times\{z: |\mathrm{Re}\, z|<p+1/2+\varepsilon/2,\ i_1+1/2-\varepsilon/2<\mathrm{Im}\, z < i_1+1/2+\varepsilon/2\}$.

On the other hand, from estimations ( ), ( ), and ( )

we obtain estimations of the functions σ_{p,i_2}'s in

the following ways.

$$(3.1.11)_2\qquad |\sigma^{0,+}_{p,i_2}| \leq 4^{5d_2+2}\cdot(\varepsilon)^{-1}\cdot d_1\cdot\{p^{3d_2+6}+i_2^{3d_2+6}+|w|^{3d_2+6}+1\}\cdot$$

$$\cdot(2p+2),$$

in $\mathcal{D}\times\{z: |\mathrm{Re}\, z|<p+1/2+\varepsilon/2,\ i_1+1/2-\varepsilon/2<\mathrm{Im}\, z<i_1+1/2+\varepsilon/2\}$

$(i = 0, 1, \cdots\cdots)$.

Repeat an entirely same arguements as above for $i \leq 0$.

Also repeat an entirely same arguement as in the part (I),

(c.f. the end of I) . Then we obtain functions $\sigma^0_{p,i}$ ($i = 0, \pm 1, \ldots$) in $\mathcal{D} \times \{z : |\mathrm{Re}\, z - p|, |\mathrm{Im}\, z - i_2'| < \frac{1}{2} + \frac{\xi}{4}\}$ so that the equation

$$(3.1.12)_1 \qquad \sigma^0_{p,i+1} - \sigma^0_{p,i} = \sigma^1_{i,i+1} \qquad (i = 0, \pm 1, \cdots)$$

as well as an estimation

$$(3.1.2)_2 \qquad |\sigma^0_{p,i}| \leqq 4^{\tilde{c}_7 d_2 + \tilde{c}_3} \times (\varepsilon')^{-2} \times d_1 \times \{p^{\tilde{c}_3 d_2 + \tilde{c}_4} + |w_2|^{\tilde{c}_3 d_2 + c_4} + |i_2|^{\tilde{c}_3 d_2 + c_4} + 1\}$$

are valid ,

let $p' > p$ be two positive integers . Then functions $\sigma_{p'p} = \sigma_{p' i_2} - \sigma_{p i_2}$'s are independent of i and defined a global function in a neighbourhood of the set $\mathcal{D} \times \{z : |\mathrm{Re}\, z| < p + \frac{1}{2}\}$. Especially we put $\sigma_p = \sigma_{p+1,p}$. The following estimation is valid .

$$(3.1.3) \qquad |\sigma_p(w,z)| \leqq \varepsilon^{-\tilde{c}_0'} c_1'^{\tilde{c}_2 d_2 + \tilde{c}_3} d_1 \{p^{\tilde{c}_3' d_2 + \tilde{c}_4'} + |i_2|^{\tilde{c}_3' d_2 + \tilde{c}_4'} + |w|^{\tilde{c}_3 d_2 + \tilde{c}_4'} + 1\}$$

so far as the inequality $|\mathrm{Im}\, z| \leqq i_2'$ is valid. Here constants ($\tilde{c}_0', \cdots \tilde{c}_4'$) are also absolutely independent.

Now we adjust functions $\sigma'^{0}_{i_2,p}$'s to obtain desired functions $\sigma^{0}_{i_2}$'s . For this purpose we expand functions σ_p's at the origin 0 in the following form.

$$(3.1.13)_1 \qquad \sigma_p = \sum_{j=0}^{\infty} \sigma_{p,j}(w)\cdot z^j$$

Then the coefficients $\sigma_{p,j}(w)$'s are estimated in the manner

$$(3.1.13)_2 \quad |\sigma_{p,j}(w)| \leqq \varepsilon^{-c_1''}\cdot c_1''^{\,c_2''d_1+c_3''}\cdot d_1\cdot\{ p^{c_4''d_2+c_5''} + |w|^{c_4''d_2+c_5''} + 1\}\cdot(p+\tfrac{1}{2})^{-j-1}$$

$c_1'', \cdots, c_5''$.

As before we define functions $\sigma_p(\mathbb{P})$ and $\sigma_p(\infty)$ by

$$(3.1.14) \quad \sigma_p(\mathbb{P}) = \sum_{j=0}^{c_4''d_2+c_5''+2} \sigma_{p,j}(w)\cdot z^j \,, \qquad \sigma_p(\infty) = \sigma_p - \sigma_p(\mathbb{P}).$$

We consider the second function $\sigma_p(\infty)$ in the set $\mathcal{D}\times\{z : 2|\mathrm{Re}\,z|, 2|\mathrm{I_m}\,z| < \sqrt{p}\}$.

Finally we define functions σ_{i_2} $(i_2=0,\pm1,\cdots)$ in the set $\mathcal{D}\times\{z : i_2-\tfrac{1}{2}-\tfrac{\varepsilon}{4} \leqq \mathrm{I_m}\,z \leqq i_2+\tfrac{1}{2}+\tfrac{\varepsilon}{4}\}$ by

$$(3.1.15) \qquad \sigma^{0}_{i_2} = \sigma^{0}_{i_2,p} + \sum_{s\geqq p} \sigma_s(\infty) - \sum_{p>s\geqq 0} \sigma_s(\mathbb{P}), \quad \text{for the points}$$

in the set $\mathcal{D}\times\{z : 2|\mathrm{Re}\,z|<\sqrt{p},\ |\mathrm{I_m}\,z - i_2| \leqq \tfrac{1}{2}+\tfrac{\varepsilon}{4}\}$ $\quad (\sqrt{p}/2 \geqq |i_2|+1)$.

It is found easily that these functions σ_{i_2}'s are defined in the set $\mathcal{D}\times\{z : |\mathrm{I_m}\,z-i_2| \leqq \tfrac{1}{2}+\tfrac{\varepsilon}{4}\}$ globally. (i.e. independent of p)

On the otherhand it is clear that the equation

$$(3.1.15)_1 \qquad \sigma^{0}_{i_2+1} - \sigma^{0}_{i_2} = \sigma^{1}_{i_2+1, i_2} \quad (i_2 = 0, \pm 1, \cdots)$$

is valid .

Estimations of these funstions are easily obtained .

To be sure we give estimations of these functions

in the following form :

$$(3.1.15)_{2,1} \qquad |\sigma^{0}_{p}(P)| \leqq \varepsilon^{-\tilde{c}''_0} \times c''_1{}^{\tilde{c}''_2 d_2 + \tilde{c}''_3} \cdot d_1 \cdot \{ p^{\tilde{c}''_4 d_2 + \tilde{c}''_5} + |w|^{\tilde{c}''_4 d_2 + \tilde{c}''_5} + 1 \} \times \{ \sum_{j=0}^{\tilde{c}''_4 d_2 + \tilde{c}''_5 + 2} |t^{j}| \},$$

$$(3.1.15)_{2,2} \qquad |\sigma^{0}_{p}(\infty)| \leqq \varepsilon^{-\tilde{c}''_0} \cdot c''_1{}^{\tilde{c}''_2 d_2 + \tilde{c}''_3} \cdot d_1 \{ p^{\tilde{c}''_4 d_2 + \tilde{c}''_5} + |w|^{\tilde{c}''_4 d_2 + \tilde{c}''_5} + 1 \} \times |z|^{\tilde{c}''_4 d_2 + \tilde{c}''_5 + 3 - \tilde{c}''_4 d''_5} \cdot (p + 1/2)$$

$$(p \geqq 1, \quad Re|z|, \ Im|z| < \sqrt{p}/2).$$

$$(3.1.15)_{2,3} \qquad |\sigma^{0}_{i_2}| \leqq |\sigma^{0}_{k_2, p}| + \varepsilon^{-\tilde{c}'''_0} \times \tilde{c}'''_1{}^{\tilde{c}'''_2 d_2 + \tilde{c}'''_3} \cdot \{ p^{\tilde{c}'''_4 d_2 + \tilde{c}'''_5} + |w|^{\tilde{c}'''_4 d_2 + \tilde{c}'''_5} + 1 \} \times \{ \sum_{j=0}^{\tilde{c}'''_4 d_2 + \tilde{c}'''_5 + 3} |z|^{j} \}$$

$$(p \geqq 1, \ |Re z|, \ |Im z| < \sqrt{p}/2),$$

In the aboves $\tilde{c}''_0, \cdots, \tilde{c}'''_5$ are absolutely independent

From the above estimations we know easily the desired

estimation $(3.1.18)_{1,2}$.

Now it is obvious that (3.1.) and $(3.1.18)_{1,2}$

suffice for our purpose .

(II) Now we shall discuss simple quantit

, which is also used in the later .

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Discussions here are also quite elementary. We shall enter into details for completeness. Our arguements will be done for two types of geometric figures. Differences of two situations considered here are quite small. However, to discuss both cases in one time causes technical confusions (to the author). (A) Our first situation is explained in the following manner : Let $\mathbb{C}^{\ell_1+\ell_2}$ be a complex Euclidean space with coordinates ($x_1, \ldots, x_{\ell_1}, x_{\ell_1+1}, \ldots, x_{\ell_1+\ell_2}$). Assume that a geometric figure $\square = \prod_{v=1}^{\ell_1+\ell_2} \square_{a_v, b_v}$ of elementary type in $C^{\ell_1+\ell_2}$ is given . For a given series of positive integers $\{m_v^i, n_v^i\}_{v=1,\ldots,\ell_1+\ell_2,\ i=1,2}$, we write indices $\{s_v^i, t_v^i\}_{v=1,\ldots,\ell_1+\ell_2}^{0\leq s_v \leq n_v, 0 \leq t_v \leq n_v}$ as (S , T). Define a neighbourhood $\square(P_{ST})$ of P_{ST}

$$P_{ST} = \{(x) : x_{v_1}(P_{ST}) = a_{v1} + s_v/m_i,\ x_{v_2}(P_{ST}) = b_{v1} + t_v/n_i\}_{v=1,\ldots,\ell_1+\ell_2} \text{ by}$$

$$\square(P_{ST}) = \{(x) = (x_1, \ldots, x_{\ell_1+\ell_2}) : |\mathrm{Re}\, z_\mu - \mathrm{Re}\, z_\mu(P_{ST})| \leq (3/4) \cdot a_v/m_v,$$

$$|\mathrm{Im}\, z_v - \mathrm{Im}\, z_v(P_{ST})| \leq (3/4) \cdot b_v/n_v\ \ (v = 1, \ldots, \ell_1+\ell_2)\}.$$

Furthermore, assume that an another figure $\square' \subset \mathbb{C}^{\ell_1+\ell_2}$ of elementary type is given. By $\mathfrak{N}(\square, \square')$, we mean the subset of $\{\square(P_{ST})\}$ composed of all the figures satisfying the following condition.

$$(3.1.16) \qquad \square(P_{ST}) \cap \square' \neq \phi$$

Obviously the open set $|\mathfrak{N}(\square, \square')| = \bigcup \square(P_{ST}) : \square(P_{ST}) < \mathfrak{N}(\square, \square')$ is also a geometric figure of elementary type. We assume that a domain $\mathfrak{D}$ in $C^{\ell_3}_{z\{w\}}$ is given. By $\tilde{\square}(P_{ST})$, ~~$\tilde{\mathfrak{N}}(\square, \square')$~~ and $\tilde{\mathfrak{N}}(\square, \square', \mathfrak{D})$

By $(\square(P_T))$, we mean the projection $P_z \square(P_{ST})$.

, we mean $\square(P_{ST}) \times \mathfrak{D}$ and $\{\square(P_{ST}) \times \mathfrak{D} : \square(P_{ST}) \in \mathfrak{N}(\square, \square')\}$ respectively.

Take a set $\tilde{\mathfrak{K}} \{\square(P_T)\}_{\mu=1,\cdots,s}$ of figures in $\mathbb{C}^{\ell_2} \times \mathbb{C}^{\ell_3}$ * composed of s elements. Henceforth such a set $\{\square(P_{T_\mu})\}_{\mu=1,\cdots,s}$ will be denoted by $\mathfrak{K}^{\ell_2}_s$. We mean by $\tilde{\mathfrak{N}}(\square, \square', \mathfrak{D}, \mathfrak{K}^{\ell_2}_s)$ the set of all the figures $\tilde{\square}(P_{ST}) \in \tilde{\mathfrak{N}}(\square, \square', \mathfrak{D})$ satisfying the following two conditions.

$$(3.1.17)\ P_z' \tilde{\square}(P_{ST}) \in \mathfrak{K}^{\ell_2}_s : \quad P_z'(\tilde{\square}(P_{ST})) = pr_{z,w}(\tilde{\square}(P_{ST}))$$

It is clear that the set $\tilde{\mathfrak{N}}(\square, \square', \mathfrak{D}, \mathfrak{K}^{\ell_2}_s)$ is an open covering of $|\tilde{\mathfrak{N}}(\square, \square', \mathfrak{D}, \mathfrak{K}^{\ell_2}_s)|$.

Our first assertion, which is quite elementary, is spoken for such a set. A q – cochain $\varphi^q \in C^q(N(\tilde{\mathfrak{N}}(\square, \square', \mathfrak{D}, \mathfrak{K}^{\ell_2}_s)), \mathcal{O})$ is said to be <u>defined already in</u> $\overset{M}{\tilde{\mathfrak{N}}}(\square, \square', \mathfrak{D}, \mathfrak{K}^{\ell_2}_s)$, if, for each set $\{\square_{S_\nu T_\nu}\}_{\nu=0,\cdots,s} : \bigcap_{\nu=0}^{s} \square_{S_\nu T_\nu} \neq \phi$, $\varphi(\bigcap_{\nu=0}^{s} \square_{S_\nu T_\nu} \times \mathfrak{D})$ is already defined in $\bigcap_{\nu=0}^{s} \overset{M}{\square}_{S_\nu T_\nu} \times \mathfrak{D}$. (Note that the above does not mean that φ^q is a restriction of a cochain $\tilde{\varphi}^q$ in $C^q(N(\tilde{\mathfrak{N}}(\square, \square', \mathfrak{D}, \mathfrak{K}^{\ell_2}_s)))$. A q – cochain φ^q is of cotype s', if $\varphi^q(\bigcap_{\nu=0}^{q} \square_{S_\nu T_\nu}) = 0$, unless $\{pr_{z,w}(\square_{S_\nu T_\nu})\}$ is composed of (at least) s' elements. Let φ^q be

* ~~$\square(P_T) = P_z(\square_{ST})$~~ : $P_z : (z_1, \cdots, z_{\ell_1}, z_{\ell_1+1}, \cdots, z_{\ell_1+\ell_2}) = (z_{\ell_1+1}, \cdots, z_{\ell_1+\ell_2})$.

a q – cochain of cotype s . In this case, a further definition will be made use of . For this purpose fix an order $\{\square_{T_\nu}\} \succ \square_{T_{\nu(\mu)}}$ of $\{\square_{T_\nu}\}_{\nu=1,\cdots,s}$ once and for all. With this fixed order, a q – cochain φ^q of cotype s is further said to be of cotype (s, k) if the following further conditions are valid,

$$(3.1.17)_2 \quad \varphi^q(\cap_{\nu=0}^{q} \square_{S_\nu T_\nu}, \mathfrak{D}) = 0, \quad \text{unless the}$$

set $\{\square_{S_\nu T_\nu}\}_{\nu=0,\cdots,q}$ has the property : The cardinal number of the $\{pr\square_{S_\nu T_\nu}\}_{\nu=0,\cdots,q}$ is (at least) k .

Using the above notations , what we make use of will be stated in the following fashion.

$(A')^{l_1,l_2}_{s,k}$ Assume that $2 \leq s \leq q$ and that a q – cocycle $\varphi^q \in C^q(N(\mathfrak{N}(s,\epsilon,\delta)), \mathfrak{F})$ is given. Furthermore, we assume that $\|\varphi^q\| = \sup_{(z',z'',w)} |\varphi^q(z',z'',w)|$ is finite. If φ^q is of cotype (s,k) , then ther exists a (q-1) – cochain $\varphi^{q-1} \in C^{q-1}(N(\mathfrak{N}(s,\epsilon,\delta)), \mathfrak{F})$ in such a manner that the following conditions are true.

$(A')^{l_1,l_2}_{s,k,1}$ φ^{q-1} is defined in $\mathfrak{N}^{M'}(\square,\square',\delta',\mathfrak{K}_s)$; $M' = e\cdot 21$, and moreover,

$$\|\varphi^{q-1}\| \leq c_1 \cdot M'^{l_2} \{\prod_{\nu=1}^{l+l_2} \max(m_\nu, \tau_\nu)\}^{c_3} \cdot \prod_{\nu=1}^{l+l_2} \min_\nu(\tilde{z}_\nu, \tilde{b}_\nu)\}^{-l_4} \cdot \|\varphi^p\| .$$

$(A')^{l_1,l_2}_{s,k,2}$ $-\delta_{\check{C}ech}(\varphi^{q-1}) + \varphi^q$ is of cotype (s, k +1) if $k < p+2-s$

is zero if k = q +2 – s,

$(A')^{l_1,l_2}_{s,k}$, φ^{q-1} is of cotype (s,k) if $k \neq q+2-s$,
3-1-19' (s, k-1) if $k = q+2-s$.

In the above, quantities $e = e^{\ell_1,\ell_2}_{s,k}$, $c_\mu = e^{\ell_1,\ell_2}_{s,k,\mu}$ $(\mu=1,2,3)$ are depending on $(\ell_1, \ell_2, q, s, k)$ only.

(B) Our secind assertion, which is easily deduced from $(A)^{\ell_1\ell_2}_{(s)}$, $)^{\ell_1\ell_2}_{s,2}$, is stated in the following manner.

$(A')^{\ell_1,\ell_2}_{s}$ Assume that $1 \leq s \leq q$. Also assume that a q-cocycle φ^{ℓ} of cotype s is given; φ^{ℓ} is already defined in $\tilde{\mathfrak{N}}^{M}(\sqsubset, \sqsubset', \mathfrak{D}, \mathfrak{f}^{\ell}_{s})$. Then we find a (q-1)-cochain $\varphi^{\ell-1}$ of cotype s so that the equation

$$(A')^{\ell_1,\ell_2}_{s,1} \quad -\delta_{\check{c}ech}(\varphi^{\ell-1}) + \varphi^{\ell} = 0; \qquad \varphi^{\ell-1} \text{ is of cotype } s$$

as well as the estimation,

$$(A')^{\ell_1\ell_2}_{s,2} \quad \|\varphi^{\ell-1}\| \leq c'_1 \cdot M^{c'_2} \cdot \{\Pi^{\ell_1+\ell_2}_{\nu=1} \max(n_\nu, n_\nu)\}^{c'_3} \times \{\Pi^{\ell_1+\ell_2}_{\nu=1} \min(\tilde{c}_\nu, \tilde{b}_\nu)\}^{-c'_4} \|\varphi^{P}\|$$

hold, so far as $M' \geq e' M > 1$.

In the above quantities $(c'_1, c'_2, c'_3, c'_4, e')$ are depending on $(\ell_1, \ell_2, \ell, s)$ only.

For the sake of completeness, we shall formulate one another problem. Let $\ell_2 = 0$, and let $\ell = \ell_1$. Then we know the following.

$(A'')^{\ell}$ There exists a set of positive numbers $\{e'', c''_\mu\}_{\mu=1,2,3,4}$ determined by (ℓ, q), with which the statement below is valid.

$(A'')^{\ell}$ For a pair $\{\sqsubset, \sqsubset'\}$ and a q-cocycle $\varphi^{\ell} \in Z^{q}(N(\mathfrak{N}(\sqsubset, \sqsubset')), \mathcal{O})$, defined in in $\mathfrak{N}^{M}(\Box, \Box')$, there exists a (q - 1) - cochain φ^{q-1} defined in $\mathfrak{N}^{M'}(\Box,\Box')$ which satisfy the equation

$$(A'')_1^{\ell} \qquad - \delta_{\check{c}ech}(\varphi^{q-1}) + \varphi^{q} = 0$$

as well as the estimation

$$((A'')_2^{\ell} \quad \|\varphi^{q-1}\| \leq c_1'' \cdot M^{c_2''} \prod_{\nu=1}^{\ell_1+\ell_2} \{max(\tilde{a}_\nu, \tilde{b}_\nu)\}^{c_3''} \cdot \{\prod_{\nu=1}^{\ell_1+\ell_2} min(\tilde{a}_\nu, \tilde{b}_\nu)\}^{-c_4''} \cdot \|\varphi^{p}\|$$

are valid, where $M' = e'' M$ is assumed to be : $M'' \geq 1$.

(Remark) In the above statement, natures of the figure $\Box'$ does not appear. We are formulating our prob problem including the figure $\Box'$,for purposes of applications.

(B) Our second concern is figures in $\Sigma_{n,N}$. our assertions here are pararell to statements $(A^{*})^{\ell_1,\ell_2}_{\Delta,\xi,1,2}$ and $(A)^{\ell_1,\ell_2}_{\Delta,1,2}$. Though arguements are pararell to (A), a care is necessary for quantitative arguements .(A care , we take of , is quite simple. (C. f .)).

In this situation we assume the estimation of φ^{q} of the following form ;

$$\|\varphi^{q}(z,w)\| \leq d_1 \cdot \{ |z|^{d_2} + |w|^{d_2} + 1 \}$$

Then, our assertions corresponding to $(A)^{\ell_1,\ell_2}_{\Delta,\xi}$ and $(A)^{\ell_1,\ell_2}_{\Delta}$ are stated in the foll wing manner: First we shall need to add some notations which are similar to ones in (A).

Let us return to the figure $\Sigma_{n,N}$. We mean by pr_z and pr_w the natural projections : $h_z(z,w) = (z)$ and $h_w(z,w) = (w)$. Assume that a figure $\tilde{\boxminus}' \subset \mathbb{C}^n \times \mathbb{C}^N$ of an elementary type is given. Also assume that an ordered set $\tilde{\mathfrak{B}}^s_\Delta$ of figures $\{\tilde{\boxminus}^s_{I_\nu J_\nu}\}_{\nu=1,\dots}$ in $\mathbb{C}^N$ of elementary types are given. We let $\mathfrak{N}(\tilde{\boxminus}', \tilde{\mathfrak{B}}^s_\Delta)$ be the set of all the sets $\{\tilde{\boxminus}^s_{IJK}\}$ which satisfy the following condition

$$(3.1.18) \qquad \tilde{\boxminus}^s_{IJK} \cap \tilde{\boxminus}' \neq \phi \qquad , \text{and } h_z(\tilde{\boxminus}^s_{IJK}) \in \tilde{\mathfrak{B}}^s_\Delta$$

Notations used in (A) are used here under the following modifications. A subset of open sets $\mathfrak{N}' = \{\tilde{\boxminus}^s_{I'J'K'}\}_{I',J',K'}$ of $\mathfrak{N}(\tilde{\boxminus}',\tilde{\mathfrak{B}}^s_\Delta)$ is of type s if the set $\{ pr(\boxminus^s_{I'J'K'}) \}_{I',J',K'}$ is composed of s elements. For a set $\mathfrak{N}'$ of type s , we say that $\mathfrak{N}'$ is of type (s , k) if the number of elements in $\mathfrak{N}'$, whose projection is $\boxminus^s_{I_\lambda J_\lambda}$, is exactly k . A cochain $\varphi^p \in C^p(\mathfrak{N}(\tilde{\boxminus}',\tilde{\mathfrak{B}}^s_\Delta),\mathcal{O})$ is said to be defined in $\mathfrak{N}^{ns}(\tilde{\boxminus}',\tilde{\mathfrak{B}}^s_\Delta)$ if $\varphi(\bigcap_{\mu=0}^{p}\tilde{\boxminus}^s_{I_\mu J_\mu K_\mu})$ is defined in $\bigcap_{\mu=0}^{p}\tilde{\boxminus}^{ns}_{I_\mu J_\mu K_\mu}$ so far as $\bigcap_{\mu=0}^{p}\tilde{\boxminus}^s_{I_\mu J_\mu K_\mu} \neq \phi$. Now assume that a p-cocycle φ^p, defined in $\mathfrak{N}^{ns}(\tilde{\boxminus}',\tilde{\mathfrak{B}}^s_\Delta)$ is given. In a similar way to (A) , we show the following two assertions respectively.

$(\mathcal{B})^{n,N}_{s,k}$ Assume the inequality $2 \leqq s \leqq q$:

moreover, we assume that a q - cocycle $\varphi^q \in C^q(N(\mathcal{U}^{\delta}_{\alpha}, \mathcal{B}^N_{k_2}), \theta)$ of cotype (s, k), defined already in $\mathcal{M}^{M\delta}(\alpha, \mathcal{B}^N_{k_2})$, is given. The following estimation of φ^q is assumed.

$$(\mathcal{B})_1 \qquad |\varphi^q(z, w)| \leqq d_1(|z|^{d_2} + M^{d}+1), \quad z = 1$$

Then, corresponding to $(A)^{l_1, l_2}_{s,k}$, we know the follow

$(\mathcal{B}_1)^{n,N}_{s,k}$ There exists a $(q-1)$ - cochain $\varphi^{q-1} \in C^{q-1}(N(\mathcal{U}^{\delta}_{\alpha}, \mathcal{B}_s), \theta)$ with which the following facts are valid. (Moreover, φ^{q-1} is assumed to be defined already in $\mathcal{M}^{M\delta}(\alpha, \mathcal{B}^N_s)$.

$(\mathcal{B}_{1'})^{n,N}_{s,k}$ According to $k < q + 2 - s$ or $k = q + 2 - s$, the relations

$$(\mathcal{B}_{1',1})^{n,N}_{s,k} \qquad \varphi^q - \delta_{\check{c}ech}(\varphi^{q-1}) = \text{of cotype } (s, k+1)$$

and φ^{q-1} is of cotype (s, k) ;
if $k < q + 2 - s$.

$$(\mathcal{B}_{1',2})^{n,N}_{s,k} \qquad \varphi^q - \delta_{\check{c}ech}(\varphi^{q-1}) = 0,$$

and φ^{q-1} is of cotype $(s, k-1)$
if $k = q + 2 - s$.

$(\mathcal{B}_{1',3})^{n,N}_{s,k}$ In both cases of $k <(=) q + 2 - s$, the following estimation of φ^{q-1} is valid,

$$(\mathcal{B}_{1',4})^{n,N}_{s,k} \quad |\varphi^{q-1}| \leqq d_1'(|z|^{d_2'} + |w|^{d_2'} + 1)$$

In the above, quantities (M', d_1', d_2') are determined by the following equations

$$(\mathcal{B}_{1,5}) \qquad \tilde{M}' = e_1^{-\tilde{e}_2 \delta_2} \cdot \tilde{M}, \qquad \alpha_1' = \mathbb{P}_1(\tilde{e}_p)^{\mathbb{P}_2(\tilde{\alpha}_3 \tilde{\delta}_2)\tilde{c}_2} \cdot M \mathbb{P}_3(\tilde{\alpha}_1, \tilde{\delta}_1) \cdot \min_\nu(\tilde{\alpha}_\nu, \tilde{b}_\nu)^{-\tilde{c}_3}$$

$$\alpha_2' = L(\alpha_2 + \delta_2)$$

,where $\tilde{e}_2 = e^{N,n}_{\delta,s,\delta_1\dot{\iota}_2} \in \mathbb{R}^+$, $\tilde{c} = \tilde{c}^{N,n}_{\delta,s,\delta_1} \in \mathbb{R}^+$, $\mathbb{P}_i$, L are depending on (N, n, q, s, k) only.

Corresponding to the assertion $(\mathcal{B})^{\ell_1 \ell_2}_{s}$, we know the following facts.

$(\mathcal{B}')^{n,N}_{s,k}$ Assume the inequality : $1 \leqq s \leqq q$

Also assume that a q- cocycle $\varphi^q \in Z^q(\mathcal{M}(\tilde{x}(\Xi, \tilde{k}_s), \mathcal{O})$ of cotype s,

defined in $\mathcal{M}^{\tilde{M}\delta}(\Xi, \tilde{k}_s)$ and estimated as

$$(\mathcal{B}'_1)^{n,N}_{s,k} \quad |\varphi^q| \leqq \tilde{\alpha}_1 \cdot (|z|^{\tilde{\alpha}_2} + 1),$$

is given . Then we obtain a (q- 1) cochain φ^{q-1}, defined in $\mathcal{M}^{\tilde{M}\delta}(\Xi', \tilde{k}_s)$,which satisfies the following conditions.

$$(\mathcal{B}'_{1,1})^{n,N}_{s,k} \quad \varphi^q - \delta_{\check{c}ech}(\varphi^{q-1}) = 0,$$

$$(\mathcal{B}'_{1,2})^{n,N}_{s,k} \quad |\varphi^{q-1}_{(z,w)}| \leqq \tilde{\alpha}_1' (|z|^{\tilde{\alpha}_2'} + 1)$$

Here quantities ($\tilde{M}'$, $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$) are determbecause the steps in (II) involve considerable details we shall outline each step taken in the second step. In the first place (I), we settle a set of neighbourhood to each stratum $S^{*j}_{\lambda_j \mu_\gamma}$. This is done by dividing the strata into first and second type, and this part is of essential (but of quite simple and elementary) nature in the remaining parts of this section. Once we fix a set of neidhbourhoods to each stratum $S^{*j}_{\lambda_j \mu_\gamma}$, we investi-igate intersection relations of neighbourhoods for a series $S^{\mu j}_{\lambda_j \mu_\alpha} \dashv S^{\lambda j}_{\lambda_\psi \mu_\tau}$. This part will be divided into three cases : (i) The case in which the series is composed of strata of first type only . (i i) The case where the series is composed of strata of second type only and finally (i i i) the mixed case where the series contains strata of both types If the intersections takes a simple form which is commodiate for our purpose, we say that the associated

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neighbourhoods are suitable, We ask quantitative conditions

in order that the associated neighbourhoods are suitable . This will be done by dividing the cases into three cases (i) , (ii) , and (iii) , and is also of quite elementary nature. In (III) the condition ( ) is quickly shown for the neighbourhoods which are suitable. Finally in (IV) we show the condition ( ). Our arguements in the last two problems will be done easily on basis of explict expressions of intersections of neighbourhoods.

Finally we note that, in the case of $\gamma = 1$ U^{γ} is a expressed as a disjoint finite union of points and one dimensional open segments . Therefore to verify all the desired conditions in propositions 3.13.5 is an obvious matter, and we do not enter into.

Actually , we first see that assertions $(Imp)_1 \sim (Imp)_5$ lead to the assertions $(A)^{\ell_1,\ell_2}_{s,k}$, $(A)^{\ell_1,\ell_2}_{s}$ and $(A)^{\ell}$.

For this purpose, note that the following series of implications are valid.

$$(\widetilde{Imp})^1 \quad (A)^{\ell'} \ (\ell' = 1,\cdots,\ell-1) \Rightarrow (A)^{\ell-1,1}_1 \Rightarrow$$
$$(A)^{\ell-1,2}_k \ (k = 1,\ldots\ldots, q+1) \Rightarrow (A)^{\ell-1,2}_q \Rightarrow (A)^{\ell-1,2}_{2,k}$$
$$\cdots \Rightarrow (A)^{\ell-1,1}_{1,k} \Rightarrow (A)^{\ell-1,1}_q \Rightarrow (A)^{\ell-1,1}_{q+1} \Rightarrow (A)^{\ell}.$$

In the above sequance , the first assertion follows from $(Imp)_3$ and the last assertion follows from $(Imp)_5$. On the otherhand, intermideate sequences are because of $(Imp)_{1,2}$.

Thus we know that the assertions $(Imp)_1 \sim (Imp)_5$ implies the validity of the assertion $(A)^{\ell}$ $(\ell = 1,\cdots)$.

Then it is clear that the assertions $(Imp)_1$, $(Imp)_2$ and $(Imp)_3$ lead to the assertions $(A)^{\ell_1,\ell_2}_{s,k}$, $(A)^{\ell_1,\ell_2}_{s}$.

In a similar reason as above, it is clear that the assertions $(Imp)'_1$, $(Imp)'_2$ and $(Imp)'_3$ imply the assert-ions $(B)^{n,N}_{s,k}$ and $(B)^{n,N}_{s}$. Now we show the assertions

$(Imp)_{1\sim 5}$ (resp. $(Imp)_{1\sim 3}$) in the following manner. At first note that it is clear that the repeated uses of $(A)^{l_1,l_2}_{s,k}$ (resp. $(B)^{n,N}_{s,k}$) lead easily to assertions $(A)^{l_1,l_2}_{s}$ (resp. $(B)^{n,N}_{s}$). On the other hand, $(Imp)_3$ $((Imp)'_3)$ is shown quickly as follows :

Take an element $\square^{l_2}_0 \subset C^{l_2}$ (resp. $\square^{s}_{I^0 J^0} \subset C^{N}$) respectively. We mean by $\mathfrak{K}^{l_2}_1$ (and $\mathfrak{K}^{N}_1$) the set composed of one element $\square^{l_2}_0$ ($\square^{s}_{I^0 J^0}$) only. Write the given figure $\square^{l_1+l_2}$ in $C^{l_1+l_2}$ (resp. $\square^{n+N}$ in C^{n+N}) as

$$\square^{l_1+l_2} = \square^{l_1} \times \square^{l_2}, \quad \square^{n+N} = \square^{n} \times \square^{N}.$$

Then $\mathfrak{N}(\square^{l_1+l_2}, \mathfrak{K}^{l_2}_1, \mathfrak{D})$ (resp. $\mathfrak{N}(\square^{n+N}, \mathfrak{K}^{N}_1)$) is composed of figures $\square^{l_1}_\lambda \times \square^{l_2}_0 \times \mathfrak{D}$ (resp. $\square^{s}_{I_\mu K_\mu} \times \square^{s}_{I^0 J^0} \times \mathfrak{D}$) which satisfy

$$(3.1.19) \quad \square^{l_1}_\mu \cap \square^{l_1} (\neq \phi) \quad (\square^{s}_{I_\mu K_\mu} \cap \square^{n} \neq \phi).$$

Denote by $\mathfrak{N}^{l_1}(\square^{l_1}, \mathfrak{K}^{l_2}_1, \mathfrak{D})$ ($\mathfrak{N}^{n}(\square^{n}, \mathfrak{K}^{N}_1)$) the set of figures $\{\square^{l_1}_\mu \times \mathfrak{D}\}$ ($\{\square^{s}_{I_\mu K_\mu}\}$) so that the relation (3.1.19) holds. Define a map i (i') by

(3.1.20) $i: C^*(N(\pi(\square^{\ell_2}_{\mathcal{A}_1}, \mathcal{D})), \mathcal{O}) \ni \varphi \to C^*(N(\pi(\square^{\ell_1}, \mathcal{A}^{\ell_2}_1, \mathcal{D})), \mathcal{O}) \ni i(\varphi)$

$: i(\varphi)(\cap_\nu \square^{\ell_1}_{\mu} \times \mathcal{D}) = \varphi(\cap_\nu \square^{\ell_1}_{\mu} \times \square^{\ell_2}_0 \times \mathcal{D})$

(3.1.20)′ $i': C^*(N(\pi(\square', \mathcal{A}^{N}_1)), \mathcal{O}) \ni \varphi' \to C^*(N(\pi(\square', \mathcal{A}^{N}_1)), \mathcal{O}) \ni i'(\varphi')$

$: i'(\varphi')(\cap_\nu \square^{\delta}_{IJK_\mu}) = i(\varphi'(\cap_\nu \square^{\delta}_{I^0J^0K_\mu}))$

The above bijective map i (i') commutes with the Cech coboundary operator $\delta_{\check{c}ech}$. Then it is clear that $(A)^{\ell'}$ ((A^{ℓ})) imply the assertion $(A)^{\ell_1\ell_2}_1$ $((B)^{n,N}_3)_1$. Therefore the remaining problem is to show assertions $(Imp)_2$ ($(Imp)'_4$, $(Imp)_5$ (and $(Imp)'_2$)). We consider the assertion $(Imp)_2$ ($(Imp)'_2$).

<u>Verifications of the assertion $(Imp)_2$. ($(Imp)'_2$).</u>

Arguements will be done by dividing the cases : $k \neq q+2-s$, and $k = q+2-s$. First we remark the following simple facts. Let M, M' be two positive numbers ($M, M' \geqq 1$). Mooeover, let $\square_1$, $\square_2$ (resp. $\square^{\delta}_{I_1J_1K_1}$, $\square^{\delta}_{I_2J_2K_2}$) be figures so that $\square_1 \cap \square_2 \neq \phi$ (resp. $\square^{\delta}_{I_1J_1K_1} \cap \square^{\delta}_{I_2J_2K_2} \neq \phi$) holds. Then there exist positive numbers $e = e_{\ell_1\ell_2}$ (resp. $e_{M} = e_{N,\mu}^{(k=1,2)}$) with which the folloeing are valid.

(3.1.20) If $M \geqq e \cdot M'$, then $\square^{M}_1 \supset \square^{M'}_2$,

(3.1.20)′ If $M \geqq e^{e_2 \cdot \delta}_1 M'$, then $\square^{M\delta}_1 \supset \square^{M'\delta}_2$

We show assertions $(Imp)_2$ ($(Imp)'_2$) as follows.

$(\text{Imp } \&)_k$ $k > q + 2 - s$: Start with a datum $(\sqcap, \sqcap', \mathcal{B}_s^{\ell_2})$ (resp. $(\tilde{\sqcap}, \tilde{\mathcal{B}}_s^N)$) as in assertions $(A)_{s,k}^{\ell,\ell_2}$ ($(B)_{s,k}^{N,n}$) We express the ordered set $\mathcal{B}_s^{\ell_2}$ (resp. $\tilde{\mathcal{B}}_s^N$) in the following way. $\mathcal{B}_s^{\ell_2} = \{\sqcap_{I_\mu J_\mu}\}_{\mu=1,\dots,s}$, $\tilde{\mathcal{B}}_s^N = \{\tilde{\sqcap}^s_{I_\mu J_\mu K_\mu}\}_{\mu=1,\dots,s}$. Take elements $(\sqcap_{I_\nu J_s})_{\nu=1,\dots,k} : \cap_\nu \sqcap_{I_\nu J_s} \neq \phi$ (resp. $(\tilde{\sqcap}^s_{I_\nu J_s K_\nu})_{\nu=1,\dots,k} : \cap_\nu \tilde{\sqcap}^s_{I_\nu J_s K_\nu} \neq \phi$) ($\nu = 1, \dots, k$)

Instead of the given figure $\sqcap \subset \mathbb{C}^{\ell+\ell_2}$ (resp. $\sqcap \subset \mathbb{C}^{N}$) we consider the figures $\sqcap^*_k = \sqcap \cap (\cap_{\nu=1}^k \sqcap_{I_\nu J_s})$ (resp. $\tilde{\sqcap}^*_k = \sqcap \cap (\cap_{\nu=1}^k \tilde{\sqcap}_{I_\nu J_s K_\nu})$). Also we start with the ordered set $\tilde{\mathcal{B}}_{s-1}^{\ell_2} = \{\sqcap_{I_\mu J_\mu}\}_{\mu=1,\dots,s-1}$ (resp. $\tilde{\mathcal{B}}_{s-1}^N = \{\tilde{\sqcap}_{I_\mu J_\mu}\}_{\mu=1,\dots,s-1}$) instead of $\mathcal{B}_s^{\ell_2}$ (resp. $\tilde{\mathcal{B}}_s^N$) . In a little while our arguement will be done with the data $(\sqcap^*, \sqcap'^*, \mathcal{B}_{s-1}^{\ell_2})$ (resp. $(\sqcap, \tilde{\sqcap}^*, \tilde{\mathcal{B}}_{s-1}^N)$). For a given cocycle $\varphi^q \in C^q(N(\mathcal{U}(\sqcap, \sqcap', \mathcal{B}_s^{\ell_2})), \mathcal{O})$ (resp. $\tilde{\varphi}^q \in C^q(N(\mathcal{U}^s(\tilde{\sqcap}, \tilde{\sqcap}', \tilde{\mathcal{B}}_s^N)), \mathcal{O})$ we obtain a $(q-k)$ - cocycle $\varphi_*^{q-k} \in C^{q-k}(N(\mathcal{U}(\sqcap, \sqcap^*, \mathcal{B}_{s-1}^{\ell_2})), \mathcal{O})$ ($\tilde{\varphi}_*^{q-k} \in C^{q-k}(N(\mathcal{U}^s(\tilde{\sqcap}, \tilde{\sqcap}^*, \tilde{\mathcal{B}}_{s-1}^N)), \mathcal{O})$) by the following equation.

$$(3.1.21) \qquad \varphi_*^{q-k}\left(\bigcap_{\tau=1}^{q-k+1} \sqcap_{I_\tau J_\tau}\right) = \varphi^q\left(\bigcap_{\tau=1}^{q-k+1} \sqcap_{I_\tau J_\tau} \bigcap_{\nu=1}^{k} \sqcap_{I_\nu J_s}\right),$$

if $pr_{x^2}\{\sqcap_{I_\tau J_\tau}\}_{\tau=1,\dots,q-k+1}$ coincide with $\mathcal{B}_{s-1}^{\ell_2}$.

$= 0,$ otherwise.

$$(3.1.21)' \qquad \tilde{\varphi}_*^{q-k}\left(\cap_{\tau=1}^{q-k+1} \sqcap_{I_\tau J_\tau K_\tau}\right) = \tilde{\varphi}^q\left(\cap_{\tau=1}^{q-k+1} \sqcap_{I_\tau J_\tau K_\tau} \cap_{\nu=1}^{k} \sqcap_{I_\nu J_s K_\nu}\right) \text{ if } P_{x^2}(\{\sqcap_{I_\tau J_\tau K_\tau}\}) = \tilde{\mathcal{B}}_{s-1}^N$$

$= 0$, otherwise

moreover, we assume that the cocycle φ^{q} (resp. $\tilde{\varphi}^{q}$ is already defined in $\mathfrak{N}^{M}(\square, \square', \mathcal{B}_{s}^{\ell_2})$ (resp. $\mathfrak{N}^{M\delta}(\square', \mathcal{B}_{s}^{N})$) . Then, in view of ( ) , we can assume that φ_{*}^{q-k} (resp. $\tilde{\varphi}_{*}^{q-k}$) is already defined in $\mathfrak{N}^{M'}(\square, \square', \mathcal{B}_{s}^{\ell_2})$ (resp. $\mathfrak{N}^{\tilde{M}'\delta}(\tilde{\square}', \tilde{\mathcal{B}}_{s}^{N})$, $M'=e^{*}M$, $\tilde{M}'=\tilde{e}_{1}^{*}\tilde{e}_{2}^{*}M$ here constants e^{*} ($\tilde{e}_{1}^{*}$, $\tilde{e}_{2}^{*}$) are depending on (ℓ_1, ℓ_2) (n, N) only . By taking M sufficiently large, we can assume the inequality $M' \geqq 1$. For the abobe cocycle φ_{*}^{q-k} (resp. $\tilde{\varphi}_{*}^{q-k}$) , we apply the assertion $(A)_{s-1}^{\ell_1,\ell_2}$ (resp. $(B)_{s-1}^{n,N}$). Then we obtain a ($q-k-1$) - cochain φ

$\varphi_{*}^{q-k-1} \in C^{q-k-1}(N(\mathfrak{N}^{M\delta}(\square, \square^{*}, \mathcal{B}_{s-1}^{\ell_2})), \mathcal{O})$ (resp. $\tilde{\varphi}_{*}^{q-k-1} \in C^{q-k-1}(N(\mathfrak{N}^{\tilde{M}\delta}(\square^{*}, \mathcal{B}_{s-1}^{N})), \mathcal{O})$, already defined in $\mathfrak{N}^{M}(\square, \square^{*}, \mathcal{B}_{s-1}^{\ell_2})$ (resp. in $\mathfrak{N}^{\tilde{M}\delta}(\hat{\square}^{*}, \mathcal{B}_{s-1}^{N})$) in such a manner that the equation

$$(3.1.22) \qquad \varphi_{*}^{q-k} - \delta_{Cech}(\varphi_{*}^{q-k-1}) = 0,$$

$$(\text{resp.} \quad \tilde{\varphi}_{*}^{q-k} - \delta_{Cech}(\tilde{\varphi}_{*}^{q-k-1}) = 0 ,)$$

holds.

The estimation of φ^{q-k-1}_{*} (resp. $\tilde{\varphi}^{q-k-1}_{*}$) is obtained from $(A)^{\ell_1,\ell_2}_{s-1}$ (resp. $(B)^{n,N}_{s-1}$) in the following fashion.

$$(3.1.23) \quad |\varphi^{q-k-1}_{*}| \leq c_1^{*}\cdot M^{c_2^{*}}\cdot|\min_{\nu}(\tilde{a}_\nu,\tilde{b}_\nu)|^{-c_3^{*}}\cdot\{\max_{\nu}(m_\nu,n_\nu)\}^{c_4^{*}}\cdot\|\varphi^{q-k}_{*}\|,$$

$$(3.1.23)' \quad |\tilde{\varphi}^{q-k-1}_{*}| \leq d_1^{*}\cdot(|z|^{d_2^{*}}+1)$$

In the aboves , quantities $(c_1^{*},\dots,c_4^{*})$ (d_1^{*} , d_2^{*}) are given by the $(A)^{\ell_1,\ell_2}_{s-1}$ (resp. $(B)^{n,N}_{s-1}$). Using the above cochain φ^{q-k-1}_{*} (resp. $\tilde{\varphi}^{q-k-1}_{*}$) , we define a (q − 1) − cochain $\varphi^{q-1} \in C^{q-1}(N(\mathfrak{M}(\square,\square',\mathcal{B}_s^{\ell_2})),\mathcal{O})$ (resp. $\tilde{\varphi}^{q-1} \in C^{q-1}(N(\mathfrak{M}(\square,\square',\mathcal{B}_s^{N})),\mathcal{O})$) by the following equation.

(3.1.24) $\varphi^{g-1}(\cap_{\nu=1}^{g}\square_{I_\nu J_\nu}) = \varphi_*^{g-k-1}(\cap_{\nu=1}^{g-s-1}\square_{I_\nu J_\nu}\cap_{\lambda=1}^{k}\square_{I_\lambda J_\lambda})$, if the set $\{\square_{I_\nu J_\nu}\}_{\nu=1,\dots,g-1}$ satisfies the following condition,

(3.1.24)′ there exists exactly k figures $\{\square_{I_{\nu_\lambda}J_{\nu_\lambda}}\}_{\lambda=1,\dots,k}$ whose projection $pr_{\nu_\lambda}(\square_{I_{\nu_\lambda}J_{\nu_\lambda}}) \supseteq \square_{J_\lambda}$, and $\{\square_{I_{\nu_\lambda}J_{\nu_\lambda}}\}$ in ( ) is $\{\square_{I_\nu J_\nu}\}_{\nu=1,\dots,s} - \{\square_{I_{\nu_\lambda}J_{\nu_\lambda}}\}_{\lambda=1,\dots,k}$ *
$=0$, otherwise.

In the case of $(B)_{s,k}^{n,N}$ we define φ^{g-1} by

$(3.1.24)^*$ $\varphi^{g-1}(\cap_{\nu=1}^{g}\square_{I_\nu J_\nu}) = \varphi_*^{g-k-1}(\cap_{\lambda=1}^{g-k}\square_{I_\lambda J_\lambda K_\lambda})$ if the set $\{\square_{I_\nu J_\nu K_\nu}\}_{\nu=1,\dots,g}$ satisfy the coldition

$(3.1.24)'^*$ there exists exactly k figures $\{\square_{I_{\nu_\lambda}J_{\nu_\lambda}}\}_{\lambda=1,\dots,k}$ whose projection $pr_{\nu_\lambda}(\square_{I_{\nu_\lambda}J_{\nu_\lambda}K_{\nu_\lambda}})$ is $\square_{I_\lambda J_\lambda}$, and in ( ), $\{\square_{I_{\nu_\lambda}J_{\nu_\lambda}K_{\nu_\lambda}}\}_{\nu=1,\dots}$ is $\{\square_{I_\nu J_\nu K_\nu}\}_{\nu=1,\dots,s} - \{\square_{I_{\nu_\lambda}J_{\nu_\lambda}K_{\nu_\lambda}}\}_{\lambda=1,\dots,k}$.
$= 0$, otherwise.

Moorover,, in view of the simple remarks (3.1.20)'s, we can assume that this cochain φ^{g-1} (resp. φ^{g-1}) is already defined in $\mathfrak{N}^{M'}(\square,\square',\mathcal{B}_s^{\ell_2})(\mathfrak{N}^{M'\delta}(\square,\square',\mathcal{B}_{\ell_2}^{N}))$, where M' is defined by $M' = e\cdot\mathcal{M}$ (resp. $M' = e_1^{-e_2\delta_2}\cdot\mathcal{M}$) with constants e (e_1, e_2) depending on (ℓ_1, ℓ_2) (resp. (N, n)) only.

On the otherhand, estimations of cochains φ^{g-1} ($\tilde{\varphi}^{g-1}$) are obtained from the induction hypothesis $(A)_{s-1}^{\ell_1,\ell_2}$ $\{(B)_{s-1}^{n,N}\}$.

Thus required quantitative properties required for cochains φ^{q-1} ($\tilde{\varphi}^{q-1}$) are satisfied. On the otherhand, it is clear from the equations ( ) , ( ) that the following relations are valid.

(3.1.25) $\varphi^{q} - \delta_{Cech}(\varphi^{q-1})$ is of cotype (s, k + 1). Finally it is also obvious , from ( ) , that the cochain φ^{q-2} is of cotype (s,k+1).

This finishes our proof in this case.

$(\mathrm{Imp}\ 2)'_2$ k = q + 2 - s ; in this case our method of reducing the problem$()^{\ell_1,\ell_2}_{s,k}$ to the ' lower ' case$()^{\ell_1,\ell_2}_{s-1}$ is not applicable as in the case ( ). We proceed as follows : Take a set of figures $\{\square_{I_\nu J_\nu}\}_{\nu=1,\cdots,s-1}$ (resp. $\{\square^{\delta}_{I_\nu J_\nu K_\nu}\}_{\nu=1,\cdots,s-1}$) so that the conditions

(3.1.26) $\bigcap_{\nu=1}^{s-1}\square_{I_\nu J_\nu} \cap \square \neq \phi$, (resp. $(\bigcap_{\nu=1}^{s-1}\square_{I_\nu J_\nu K_\nu}) \cap \square \neq \phi$)

(3.1.26)′ $\{Pr_{x}(\square_{I_\nu J_\nu})\}_{\nu=1,\cdots,s-1} = \{\square_{J_\mu}\}_{\mu=1,\cdots,s-1}$, $\{Pr(\square^{\delta}_{I_\nu J_\nu K_\nu})\}_{\nu=1,\cdots,s-1} = \{\square^{\delta}_{I_\mu J_\mu}\}_{\mu=1,\cdots,s-1}$

are valid.

In this case we fix a set $\{\square_{I_\nu J_\nu K_\nu}\}_{\nu=1,\cdots,s-1}$ (resp. $\{\square^{\delta}_{I_\nu J_\nu K_\nu}\}_{\nu=1,\cdots,s-1}$)

at a moment. Define a figure $\square'^{*}_{s-1}$ (resp. $\tilde{\square}'^{*}_{s-1}$) by $\square'^{*}_{s-1} = \square' \cap (\cap_{\nu=1}^{s-1} \square_{I_\nu J_\nu})$ (resp. $\tilde{\square}'^{*}_{s-1} = \tilde{\square}'^{*} \cap (\cap_{\nu=1}^{s-1} \square_{I_\nu J_\nu})$).

Moreover, define a set of figures $\mathcal{N}(\square, \square'^{*}_{s-1} | \square_{J_s})$ by $\mathcal{N}(\square, \square, \square_{J_s}) = \{ \square : b_{x}(\square) = \square_{J_s}, \square \cap \square'^{*}_{s-1} \neq \phi \}$ (resp. $\mathcal{N}(\square, \tilde{\square}'^{*}_{s-1} | \tilde{\square}^{s}_{I_s J_s}) = \{ \square^{s}_{IJK} ; B_z(\square_{IJK}) = \square_{I_s J_s}, \square_{IJK} \cap \tilde{\square}'^{*}_{s-1} \neq \phi \}$). Then

it is clear that a $(q-s+1)$ - cocycle $\varphi^{q-(s-1)}_{*} \in C^{q-(s-1)}(\mathcal{N}(\square, \square, \square_{J_s}), \mathcal{O})$ (resp. $\tilde{\varphi}^{q-s+1}_{*} \in C^{q-s+1}(\mathcal{N}(\square, \square, \square), \mathcal{O})$) is obtained from a q - cocycle $\varphi^{q} \in C^{q}(\mathcal{N}(\square, \square, \square_{J_s}), \mathcal{O})$ ($\varphi^{q} \in C^{q}(\mathcal{N}(\square, \square, \square), \mathcal{O})$) defined already in $\mathcal{N}^{M}(\square, \square, \square_{J_s})$ ($\mathcal{N}^{M, s}(\square, \square, \square_{I_s J_s})$) by the following formula.

$$(3.1.27) \qquad \varphi^{q-s+1}_{*}(\cap_{\nu'=1}^{q-s+2} \square_{I_{\nu'} J_s}) = \varphi^{q}(\cap_{\nu'=1}^{q-s+2} \square_{I_{\nu'} J_s} \cap_{\nu=1}^{s-1} \square_{I_\nu J_\nu}),$$

$$(3.1.27)' \qquad \tilde{\varphi}^{q-s+1}_{*}(\cap_{\nu'=1}^{q-s+2} \square_{I_s J_s K'_{\nu}}) = \varphi^{q}(\cap_{\nu'=1}^{q-s+2} \square_{I_s J_s K'_{\nu}} \cap_{\nu=1}^{s-1} \square_{I_\nu J_\nu K_\nu})$$

In the above, we can assume that φ^{q-s+1}_{*} (resp $\tilde{\varphi}^{q-s+1}_{*}$) is defined in $\mathcal{N}^{M'}(\square, \square'^{*} | \square_{J_s})$ (resp. $\mathcal{N}^{M', s}(\tilde{\square}^{*}, |\tilde{\square}^{s}_{I_s J_s})$), where M' is determined by $M' = e_{*} \cdot M$ (resp. $M' = \tilde{e}^{*}_{1}{}^{\tilde{e}^{*}_{2} s_2} \cdot M$) with constants e_{*} ($\tilde{e}^{*}_{1}, \tilde{e}^{*}_{2}$) depending on $(\ell_1, \ell_2, \delta, q)$ ((N, n, s, q)) only.

(The existence of constants e_{*} ($\tilde{e}^{*}_{1}, \tilde{e}^{*}_{2}$) are obvious in view of (3.1.19). Also our arguements will be done by assuming that $M' \geqq 1$. Here we remark that φ^{q-s+1}_{*} (resp. $\tilde{\varphi}^{q-s+1}_{*}$) is <u>independent of $\{\square_{I_\nu J_\nu}\}_{\nu=1,\cdots,s-1}$ (resp. $\{\square_{I_\nu J_\nu K_\nu}\}_{\nu=1,\cdots,s-1}$)</u> in the sense that

$\varphi^{q-s+1}(\cap_{\nu=1}^{s-1}\square_{I_\nu J_\nu}\cap_{\nu=1}^{q+2-s}\square_{I'_\nu J'_\nu}) = \varphi^{q-s+1}(\cap_{\nu=1}^{s-1}\square_{I_\nu \tilde{J}_\nu}\cap_{\nu=1}^{q+2-s}\square_{I'_\nu J'_\nu})$ (resp. $\varphi^{q-s+1}(\cap_{\nu=1}^{s}\square_{I_\nu J_\nu K_\nu}\cap_{\nu=1}^{q+2-s}\square_{I_\nu J_\nu K_\nu}) = \varphi^{q-s+1}(\cap_{\nu=1}^{s}\square_{I_\nu J_\nu K_\nu}\cap_{\nu=1}^{q+2-s}\square_{I_\nu J_\nu \tilde{K}_\nu})$)

holds. This independenceness follows from the fact that the cocycle φ^q (resp. φ^q) is of cotype (s, q + 2 - s) once we note the following simple fact: (C_h) Let us assume that a figure of elementary type $\square_0 < \mathbb{C}^{\ell_1+\ell_2}$ ($\tilde{\square}_0 < \mathbb{C}^{n+N}$) and figures $\square_{IJ}$, $\square_{I'J'}$ ($\square^{\delta}_{IJK}$, $\square^{\delta}_{I'J'K'}$) are given. Assume that the follow--ing condition $\square^0_{\cap}\square_{IJ}(\square^0_{\cap}\square_{I'J'}) \neq \phi$ (resp. $\square_{\cap}\square^{\delta}_{IJK'} \neq \phi$) is valid.

. A chain of figures $\{\{\square_{I_\tau J_\tau}\}_{\tau=1,\ldots,\tau_0}(\{\square_{I_\tau J_\tau K_\tau}\}_{\tau=1,\ldots,\tau_0})$ is defined to be an ordered set of figures $\square_{I_1J_1}, \ldots, \square_{I_{\tau_0}J_{\tau_0}}$ (resp. $\square^{\delta}_{I_1J_1K_1}, \ldots\ldots, \square^{\delta}_{I_{\tau_0}J_{\tau_0}K_{\tau_0}}$) in such a manner that the conditions

$$\square_{IJ}\cap\square_{I_1J_1}\neq\phi, \ldots; \square_{I_{\tau-1}J_{\tau-1}}\cap\square_{I_\tau J_\tau}\neq\phi, \ldots, \square_{I_{\tau_0}J_{\tau_0}}\cap\square\neq\phi, \quad \square_{I_\tau J_\tau}\cap\square\neq\phi(\tau=1,\ldots,\tau_0),$$

$$(\square_{IJK}\cap\square_{I_1J_1K_1}\neq\phi, \ldots, \square_{I_{\tau-1}J_{\tau-1}K_{\tau-1}}\cap\square_{I_\tau J_\tau K_\tau}\neq\phi \ldots, \square_{I_{\tau_0}J_{\tau_0}K_{\tau_0}}\cap\square\neq\phi, \quad \square_{I_\tau J_\tau K_\tau}\cap\square\neq\phi(\tau=1,\ldots,\tau_0))$$

hold. A simple observstion leads easily to the existence of a chain for an arbitrarily given figures $(\square^0, \square_{IJ}, \square_{I'J'})$ (resp. $(\tilde{\square}^0, \square^{\delta}_{IJK}, \square^{\delta}_{I'J'K'})$).

From the above remark, the independenceness follows quite easily. (Because a verification is qutte simple , we omit details.)

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Apply the induction hypothesis ($)^{\ell_1,\ell_2}_{1}$ (resp. ($)^{n,N}_{1}$) to the cocycle φ^{q-s+1}_{*} (resp. $\tilde{\varphi}^{q-s+1}_{*}$). Then we obtain a (q - s) cochain $\varphi^{q-s}_{*} \in C^{q-s}(N(\mathfrak{N}(\square, \square^{*}_{s-1}, (\square_{I_s J_s})), \mathcal{O})$ (resp. $\tilde{\varphi}^{q-s}_{*} \in C^{q-s}(N(\mathfrak{N}^{N,S}(\square^{*}, (\square^{s}_{I_s J_s K_s})), \mathcal{O})$) which satisfy the following equatuon.

$$(3.1.28) \quad \delta_{\check{c}ech}(\varphi^{q-s}_{*}) - \varphi^{q-s+1}_{*} = 0 .$$

$$(\text{resp.} \quad (\quad) \; \delta_{\check{c}ech}(\tilde{\varphi}^{q-s}_{*}) - \tilde{\varphi}^{q-s}_{*} = 0 \quad .)$$

By (A $)^{\ell_1,\ell_2}_{1}$, (resp. (B $)^{n,N}_{1}$) we know necessary quantitative properties : φ^{q-s}_{*} (resp. $\tilde{\varphi}^{q-s}_{*}$) is defined in $\mathfrak{N}^{N}(\square, \square^{*}_{s-1}, \square_{J_s})$ (resp. $\mathfrak{N}^{N,S}(\square^{*}_{s-1}, (\square_{I_s J_s}))$) while the estimation of φ^{q-s}_{*} (resp. $\tilde{\varphi}^{q-s}_{*}$) is given by (A $)^{\ell_1,\ell_2}_{1}$, (B $)^{n,N}_{1}$). Using the above cochain $\varphi^{q-s}(\varphi^{q-s})$, we uefine a (q - 1) - cochain $\varphi^{q-1} \in C^{q-1}(N(\quad), \mathcal{O})$ by the following equation.

①

$$(3.1.29) \quad \varphi^{q-1}\left(\bigcap_{\nu=1}^{s-1} \square_{I_\nu J_\nu K_\nu}, \bigcap_{\nu=1}^{q+\ell-s} \square_{I_\nu J_\nu}\right) = \varphi\left(\bigcap_{\nu=1}^{q+\ell-s} \square_{I_\nu J_\nu}\right) \text{ if } \{\square_{I_\nu J_\nu K_\nu}\} \text{ is of type } (s, \ell-1)$$

$$= 0 \quad , \text{ otherwise}$$

(resp. $\tilde{\varphi}^{q-1} \in C^{q-1}(N(\mathfrak{N}(\quad)), \mathcal{O})$ by $\varphi^{q-1}\left(\bigcap_{\nu=1}^{s-1} \square^{s}_{I_\nu J_\nu}, \bigcap_{\nu=1}^{q+\ell-s} \square^{s}_{I,J}\right) =$

In the above N (resp. N$'$) is given

in the following form (in view of ( )).

$$(3.1.29)' \quad M' = \mathcal{C}^* M_1', \quad M' = \mathcal{C}_1^{*-\tilde{e}^{*}\delta_2} M''.$$

with constants $\mathcal{C}^*$ ($\mathcal{C}_1^*$, $\mathcal{C}_2^*$) depending on ($\mathcal{L}_1$, $\mathcal{L}_2$)

((resp. (n, N)) only .

Now it is easy to see that the following equation

$$(3.1.30) \quad \varphi^{g} - \delta_{Čech}(\varphi^{g-1}) = 0,$$

$$(\text{resp. } \tilde{\varphi}^{g} - \delta_{Čech}(\tilde{\varphi}^{g-1}) = 0,)$$

holds:

First it is clear that $\varphi^{g} - \delta_{Čech}(\varphi^{g-1}) = 0$ ($\tilde{\varphi}^{g} - \delta_{Čech}(\tilde{\varphi}^{g-1}) = 0$,)

except the cases where $\{\square_{I_\tau J_\tau}\}_{\tau=1,\dots,g+1}$ ($\{\square^{\delta}_{I_\tau J_\tau K_\tau}\}_{\tau=1,\dots,g+1}$) satisfy the conditions

$\{\square_{I_\tau J_\tau}\}_{\tau=1,\dots,g+1}$ ($\{\square^{\delta}_{I_\tau J_\tau K_\tau}\}_{\tau=1,\dots,g+1}$) is of type s, and of type (s, k) or $(s, k+1)$

Next, if $\{\square_{I_\tau J_\tau}\}$ $\{\square^{\delta}_{I_\tau J_\tau K_\tau}\}$ is of type (s, k) , then

( ) implies that $\varphi^{g} - \delta_{Čech}\varphi^{g-1} = 0$ (resp.

$\tilde{\varphi}^{g} - \delta_{Čech}(\tilde{\varphi}^{g-1}) = 0$). Thirdly, for $(\square_{I_\tau J_\tau})$ $(\square^{\delta}_{I_\tau J_\tau K_\tau})$

of type $(s, k-1)$, $\varphi^{g} - \delta_{Čech}\varphi^{g-1} = 0$ ($\tilde{\varphi}^{g} - \delta_{Čech}\tilde{\varphi}^{g-1} = 0$)

because of the equation (3.1.29) .

This finishes our proof. q.e.d.

Remainin- two assertions (Imp)$_4$ and (Imp)$_5$ are quite easy and are proven in the folloeing ways.

($\mathcal{H}$)$^{l_1 l_2}$ Concerning (Imp)$_4$ we first note that the relation φ^q is of cotype (q+1) implies the existence of a q – cocycle $\varphi'^q \in C^q(N(\square_1^M \times \mathcal{D}, \square_2'), \mathcal{O})$

$$(3.1.31) \qquad \varphi'^q(\square_1^M \times \bigcap_{\nu=1}^{q+1} \square_{2\nu} \times \mathcal{D}) = \varphi^q(\bigcap_{\nu=1}^{q+1} \square_{1\nu} \times \square_{2\nu} \times \mathcal{D}) \quad \square_1^M = \bigcup Pr_1(\square), \;^*$$

$$\square_2' = Pr_2(\square') \;;\; Pr_1(x_1, \cdots, x_{l_1}, x_{l_1+1}) = (x_1, \cdots x_{l_1}),\; Pr_2(x_1, \cdots, x_{l_1}, x_{l_1+1}) = x_{l_1+1}$$

Thus the problem is reduced to the one dimansional case. (The existence of φ'^q is a consequence of the relation : φ^q is of cotype (q+1) and of the existence of a chain remarked in (3.1.18)) . we find a q – 1 cochain $\varphi'^{q-1} \in C^{q-1}(N(\square_1^M \times \mathcal{D}, \square_2'), \mathcal{O})$ so that the equation $\varphi'^q - \delta_{\check{C}ech}(\varphi'^{q-1}) = 0$ holds. Moreover, we define a$(q-1)$– cochain φ^{q-1} by

$$(3.1.32) \qquad \varphi^{q-1}(\bigcap_{\nu=1}^{q} \square_{1\nu} \times \square_{2\nu} \times \mathcal{D}) = \varphi'^{q-1}(\square_1^M \times \bigcap_\nu \square_{2\nu} \times \mathcal{D})$$

it is clear that (Imp)$_4$ is assured with this cochain φ^{q-1}.

(A)^1 The assertion is quite clear and we omit it

* $\square_{\nu 1}$, $\square_{\nu 2}$ are projections $Pr_1(\square_\nu)$, $Pr_2(\square_\nu)$ and the left side of (3.1.31) is the restriction to the set $(\bigcap_{\nu=1}^{q+1} \square_{\nu 1}) \times (\bigcap_{\nu=1}^{q+1} \square_{\nu 2}) \times \mathcal{D}$

(III) After the above elementary preparations, we sho[w] quickly our assertion : Lemma 3. 1. 1'. We start with the following.

(E . L)$_K$ ....Eliminations of the indices K: There exists a datum $\{ \mathcal{L}, P_1, P_2, L, \tilde{c} \}$, depending on (N, n, b) only, ~~and compo~~ of a map $\mathcal{L}$ of e-L-type, $c \in \mathbb{Z}^+$, polynomials P_i $(i=1,2)$, and a linear functions L so that the following assertion is valid.

(E . L)^1_K There exists a (q - 1) - cochain φ^{q-1} $\in C^{q-1}(N(\mathfrak{N}^{\delta'}(\Xi_{n,N})), \mathcal{O})$ with algebraic growth (α_1', α_2') in such a manner that the relation

(E L)^2_K $\varphi^q - \delta_{Cech}(\varphi^{q-1})$ is of cotype (q + 1), holds. (Here the right side φ^q is understood to be an element in $C^q(N^q(\mathfrak{N}^{\delta'}(\Xi_{n,N})), \mathcal{O})$ by means of a suitable refinement map.)

Moreover, quantities (δ'), (α_1', α_2') are determined by

~~(E L)^3_K (δ') = $\mathcal{L}$(δ) , (α_1', α_2') = (...)~~

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$$(\delta_1') = \mathcal{L}(\delta), \quad \partial_1 = \mathbb{P}_1 (\partial_1, \delta_1)\, e^{\mathbb{P}_2(\partial_2,\delta_2)} \min(\tilde{a}_\nu, b_\nu)^{-c},$$
$$\partial_2' = Z(\partial_2 + \delta_2).$$

In concluding this part we note the following :

For a given couple (δ) ,we define an another type of an open covering of $\mathbb{C}^N \times U$: Namely , by $\widetilde{\mathfrak{N}}^s(C^N \times U)$ (Note that we are writing $C^N \times U$ instead of) the set of all the open sets $\{ \square^s(P_{IJ}) \times U \}_{I,J}$. Assume that a cocycle $\varphi^q \in \mathfrak{N}^s(\Xi_{n,N})$ of type (q+ 1) is given. Then there exists an uniquely defined cocycle $\tilde{\varphi}^q \in \widetilde{\mathfrak{N}}^s(\mathbb{C}^N \times U)$ so that the equations

$$(3.1.31) \qquad \tilde{\varphi}^q(\cap_\nu \square^s_{I_\nu J_\nu} \times U) = \varphi^q(\cap_\nu \square^s_{I_\nu J_\nu K_\nu}) \quad \text{in } \mathbb{C}^N \times U$$

for any indices (I , J) and K , hold.

This uniquely determined cocycle $\tilde{\varphi}$ will be called the restriction of φ^q to $\mathbb{C}^N \times U$ and will be denoted by $\varphi^q_{|\mathbb{C}^N \times U}$.

(E.L. J) Elimimation of the index J : Let $\tilde{\varphi}^s$ be a cochain in $G^q(\mathcal{N}(\widetilde{\mathfrak{N}}(\mathbb{C}^N \times U)), \mathcal{O})$. such a cochain $\tilde{\varphi}^q$ is of type s if $\tilde{\varphi}^{q,q+1}(\cap_{\nu=1}^{s} \square^s_{I_\nu J_\nu} \times U)=0$ for any sets $\{ I_\nu, J_\nu \}_{\nu=1,\dots,q+1}$ so that the cardinal number of $\{ I_\nu, J_\nu \}_{\nu=1,\dots,q+1}$ is at most s . A cochain $\tilde{\varphi}^q$ is of algebraic growth (∂_1, ∂_2) if the asymptotic behavior ( ) is true . Then we know that there exists a datum ($\mathbb{P}_1'$, $\mathbb{P}_2'$, L', c', $\mathcal{L}'$) depending on (N, n, q) only , so that the following are valid.

(E.L.J) There exists a (q-1) - cochain $\overset{q-1}{C}$(N ($\tilde{\mathfrak{X}}^{\delta'}(\mathbb{C}^N \times U)$), $\mathcal{O}$) of type $q+1$ in such a manner that

(E. L. J)$_1$ $\varphi^q - \delta_{Cech}(\varphi^{q-1})$ is of cotype $(q+1)$,

as well as estimations

(E. L. J)$_2$ $|\varphi^{q-1}| \leqq \partial_1' \cdot (|z|^{\partial_2'} + |w|^{\partial_2'} + 1)$;

$$\partial_1' = \mathbb{P}_1'(\partial_1, \delta_1) \cdot e^{\mathbb{P}_2(\partial_2, \delta_2) \cdot \{\min(a_\nu, b_\nu)\}^{-c'}}, \quad \partial_2' = L'(\partial_2 + \delta_2), \ \delta' = \mathcal{L}(\delta)$$

are true.

The above fact is shown by a similar and an easier manner than in ( ). Therefore we omit any details of them . We also note that if a cocycle $\tilde{\varphi}^q$ is of cotype (q + 1) , then we can regard $\tilde{\varphi}^q$ as an element of $C^q(N(\mathfrak{U}^0_{\frac{3}{4}}(\mathbb{C}^N \times U), \mathcal{O})$ ($\mathfrak{U}^0_{\frac{3}{4}+\varepsilon}(\mathbb{C}^N \times U)$ = an open covering of $\mathbb{C}^N \times U$ composed of all the open sets $\{\square_{\frac{3}{4}+\varepsilon}(\mathbb{P}_I) \times U\}$). More precisely there exists a cocycle $\varphi^q_{(I)} \in C^q(N(\mathfrak{U}^0_{3/4}(\mathbb{C}^N \times U), \mathcal{O})$ so that the following equations are valid for any indices (I_ν) and (J_ν).

(   )   =

Thirdly we consider the following

(E.L. I) Elimination of the indices I :

An element $\varphi^{q}_{(I)}$ in $C^{q}(N(\mathfrak{M}^{0}_{\xi,t_2}(\mathbb{C}^N\times 0)), \mathcal{O})$ is of algebraic growth (∂_1, ∂_2) if an asymptotic behavior (   ) is valid . Then we know the following fact from (   ), (   ) and an elementary arguement like (   ).

(E.L. I) There exists a datum $(\mathbb{P}_1, \mathbb{P}_2, L, c)$, deoending on (N, n, q) only, so that the following are true .

(3.1.32) For a cocycle φ^{q} of growth (∂_1, ∂_2), there exists a cochain $\varphi^{q-1}_{(I)}$ $C^{q-1}(N(\mathfrak{M}^{0}_{\xi,t_2}(\mathbb{C}^N\times 0)), \mathcal{O})$ of growth $(\partial_1', \partial_2')$ so that

(3.1.33)$_1$
$$\delta_{\text{Čech}}(\varphi^{q-1}_{(I)}) = \varphi^{q}_{(I)},$$

as well as

(3.1.33)$_2$
$$|\varphi^{q-1}_{(I)}| \leq \partial_1'(|z|^{\partial_2'} + |w|^{\partial_2'} + 1): \quad \partial_1' = \mathbb{P}_1(\partial_1)\cdot e^{\mathbb{P}_2(\partial_2)}, \quad \partial_2' = L(\partial_2 + \delta_2).$$

hold.

(   ) Now it is easy to deduce the original lrmma from the above arguements : This part is quite obvious and might be excouded . However we discuss for sureness . Start with a cocycle $\varphi^{q} \in C^{q}(\mathcal{N}(\mathfrak{U}^{\delta}(\bar{Z}_{n,N}), \mathcal{O})$ of growth (∂_1 , ∂_2) . Then, from (E.L. K), we know the existence of (δ^1), (∂^1) and a refinement map $r_K : \mathfrak{U}^{\delta^1}(\bar{Z}_{n,N}) \hookrightarrow \mathfrak{U}^{\delta}(\bar{Z}_{n,N})$ so that the following equation

$$(3.1 . \quad) \qquad r_K^*(\varphi^q) = \delta_{\check{C}ech} \vartheta_1^{q-1} + \varphi_1^q$$

holds. Here $\varphi_1^{q-1}, \varphi_1^q$ are in $C^{*}(\mathcal{N}(\mathfrak{U}^{\delta^1}(\bar{Z}_{n,N})), \mathcal{O})$ and (δ^1), (∂^1) are determined by (日, ㄷ, J) . Restrict φ_1^q to $C^N \times U$, and apply (E.L.J.) . Then we obtain a cochain $\tilde{\varphi}^{q-1}$ in $C(\mathcal{N}(\tilde{\mathfrak{U}}^{2}(C^N \times U)), \mathcal{O})$ so that

$$(3.1 . \quad) \qquad \tilde{r}_J(\tilde{\varphi}_1^q |_{C^N \times U}) - \delta_{\check{C}ech}(\tilde{\varphi}_2^{q-1}) = \tilde{\varphi}_2^q$$

holds, where $\tilde{\varphi}_2^q$ is of cotype $(q+1)$ (w. r. t. J) ; $\tilde{r}_J : \tilde{\mathfrak{U}}^{\delta^2}(C^N \times U) \hookrightarrow \tilde{\mathfrak{U}}^{\delta^1}(C^N \times U)$. Define a refinement map r_J : $\mathfrak{U}^{\delta^2}(\bar{Z}_{n,N}) \hookrightarrow \mathfrak{U}^{\delta^1}(\bar{Z}_{n,N})$ so that the following conditions are valid.

$$(3.1 \quad) \qquad r_J(\square_{I^2J^2K^2}) = \square_{I^1J^1K^1}, \quad \tilde{r}_J(\square_{I^2J^2} \times U) = \square_{\tilde{I}^1\tilde{J}^1} \times U \Rightarrow (I^1, J^1) = (\tilde{I}^1, \tilde{J}^1)$$

For a couple (δ_1 , δ_2) , $\mathfrak{U}^{\delta^*}(\bar{Z}_{n,N})$ means the open covering of $\bar{Z}_{n,N} = C^N \times U$ of open sets $\{\square_{IJK} \cap U\}_{IJK}$.

3-1-44.

(   ) Now it is quite easy to deduce the lemma from the above three steps : (E.L.I), (E.L.J) and (E.L.K).

We start with a cocycle $\varphi^q \in C^q(N(\mathfrak{M}^{\delta}(\Xi_{n,N})),\mathcal{O})$ of algebraic growth (d_1, d_2). Then from (E.L.K), we know the existence of (δ'), (d_1', d_2') and a refinement map $\Upsilon_1 : \mathfrak{M}^{\delta'}(\Xi_{n,N}) \hookrightarrow \mathfrak{M}^{\delta}(\Xi_{n,N})$ so that the following relation

$$(3.1.36) \qquad \Upsilon_1^*(\varphi^q) - \delta_{\check{C}ech}(\varphi^{q-1}_{(1)}) = \varphi^q_{(1)} \text{ is of type } (q+1),$$

is valid. In the above $\varphi^{q-1}_{(1)}$ is a cochain in $C^q(N(\mathfrak{M}^{\delta'}(\Xi_{n,N})))$ of algebraic growth (d_1', d_2'). Of course (d_1', d_2') is determined by (E.L.K). We write the uniquely determined cocycle $\varphi^q_{(1)}|_{\mathbb{C}^N\times U}$ in $C^q(N(\mathfrak{M}^{\delta'}(\mathbb{C}^N\times U)),\mathcal{O})$ by (3.1.32). Apply (E.L.J) to obtain a

a cochain $\varphi^{q-1}_{(2)}|_{\mathbb{C}^N\times U}$ in $C^{q-1}(N(\mathfrak{M}^{\delta''}(\mathbb{C}^N\times U)))$ so that the relation

$$(3.1.37) \qquad \Upsilon_2^* \, \varphi^q_{(1)}|_{\mathbb{C}^N\times U} - \delta\varphi^{q-1}_{(2)}|_{\mathbb{C}^N\times U} = \varphi^q_{(2)} \text{ is of cotype } (q+1)$$

is valid in the sense of (E.L.J), with a suitable infinement map $\tilde{\Upsilon}_2 : \mathfrak{M}^{\delta''}(\mathbb{C}^N\times U) \hookrightarrow \mathfrak{M}^{\delta'}(\mathbb{C}^N\times U)$.

Then we define a refinement map $\Upsilon_2 : \mathfrak{M}^{\delta''}(\Xi_{n,N}) \hookrightarrow \mathfrak{M}^{\delta'}(\Xi_{n,N})$ so that the following condition is valid.

(3.1.38) If $\Upsilon_2(\square^{\delta''}_{IJK}) = \square^{\delta'}_{I'J'K'}$, then the relation

$$(I', J') = (\tilde{I}', \tilde{J}') : \quad \tilde{\Upsilon}_2(\square^{\delta''}_{I'J'}\times U) \hookrightarrow \square^{\delta'}_{I'J'}\times U$$

is valid.

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We let $\square^{*,\delta}_{IJK} = \square^{\delta}_{IJK} \cap \mathbb{C}^N \times U$. For a refinement map $r_J : \mathfrak{N}^{\delta^2} \hookrightarrow \mathfrak{N}^{\delta^1}$, we define a map $r_J(*) : \mathfrak{N}^{*,\delta^2}(\mathbb{C}^N \times U) \hookrightarrow \mathfrak{N}^{\delta^1}(Z_{n,N})$ by

$$(3.1. \quad) \qquad r_J(*)(\square^{*,\delta}_{I,J,K}) = r_J(\square^{\delta}_{I,J,K})$$

Moreover define cochains $\varphi^{*\,q-1}_2, \varphi^{*\,q}_2 \in C^*(N(\mathfrak{N}^{*}))$ by the equations

$$(3.1. \quad) \qquad \varphi^{*}_2(\cap_\nu \square^{\delta}_{I^2_\nu J^2_\nu K^2_\nu} \cap U) = \tilde{\varphi}_2(\cap_\nu \square^{\delta}_{I_\nu J_\nu} \times U)$$

Then we obtain the following equation

$$(3.1 \quad) \qquad r^*_J(*) \cdot r^*_K(\varphi^q) = \delta_{\check{C}ech}(r_J(*)(\varphi^{q-1}_1)) + r_J(*)\varphi^q_1$$

$$= \delta_{\check{C}ech}(r_J(*)(\varphi^{q-1}_1)) + \delta_{\check{C}ech}(\varphi^{*\,q-1}_2) + \varphi^q_2(*)$$

Finally apply (E.L.I) to $\tilde{\varphi}^q_2(I)$. Then we obtain a cochain $\tilde{\varphi}^{q-1}_3(I) \in C^{q-1}(N(\mathfrak{N}'_{1/2\varepsilon}(\mathbb{C}^N \times U)), \mathcal{O})$ so that

$$(3.1. \quad) \qquad \tilde{\varphi}^q_2(I) = \delta_{\check{C}ech}(\tilde{\varphi}^{q-1}_3(I))$$

is valid. Quantitative properties of $\tilde{\varphi}^{q-1}_3 : (\tilde{3}), (\mathfrak{E})$ are determi

ned by (E.L.I)
ned by (E.L.I) . Finally we proceed as follows :

Define a covering $\mathfrak{N}^{\delta^2,*}_\varepsilon(\mathbb{C}^N \times U)$ of $\mathbb{C}^N \times U$ to be

$$\mathfrak{N}^{\delta^2,*}_\varepsilon(\mathbb{C}^N \times U) \; (\subset \mathfrak{N}^{\delta,*}(\mathbb{C}^N \times U)) = \{ \square^{\delta^2,*}_{I,J,K}, \; Pr_x(\square^{\delta^2}_{I,J}) \subset \square_{\frac{1}{2}\varepsilon}(P_I) \}.$$

The map $i(*)$ stands for the injection map : $i(*) : \mathfrak{N}^{\delta^2,*}_\varepsilon(\mathbb{C}^N \times U) \hookrightarrow \mathfrak{N}^{\delta^2,*}(\mathbb{C}^N \times U)$. Then, from ( ), we obtain the following equation

$$(\qquad) \qquad (i(*))^* \cdot r^*_J(*) \cdot r^*_K(\varphi^q) = \delta_{\check{C}ech}(i(*)^*(\varphi^{q-1}_1)) + \delta_{\check{C}ech}(i^*(*)(\varphi^{*\,q-1}_2))$$

$$+ \; i(*) \cdot (\varphi^q_2(*))$$

Moreover define a cochain $\varphi_3^{q-1} \in C^{q-1}(N(\pi_2^{\delta,*}(\mathbb{C}^n \times U)), \mathcal{O})$ by $\varphi_3^{q-1}(\sigma_{I,J,K}^{\delta,*}) = \varphi^{q-1}(I)$.

Then we know the relation $\lambda(x) \cdot \varphi_2^q(x) = \delta_{\text{čech}}(\varphi_3^{q-1})$.

Then, finally we obtain an equality of the following form.

(3.1.) $\{\lambda(x)\}^* \cdot \pi_J^* \cdot \pi_K^* (\varphi^q) = \delta_{\text{čech}}(\varphi_2^{q-1,*})$,

where $\varphi_2^{q-1,*}$ is an element in $C^{q-1}(N(\pi_2^{\delta,*}(\mathbb{C}^n \times U)), \mathcal{O})$ of algebraic growth ($\tilde{\alpha}_1$, $\tilde{\alpha}_2$). Quantities ($\tilde{\alpha}_1$, $\tilde{\alpha}_2$) are clearly of the form ( ).

our assertion (of both lemmas 3.1.1 and 3.1.1' follow from (3.1.) easily. q.e.d.

n. . Next we consider a couple (U, D) of a bounded domain U in C^n and a divisor D in U defined by h = 0 . A canonical lifting $\tilde{U}-D$ of U − D in $\sum_{n,1} = C^n \times U$ is a divisor in $C \times U$ defined by $S = 1 - w\,h(x)$. Fix a point P in D and by P we mean a point near P . Now we show the following

Lemma3.2.3. There exists a datum ( ) depending on (D, h) only in such a manner that the following are valid .

For a small positive number r, define an open covering $\mathfrak{M}^{\delta}(\Delta_{\gamma}(P)-D)$ by $\mathfrak{M}^{\delta}(\overline{\Delta_{\gamma}(P)}-D)=\{\Delta_{\delta}(Q:D)\}_{Q\in\overline{\Delta_{\gamma}(P)}-D}$. An element $\varphi^{q}\in C^{q}(N(\mathfrak{M}^{\delta}(\overline{\Delta_{\gamma}(P)}-D)),\mathcal{O})$ is of algebraic growth (∂_1,∂_2) if φ^{q} is estimated as

$$(3.1.\ )\qquad |\varphi^{q}(Q)|\leqq \partial_1\cdot d(Q:D)^{-\partial_2}$$

Now we show the following

<u>Lemma 3.1.3. There exists a datum $(r_0,\mathcal{L},M_1,M_2,P_1,P_2,L)$, depending on (D,h) only so that the following are true.</u>

<u>(3.1.) For a given cocycle $\varphi^{q}\in C^{q}(N(\mathfrak{M}^{\delta}(\overline{\Delta_{\gamma}(P)}-D)),\mathcal{O})$ of algebraic growth (∂_1,∂_2) there exists a cochain $\varphi^{q-1}\in C^{q}(N(\mathfrak{M}^{\delta'}(\Delta_{\gamma'}-D)),\mathcal{O})$ of algebraic growth $(\partial_1',\partial_2')$ so that</u>

$$(3.1.\ )_1\qquad \varphi^{q}=\delta_{\check{C}ech}(\varphi^{q-1})$$

<u>and</u>

$$(3.1\ )_2\qquad r'=M_1(r),\quad \delta'=\mathcal{L}(\delta),\quad \partial_1'=M_2(r)^{-1}\cdot P_1(\partial,\delta)e^{P_2(\partial,\delta_2)},\quad \partial_2'=L(\partial_2+\delta)$$

<u>hold, so far as $r<r_0$ is valid.</u>

(Remark) In the above our assertion of independeseness of data $(\delta'),(\partial')$ on points P is important. This will be called uniformity of v.a.

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Proof . The above fact is easily shown by means of lemma 3.1.1. and the inequamity of Lojasiewicz .

(i) Note that the derivative $\partial w/\partial x \cdot w = 1/h(x)$ is estimated as

(3.1.) $\left|\frac{\partial w}{\partial x}(\theta)\right| \leq a_1 \cdot d(\theta, D)^{-a_2}$: a_1, a_2 are depending on D only.

For a point $\tilde{Q}$ on $\widetilde{U - D}$, we consider a neighbourhood $\Delta_{r,r^2}(\tilde{\theta}) = \Delta_r(h\theta) \times \{w : |w - w(\theta)| < r^2\}$ rather than $\Delta_r(\tilde{\theta})$.

A neighbourhood $\Delta_{r,r'}(\tilde{\theta})$ is adequate if $\widetilde{U-D}$ is parametrized on $\Delta_{rr^2}(\tilde{\theta})$ as $w = 1/h(x)$: If $\Delta_{rr^2}(\tilde{\theta})$ is adequate then there exist a unique projection $\tilde{\pi} : \Delta_{rr^2}(\tilde{\theta}) \to \Delta_{r,r^2}(\tilde{\theta}) \cap \widetilde{U-D}$ sot that $\pi(\theta) = \pi \circ \tilde{\pi}(\theta')$ holds . From (3.1) it is clear that the following is valid.

(3.1.) There exists a couple (δ_1^0, δ_2^0) , a monomial M in such a manner that the statement below is true .

(3.1.) If $r < \tilde{\delta}_1^0 |w(\theta)|^{\tilde{\delta}_2^0}$ then $\Delta_{r r'}(\tilde{\theta})$ is adequate : $r' = M(r)$. Here (δ_1^0, δ_2^0) , M are depending on only.

Take a point P (near P_0) and consider a neighbourhood $U_r(P)$. Given a cocycle $\varphi^q \in N(\mathcal{U}^1(U_r(P)-D)\mathcal{O})$ of algebraic growth (∂_1, ∂_2), we associate a cochain $\tilde{\varphi}^q \in C(N(\tilde{\mathcal{U}}...$

φ) as follows .

$$(3.1.\quad)\quad \tilde{\varphi}^{g}(\cap_{\nu=1}^{g+1}\Delta_{\delta,\delta}(\tilde{Q}_{\nu})) = \Pi^{*}\cdot\varphi^{g}(\cap_{\nu=1}^{g+1}\Delta_{\delta}(Q_{\nu})) ,$$

where $\tilde{Q}$'s are poins on $\widetilde{U-D}$ while Q's are projections of $\tilde{Q}$'s : $\tilde{\delta}=\mathcal{L}(\delta)$ with $\mathcal{L}$ depending on (D,h) only.

From the inequality of Lojasiewicz , we obtain estimations

$$(3.1.\quad)\quad |\tilde{\varphi}^{g}| \leq \partial_1'\cdot|w|^{\partial_2'} : (\partial_1', \partial_2') = \mathcal{L}(\partial_1, \partial_2)$$

with ($\mathcal{L}$) depending on (D, h) only .

Fix neighbourhoods $T_{\delta,\delta}(\tilde{Q}) : Q\in\widetilde{U-D}$ and let $T_{\delta,\delta}(\widetilde{U-D})$ be the union $\bigcup_{Q\in\widetilde{U-D}} T_{\delta,\delta}(Q)$

Then .it is easy to see the following

(3.1.) There exist maps $\mathcal{L}^1, \mathcal{L}^2$ so that if Q is in $T_{\delta^1}(\widetilde{U-D}) : \delta^1=\mathcal{L}(\delta)$ then there exists a point $\tilde{Q}\in\widetilde{U-D}$ with a property : $\Delta_{\delta}(\tilde{Q}) \supset \Delta_{\delta^2}(Q) : \mathcal{L}^i(\delta)=\delta^i$.

On the ptherhand it is also easy to see the following

(3.1.) There exists a map $\hat{\mathcal{L}}$ with which we know the satement below .

( ) For a point $Q\in T_{\delta}(\widetilde{U-D}) - T_{\delta'}(\widetilde{U-D})$,

$T_{\tilde{\delta}^1}(Q\quad) \cap T_{\delta^1}(\widetilde{V-D}) = \phi$: $\hat{\delta}\hat{\delta}(\delta)$ and moreover, at any point $Q' \in T_{\tilde{\delta}^1}(Q)$, an estimation of the form : $\delta(Q) \geq \hat{a}_1 \cdot d(Q)^{-\hat{a}_2}$.

Here ($\hat{a}_1$, $\hat{a}_2$) is given in a form : $(\hat{a}_1, \hat{a}_2) = \mathcal{L}^*(\delta_1', \delta_2')$.

In the aboves one can take ($\mathcal{L}$, $\mathcal{L}$, $\mathcal{L}$) to be absolutely independent while is depending on $(D, \mathcal{L})$ only

After the above observation, we proceed as follows :

(i) For a given cocycle of algebraic growth (,), define a cochain in the following manner .

(3. 1.)$_1$ For points $\tilde{Q} \in \widetilde{V-D}$, fix a set of neighbourhoods $T_\delta(\tilde{Q})$: $\tilde{\delta} = \mathcal{L}(\delta)$,

(3. 1)$_2$ For a set $T_{\tilde{\delta}}(\widetilde{V-D}) - T_\delta$, define a map i : $Q \in T_{\tilde{\delta}^1}(\widetilde{V-D}) \to \tilde{Q} \in T_{\tilde{\delta}}(\widetilde{V-D})$: $\tilde{\delta}^1 = \mathcal{L}^1(\tilde{\delta})$, so that $T_{\tilde{\delta}^2}(Q) \subset T_{\tilde{\delta}}(Q)$ holds ; $\tilde{\delta}^2 = \mathcal{L}^2(\tilde{\delta})$

(3. 1)$_3$ For a point Q' in $\Sigma_{1,n} - T_{\tilde{\delta}^1}(\widetilde{V-D})$, define a neighbourhood $T_{\tilde{\delta}^3}(Q')$ so that (3. 1.) is valid with $(\tilde{\delta}^3)$ in $T_\delta(\widetilde{V-D})$. Also we can assume that $T_{\tilde{\delta}^3}(Q'') \cap T_{\tilde{\delta}^2}(Q) = \phi$ if $Q'' \in \Sigma_{1,n} - T_\delta(\widetilde{V-D})$, $Q \in T_{\delta^1}(\widetilde{V-D})$.

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Of course we can assume that δ, δ^1, δ^2, δ^3 are depending on (k, D) only and is particularilly independent of P : Define a cochain $\hat{\varphi}^q \in C^q(N(\mathfrak{U}^{\tilde{\delta}_4}(Z_{r,N})), \mathcal{O})$ ($\tilde{\delta}_4$ = min ((δ_2), $(\tilde{\delta}_3)$)) in the following manner.

$$(3.1.\ )_1 \quad \hat{\varphi}^q(Q_0, \dots, Q_q) = i^*\hat{\varphi}^q(Q_0, \dots, Q_q) \text{ if } Q_0, \dots\dots, Q_q \in \mathcal{I}_{\delta'}(\widehat{U-D}),$$

$$(3.1.\ )_2 \quad \hat{\varphi}^q(Q_0, \dots, Q_q) = 0 \text{ , otherwise .}$$

Take the coboundary $\hat{\varphi}^{q+1} = \delta_{\check{c}ech}(\hat{\varphi}^q)$ of $\hat{\varphi}^q$. The cocycle condition imposed on φ^q implies that $\hat{\varphi}^{q+1}$ is zero on $\widehat{U-D}$. More precisely the following equations are valid.

$$(3.1.\ )_1 \quad \hat{\varphi}^{q+1}(Q_0, \dots Q_{q+1}) = 0 \cdot S \text{ if } Q_0, \dots, Q_{q+1} \text{ are in } \mathcal{I}_{\delta'}(\widehat{U-D})$$

$$(3.1.\ )_2 \quad \hat{\varphi}^{q+1}(Q_0, \dots, Q_{q+1}) = \hat{\varphi}'^{q+1}(Q_0, \dots Q_{q+1}) \cdot S \text{ if a point } Q_0 \text{ is in}$$

but other points $Q_1, \dots, Q_q$ are outside $(\widehat{U-D})$.

$$(3.1.3) \quad \hat{\varphi}^{q+1}(Q_0, \dots Q_{q+1}) = 0 \cdot S \text{ other wise .}$$

In the above $\hat{\varphi}'^{q+1}_{(Q_0, \dots Q_{q+1})} \in \tilde{\mathcal{J}}^{\tilde{\delta}'}$ with $(\tilde{\delta}'_1, \tilde{\delta}'_2)$ = $(\min(\tilde{\delta}_1, \tilde{\delta}_2) R(Q_0)e^{-c}, \mathcal{I}(\delta_2, \delta_2))$
It is clear that the collection $\{\hat{\varphi}'^{q+1}_{(Q_0, \dots Q_{q+1})}, 0\}$ as above defines a cocycle $\hat{\varphi}^{q+1} \in C^{q+1}(N(\hat{\mathfrak{U}}^{\tilde{\delta}_4}(Z_{r,N})), \mathcal{O})$. Apply lemma 3.1.1. to decomose $\hat{\varphi}^{q+1}$ as

$$(3.1.4) \quad \hat{\varphi}^{q+1} = \delta_{\check{c}ech}\, \hat{\varphi}^{q}$$

Then $\bar{\varphi}^q = \hat{\varphi}^q + (\hat{\varphi}^q) \cdot S$ is again cocycle and we apply lemma 3.1.31. once more . We obtain the equation

$$(3.1.5) \quad \bar{\varphi}^q = \delta_{\check{c}ech}(\bar{\varphi}^{q-1}) \quad 3-1-5$$

Restricting the above equation to $\widetilde{U - D}$ and using the biholomorphic property of $\pi : \widetilde{U-D} \to U - D$, our proof is finished .

Next we discuss asymrtotic behaviors concerning coherent sheaves : We start with a domain U in C and a variety V in U . Assume that the dimension of V is $N - \ell$. Moreover we assu e that a set of functions $(g_1, \ldots\ldots, g_\ell)$ vanishing on V and a su briety V of V is given so that the following condition

$$(J) \qquad \frac{\partial(g_1, \ldots, g_\ell)}{\partial(x_{N-\ell+1}, \ldots, x_N)} \neq 0$$

is valid at each point $P \in V - V'$.

In a small neighbourhood $T_\delta(V : V')$ of $V - V$, a holomorphic function h in $T_\delta(V : V')$ is expanded in the form

$$(3.1) \qquad h = \Sigma_J h_J \cdot g^J$$

where h_J's are holomorphic functions on $V - V'$.

We maan by $\nu(h)$ the order of h w.r.t. V :

$\nu(h) = \min_{\nu = |J|} h_J : h_J \neq 0$. We say that h is of mer morphic type if h_J's are meromorphic on $V - V'$ (More strictly ,

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there exists a divisor D in U so that h_J 's are restrictions of meromorphic functions in U - D on V)

Such a function h is of growth (∂_1, ∂_2) if the following estimation

$$(3.1.) \qquad h(\alpha) \leqq \partial_1 \cdot d(\alpha, V')^{-\partial_2}$$

is valid. If a function h is of meromorphic type and if an expansion of the form (3.1.) is valid with a suitable (∂_1, ∂_2), we say thtt h is of type (a) .

Given a matrix K of type (a) (i.e. whose coefficients are of type (a)) , we examine a simple property of ~~matrices K of~~ An ellementary observation which we make here is as follows : By $\mathcal{O}_P^*$ we mean the sheaf over V - V whose stalk at is characterized by $\mathcal{O}_P^* = \{ h_P^i =$ the germ of a holomorphic function in $T_\delta(V : V')$. The sheaf $\mathcal{R}$ is the kernel sheaf over V - V of the equation $K \cdot f = 0$.

The first simple assertion which we use later is as follows

(3.1) There exists a couple (of a subvariety $V'' \subset V$, polynomials P_1, P_2 and a linear function L (all these data are depending on K only) with which we know the following

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(3.1) There exists a finite set of functions

$\eta_1, \dots, \eta_s \in \Gamma(\Gamma(V: V^0)$ of type (a) which generate the stalk at each point P in V- V^0 .

(3.1) $\eta_1, \dots, \eta_s$ are estimated as

$$|\eta_j(Q)| \leq \delta_1' \cdot d(Q, V'')^{-\delta_2'} : \delta_1' = P_1(\delta, \delta_1^0) e^{P_2(\delta_2^0, \delta)}, \delta_2' = L(\delta, \delta_2^0)$$

with .

The above statement is concerned with a simple property of the sheaf . We ask an equation : K ξ = η : with a given element η . Concerning such a problem we use the following simple

(3.1) There exists a datum $((\delta), V_1''$, M_1 , M_2 , P_1 , P_2, L) with which the following are valid.

(3.1) For each point $P \in V - V''$ and an element $\eta \in \mathcal{O}_P^i$, defined already in $\Delta_{\gamma}(\Delta(P), \mathcal{O}^i)$: $\gamma < \delta(P)$ and satisfying the relation $\eta^i \in R(\mathcal{O}_P^i)^t$, we find an element $\xi^i \in \Gamma(\Delta_{\gamma'}(P), \mathcal{O}^i)$

with the following properties

(3.1) K ξ^i = η^i ,

((3.1) $|\xi^i| \leq \delta_1'' \cdot d(P, V')^{-\delta_2''} \times |\eta|$

with $\gamma' = M_1(\gamma)$, $\delta_1' = M_2(\gamma)^{-1} P_1(\delta) e^{P_2(\delta_2^0)}$, $\delta_2'' = L(\delta_2')$)

(For a couple)

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The above two statements are easily shown along standard lines to argue syxygy or coherentnessof holomorphic function ; namely let k be the number of the equation of K (= the length of K), n be the dimension of V and i be the dimension associated to T^i ( ) . It is easy to see (3.1) and (3.1) inductively on these indices, and we shall omit arguements . Assume that a divisor D in U is given and that a matrix K whose coefficients are meromorphic in U (with the pole $D: h=0$) . Further-more assume that a variety V ($\supset$ D) is given .

Now we use the following elementary

Proposition . 3.1.2. There existsa datum ((δ^0) , Q^0 M) , depending on (D,h,K), with which the following are valid.

(( ) For a point Q and for an element $g^i \in \Gamma(\Delta_r(Q) : \mathcal{O}^i)$; $r < \delta_1^0\, d(Q, V)^{\delta_2^0}$ satisfying the condiyon

$$f_Q^i \in K \cdot \mathcal{O}_Q^k$$

there exists an element $g_Q \in \Gamma(\Delta_r(Q), \mathcal{O}^k)$ so that

$$(3.i\ )\qquad K \cdot g_Q = f_Q .$$

and

$$(\quad)\qquad |g_Q| \leq M(r) \cdot \delta_1^0 \cdot d(Q : V)^{-\delta_2^0} \cdot |f_Q|$$

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hold .

Proof The above assertion follows immeadiately from ( ), ( ) and ( ).

Let us consider a matrix whose coefficients are meremorphic functions in U (with a pole D), and let

$$(\quad) \qquad 0 \to \mathcal{O}^{l_\mu} \xrightarrow{K_{q-1}} \cdots\cdots \to \mathcal{O}^{l_1} \xrightarrow{K} \mathfrak{F} \to 0$$

be a resolution of $\mathfrak{F}$. A cochain $\varphi^q \in C^q(N(\hat{\mathfrak{R}}^s(\Delta_s(D)\cdot D)), \mathfrak{F})$ is called of growth ($\mathfrak{T}_1$, $\mathfrak{T}_2$) if has an expression

$$(3 \quad) \qquad |\varphi^q(Q_0, \cdots, Q_q; \mathfrak{a})| \leq \mathfrak{T}_1 \cdot d(\mathfrak{a} : D)^{-\mathfrak{T}_2},$$

where $\varphi^q(Q_0, \cdots Q_q)$ is in $\Gamma(\hat{\mathfrak{R}}^s(Q_0, \cdots Q_q), \mathcal{O})$ so that, at each point $\mathfrak{a} \in \bigcap_{\mu=0}^{q} \Delta_s(Q_\mu)$, the relation $\varphi^q(Q_0, \cdots Q_q; \mathfrak{a}) \in \mathfrak{F}_\mathfrak{a}$ holds.

Then we show the following

Theorem 3.1. (V.a) There exists a datum (r_0, s_0, $\mathcal{L}$, P_1, P_2, L) depending on (more strictly on a resolution of) only, so that the following are valid.

(3 1) For a given cocycle $\varphi^q \in C^q(N(\mathfrak{R}(\Delta_s \cdot D)), \mathfrak{F})$ algebraic growth ($\mathfrak{T}_1$, $\mathfrak{T}_2$), there exists a (q-1)-cochain $\varphi^{q-1} \in C^{q-1}(N(\mathfrak{R}^{s'}(\Delta_{s'} \cdot D)), \mathfrak{F})$ of algebraic growth ($\mathfrak{T}_1'$, $\mathfrak{T}_2'$) so that the equality

$$(3 \ 1 \quad) \qquad \varphi^q = \delta_{Cech}(\varphi^{q-1}) \qquad 3-1-58$$

and estimations

(3.1.) $\tilde{\partial}_1' = M(x)^{-1} \cdot P_1(\delta_1, \tilde{\delta}_1) e^{P_2(\tilde{\delta}_2, \tilde{\delta}_2)}$, $\tilde{\partial}_2' = L(\partial_2, \delta_2)$

are valid , so far as $r < r_0$, $\delta < \delta_0$; P is a point near P_0.

proof . this assertion is easily proven inductively on the length of a resolution of . Keys are to use proposition 3.1.2. in order to reduce the present problem to lemma 3.1.1.. Because oue reduction is done along standard arguements we do not enter into details.

We derive an immeadiate cororally of the above theorem. Let us start with a datum (X, D, $\mathcal{O}_X$, (h)) so that the condition ( ) is true. For a point P near P_1, an element will be called of groth (α_1, α_2) so if an estimation

( ) $|\varphi^{s}(\theta, \ldots, \theta_s; \alpha)| \leqq \alpha_1 \cdot d(\theta : D)^{-\alpha_2}$

is valid . The following is derived from theorem 3.1.L.

Cororrally 3.1.1. : there exists a datum $(r_0, \delta_0, M, L, P_1, P_2, T_r)$ with which assertions ( ) , ( ) for the sheaf $\mathcal{O}_X$ is valid.

proof. Here we reduce the assertions in this to theorem 3.1.. This part is quite similar to prodedures in lemma 3.1.2. We choose maps $\alpha, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ depending on $(X, D, \mathfrak{g}_t)$ only with which the following are valid

(2.1.) If P is in $N_{\delta^1}(X, \partial X)$ then there exists a point $\tilde{P} \in X$ so that $N_{\delta^2}(\tilde{P}) \ni N_{\delta}(P)$ holds.

(2.1.) If a point P is not in $N_{\delta}(X, \partial X)$ then $N_{\delta^4}(P; \partial)$ is outside $N_{\delta^5}(X : D \cap X)$.

In the above , $(\delta), (\delta^2), (\delta^3), (\delta^4), (\delta^5)$ are determined as $(\delta^i) = \alpha_i(\delta)$. Define a cochain $\varphi^{*q} \in C^q(M(X), \theta)$ as in :

(2.1.) If $Q_0, \ldots, Q_q$ are in $N_{\delta}(X, X \cap D)$,

$$\varphi^{*q}(Q_0, \ldots Q_q) = \varphi^q(\tilde{Q}_0, \ldots, \tilde{Q}_q) ,$$

( ) φ^{*q} = 0 , otherwise.

Remark that φ^{*q} is zero on X .

Then, from proposition 3.1. , we obtain an expression

( ) $\varphi^*(Q_0, \ldots Q_q) = \sum_j a_j f_j$,

where a_j is in $\Delta_{\delta}(Q_0, Q_i, \partial X)$ of growth ($\tilde{\partial}_1'', \tilde{\sigma}_2''$) .

Here $(\delta''), (\partial'')$ are determined as $(\delta'') = \alpha''(\delta)$, $(\partial'') = P_1''(\partial, \delta_1) \, e^{P_2''(\partial, \delta_1)}$, $\partial = \partial(\delta)$

with α'', P'' 's depending on $(X, D, \mathfrak{g}_t)$ only.

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Applying the theorem 3.1. and following the procedure in the lemma 3 .1.2, we knee easily our assertion. q.e.d

We have discussed the cases lemma 3.1.2, corollary 3.1. because these two results are made use of in 3.2. we discuss quickly an another result ~~which can~~

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3. 2 . Cohomology with algebraic division and algebraic growth condition.

The arguements given in the section 3. 1 provides a sufficient analytic basis when we are concerned with a theory of differential forms for a triple ($\bar{X}$, V, D) where V - D is smooth. On the otherhand, if we ask possible theories of differential forms for the triple (X, V, D) in which V - D is (generally) not smooth, we should ask more materials on analytic sides. The purpose of this section is to propose a cohomology theory for this purpose and discuss our proposed subjects in details. A new essential feature of this section is that the powers of the ideal sheaf $\mathcal{J}$ of V * enters into our problems besides the asymptotic behaviors w.r.t. the pole divisor D.

* In our actual arguements, the powers of ideals are discussed in a somewhwt modified form.

$\{ g^{m,i-1}_{\nu_1 \cdots \nu_i} \}_{1 \le \nu_1 < \cdots < \nu_i \le \ell}$. (i.e. $\mathfrak{X}^{m,i-1}_P = \mathcal{O}_P \{ (g^{m,i-1}_{\nu_1 \cdots \nu_i})_{1 \le \nu_1 < \cdots < \nu_i \le \ell} \}$.

Especially the submodule $\mathfrak{X}^{m,0}_P$ of $\mathcal{O}_P$ stands for the one : $\mathfrak{X}^{m,0}_P = \mathcal{O}_P (g_1^m, \ldots\ldots, g_\ell^m)$ .Moreover, we put $\mathfrak{X}^{m,\ell}_P = \mathcal{O}_P \{ g_{1 \cdots \ell}^{m} \} \cong \mathcal{O}_P$. We define

$\mathcal{O}_P$ - homomorphisms K (m, i) (i =1, ..., ℓ) by the following formulas.

$$(3.2.1) \quad K(m, i) : \mathfrak{f}^i = (f_{\nu_1 \cdots \nu_i})_{1 \le \nu_1 < \cdots < \nu_i \le \ell}$$

$$\longrightarrow K(m,i)(\mathfrak{f}^i) = \sum_{1 \le \nu_1 < \cdots < \nu_i \le \ell} f_{\nu_1 \cdots \nu_i} \, g^{m,i-1}_{\nu_1 \cdots \nu_i}$$

It is quite easy to see that the equation K (m, i-1) · K (m, i) = 0 holds. We will be concerned here with submodules $\mathfrak{X}^{m,i}$'s defined above. Let $\mathfrak{f}^i = (f_{\nu_1 \cdots \nu_i})_{1 \le \nu_1 < \cdots < \nu_i \le \ell}$ be a vector in $\mathcal{O}_P^{\binom{\ell}{i}}$ which satisfies the following equation.

$$(3.2.2)_1 \qquad K(m, i) \cdot \mathfrak{f}^i = 0,$$

We express $f_{\nu_1 \cdots \nu_i}$'s in the following explict form.

$$(3.2.2)_2 \qquad f_{\nu_1 \cdots \nu_i} = f^{(m)}_{\nu_1 \cdots \nu_i} \cdot g_1^m + \sum_{s=0}^{m-1} f^{(s)}_{\nu_1 \cdots \nu_i} \cdot g_1^s$$

where $f^{(s)}_{\nu_1 \cdots \nu_i}$ ($s = 0, \ldots\ldots, m-1$) does not contain the term g_1 at all .

Define a vector $\mathfrak{y}'^{i+1}$ by

$$(3.2.3)_1 \quad \mathfrak{y}'^{i+1}_{\nu_1'\cdots\nu_{i+1}'} = \begin{cases} f^{(m)}_{\nu_1\cdots\nu_i}, & \text{if } (\nu_1',\cdots\nu_{i+1}') = (1,\nu_1,\cdots,\nu_i) \\ 0, & \text{otherwise.} \end{cases}$$

We put $\mathfrak{y}'^{i} = \mathfrak{y}^{i} - K(m, i)\cdot\mathfrak{y}'^{i+1}$ and write the vector $\mathfrak{y}'^{i}$ in the form : $\mathfrak{y}'^{i} = (\mathfrak{f}_{\mu_1\cdots\mu_i})$

It is clear that the equation

$$(3.2.3)_2 \quad \mathfrak{f}'_{\nu_1\cdots\nu_i} = \sum_{s=0}^{m-1} f^{(s)}_{\nu_1\cdots\nu_i}\cdot g_1^{s}$$

holds for $2\leqq \nu_1 < \cdots \nu_i \leqq l$.

Taking account into the fact that $(\mu_1', \ldots\ldots, \mu_{i-1}')$-component $(2\leqq\mu_1' < \cdots \mu_{i-1}'\leqq l)$ of the terms $\mathfrak{f}'_{\mu_1'\cdots\mu_{i-1}'}\cdot\mathfrak{g}_{1,\mu_1'\cdots\mu_{i-1}'}$ are divisible by g_1^{m}, we know that the following equations are valid.

$$(3.2.3)_3 \quad \sum_{2\leqq\nu_1<\cdots<\nu_i\leqq l} \mathfrak{f}'_{\nu_1\cdots\nu_i}\cdot\mathfrak{g}_{\nu_1\cdots\nu_i} = 0.$$

Note that the above equality is equivalent to

$$(3.2.3)_4 \quad \sum_{2\leqq\nu_1<\cdots<\nu_i\leqq l} \mathfrak{f}^{s}_{\nu_1\cdots\nu_i}\cdot\mathfrak{g}_{\nu_1\cdots\nu_i} = 0 \quad (s=0,\ldots, m-1)$$

On the otherhand, we note that if a vector $(g''_{\nu_1 \cdots \nu_j})_{1 \le \nu_1 < \cdots < \nu_j \le \ell}$

has the following properties

$$(3.2.4) \quad \sum_{1 \le \nu_1 < \cdots < \nu_j \le \ell} g''_{\nu_1 \cdots \nu_j} \cdot g_{\nu_1 \cdots \nu_j} = 0, \qquad g''_{\nu_1 \cdots \nu_j} = 0 \; (2 \le \nu_1 < \cdots < \nu_j \le \ell)$$

then $(g''_{\nu_1 \cdots \nu_j})_{1 \le \nu_1 < \cdots < \nu_j \le \ell} = 0$.

From the relations $(3.2.3)_2$, $(3.2.3)_4$ and from the above remark, we know that inductive arguements (on the number ℓ of the functions g_μ's) lead to the following conclusion.

(3.2; I) The solution spaces of the equation $K(m, i)\, f^{i} = 0$: are spanned by the vectors $\{ g^{m,i}_{\nu_1 \cdots \nu_{i+1}} \}_{1 \le \nu_1 < \cdots < \nu_{i+1} \le \ell}$.

Thus we obtain the following exact sequence

$$(3.2 : I)' \quad 0 \to \mathcal{O}_P^{\binom{\ell}{\ell}} \xrightarrow{K(m,\ell)} \mathcal{O}_P^{\binom{\ell}{\ell-1}} \longrightarrow \cdots \mathcal{O}_P^{\binom{\ell}{2}} \xrightarrow{K(m,2)} \mathcal{O}_P^{\binom{\ell}{1}} \xrightarrow{K(m,1)} \mathfrak{X}^{m,0} \to 0$$

This sequence will be called a canonical syzygy attached to $\mathfrak{X}^{m,0}$.

(Remark) (The exact sequence (3.2.I)' belongs, the author believes, to the realem of well known facts. Because, in an quantitative arguement done soon below, we make use of the above procedures in details, we made a detailed arguement as above.)

n°12 Let us continue elementary arguements in local levels. In this case quantitative properties will be included.

Given a domain $\mathfrak{D}'$ in $\mathbb{C}^{\ell'} = (z_1', \dots, z_{\ell'}')$ and a polydisc $\Delta_r(0;(g)) = \{(g) = (g_1, \dots, g_\ell); |g_\mu| \leq r\ (\mu = 1, \dots, \ell)\}$ in $\mathbb{C}^\ell$, we assume that additional holomorphic functions $(g_{\ell+1}, \dots, g_{\ell+\tilde{\ell}})$ in $\mathfrak{D}' \times \Delta_r(0:(g))$ are given. Furthermore we assume that $g_{\ell+\mu}$'s vanish on the manifold defined by $g_1 = \dots = g_\ell = 0$ in $\mathfrak{D}' \times \Delta_r((0);(g))$. We fix the following explict expressions of $g_{\ell+j}$'s.

$$(3.2.5)_1 \qquad g_{\ell+j} = \sum_{\mu=1}^{\ell} g'_{j\mu} \cdot g_\mu, \text{ where } g'_{j\mu}\text{'s}$$

are holomorphic in $\mathfrak{D}' \times \Delta_r(0:(g))$.

Let B be a positive number with which the inequality

$$(3.2.5)_2 \qquad B \geqq |g'_{j\mu}|, \text{ holds}$$

for each $g'_{j\mu}$ in $\mathfrak{D}' \times \Delta_r(0,(g))$.

In an analogous way to (n° 11), we let $\mathfrak{F}'^{m,\ell-1}_{\nu_1 \dots \nu_\lambda}$ $(1 \leqq \nu_1 < \dots < \nu_\lambda \leqq \ell + \tilde{\ell})$ be the vector in $\Gamma(\mathfrak{D}' \times \Delta_r((0);(g)), \mathcal{O}$

5)

which is characterized in the following manner.

$$(3.2.6)\quad \mathfrak{Y}'^{m,i-1}_{\nu_1\cdots\nu_i}(1\le\mu_1\le\cdots\le\mu_{i-1}\le\ell) = \begin{cases} 0, & \text{if } (\mu_1\cdots\mu_{i-1}) \notin (\nu_1\cdots\nu_i) \\ (-1)^{k-1}\cdot g_{k}\,;\ (\mu_1\cdots\hat{\nu}_k\cdots\nu_i) \end{cases}$$

Also the submodule $\mathfrak{X}'^{m,i-1}$ of $\mathcal{O}^{\binom{\ell+\ell}{i-1}}$ will be defined to be the one spanned by $\mathfrak{Y}'^{m,i-1}_{\nu_1,\cdots,\nu_i}$'s .

Moreover, we define sheaf homomorphisms $K(m,i)$ from $\mathcal{O}^{\binom{\ell+\ell'}{i}}$ to $\mathcal{O}^{\binom{\ell+\ell'}{i-1}}$ by

$$(3.2.7)\quad K'(m,i)(f_{\nu_1\cdots\nu_i})_{1\le\nu_1<\cdots<\nu_i\le\ell} = \sum_{1\le\nu_1<\cdots<\nu_i\le\ell} f_{\nu_1\cdots\nu_i}\cdot\mathfrak{Y}'^{m,i-1}_{\nu_1\cdots\nu_i}$$

It is clear that $K'(m,i)\circ K'(m,i+1)=0$ is valid.

But it is not true in general that the exact sequence $(3.2.I)'$ is valid for homomorphisms $K(m,i)'$ s .

We ask a simple condition in order that a given vector $\mathfrak{f}^{i-1}\in\Gamma(\omega\times\bar{\Delta}(m;\delta),\mathcal{O}^{\binom{\ell}{i-1}})$ is in $\Gamma(\omega\times\bar{\Delta}(m;\delta),\mathfrak{X}^{m,i-1})$. Assume that a vector $\mathfrak{f}^{i}=\{f_{\nu_1\cdots\nu_i}\}_{1\le\nu_1<\cdots<\nu_i\le\ell}\in\Gamma(\omega\times\Delta(m;\delta),\mathcal{O})$ satisfies the following two conditions.

$$(3.2.7)_1\quad K'(m,i)(\mathfrak{f}^{i})=0,\ (i=1,\cdots,\ell-1),$$

$$(3.2.7)_2\quad |f_{\nu_1\cdots\nu_i}|_{|\delta_1|=\cdots=|\delta_\ell|=d} \le \ell\cdot d^{d_0}\ ;\ \ell, d_0\in\mathbb{R}^+$$

Now we show the following

Proposition. 3. 2. 1. There exist positive numbers $(\beta'_{\mu}) = (\beta'_{\ell,\tilde{\ell},i,\mu})$ $(\mu = 1,2,3,4)$ and linear functions $\mathbb{L}'_{\mu} = \{\mathbb{L}'_{\ell,\tilde{\ell},i,\mu}\}$ $(\mu=1,4)$ with which the following are valid.

$(3.2.8)_1$ Data β'_{μ}'s as well as $\mathbb{L}'_{\mu}$'s depend on $(\ell, \tilde{\ell}, i)$ only (and so independent of (ϑ_λ))

$(3.2.8)_2$ For a vector $y^{i} \in \Gamma(\mathcal{D}'_x \overline{\Delta}_r((\omega:\mathfrak{g})), \vartheta^{(\ell)})$ satisfying the conditions $(3.2.7)_1$, $(3.2.7)_2$ we find a vector $y^{i+1} \in \Gamma(\mathcal{D}'_x\overline{\Delta}_r((\omega:\mathfrak{g})), \vartheta^{(\ell)})$ so that the following two conditions are valid.

$$(3.2.8)_3 \qquad K(m, i+1)(y^{i+1}) = (\cdots y^{i})_{(i=1,\cdots,\ell)}$$

$$(3.2.8)_4 \qquad |y^{i+1}| \leq (\beta'_1)^{\beta'_2 d'_3} \cdot b \cdot d^{\mathbb{L}'_1(d'_3)-\mathbb{L}'_2(m)} \quad (|\mathfrak{g}_1| = \cdots = |\mathfrak{g}_\ell| = d)$$

if $d'_3 \geq \mathbb{L}'_3(m)$ holds.

Proof of Proposition 3.2.1. Our proof will be done inductively on $\tilde{\ell}$: $(i)_{\tilde{\ell}=0}$: First we show our assertion inductively on double indices $(\ell, i' = \ell - i)$:

Concerning inductive procedures taken here, we note a simple quantitative fact : Let h be a holomorphic function in $\mathcal{D}' \times \overline{\Delta_r(0;\delta)}$. We assume the following estimation.

$$(3.2.9)_1 \qquad |h|_{|g_1|=\cdots=|g_\ell|=d} \leqq b'' d^{n''} \quad ; \; b'' \in \mathbb{R}^+, \; n'' \in \mathbb{Z}^+ .$$

we expand h in $\mathcal{D}' \times \overline{\Delta_r(0;\delta)}$ in the following (unique) way :

$$(3.2.9)_2 \qquad h = h^{(n'')} \cdot g_1^{n''} + \sum_{j=0}^{n''-1} h_j \cdot g_1^{j} \quad ; \; h_j \; (j=0,\cdots,n''-1)$$

does not contain the term g_1 at all.

At a point $Q \in \mathcal{D}' \overline{\Delta_r(0;\delta)}$ with coordinates(g) : $|g_1| = |g_2| = \cdots |g_\ell| = d$, estimations of $h^{(n'')}, h_j$'s are given by

$$(3.2.9)_3 \qquad |h_0| \leqq \int_{|g_1|=d;\,|g_2|=\cdots=|g_\ell|=d} \frac{|h|}{|g_1|} |dg_1|, \quad h_1 \leqq \int_{|g_1|=d;\,|g_2|=\cdots=|g_\ell|=d} \frac{h-h_0}{|g_1^2|} dg_1, \cdots h_s = \int_{|g_1|=\cdots=|g_\ell|=d} \frac{h - h_0 g_1 - \cdots - h_{s-1} g_1^{s-1}}{|g_1^{s+1}|} dg_1 \cdots$$

Then , by a quite simple calculation. we obtain the following estimation of $h^{(n'')}, h_j$.

$$(3.2.9)_4 \qquad |h_s|_{|g_1|=\cdots=|g_\ell|=d} \leqq (4\pi\sqrt{-1})^{s} \cdot d^{n''-s} \cdot b'', \qquad h^{(n'')} \leqq (4\pi\sqrt{-1})^{n''} \cdot b''.$$

From $(3.2.9)_4$, we obtain easily an expansion of h of the following form .

$$(3.2.9)_5 \qquad h = \sum_{|J|=n''} h_J \cdot g^J \quad : \; J = (j_1, \cdots, j_\ell),$$

$$(3.2.9)_6 \qquad |h_J| \leqq c_\ell^{n''} \cdot b'' \quad ; \; c_\ell (\in \mathbb{R}^+) \text{ depends on } \ell \text{ only}.$$

Now it is quite easy to deduce desired results from $(3.2.9)_{4,5,6}$ and procedures to find $\mathfrak{y}^{m,i+1} \in \Gamma(\mathcal{D}\times\overline{\Delta_\chi(0;\theta)}, \theta^{(\ell_1)})$.

To be sure, we proceed as follows : Define $\mathfrak{y}^{,m,i+1} \in \Gamma(\mathcal{D}\times\overline{\Delta_\chi(0;\theta)}, \theta^{(\ell_1)})$ by (3.2.6). Then the vector $\mathfrak{y}^{,m,i+1}$ is defined in $\Delta_\chi(\mathcal{D}\times\overline{\Delta_\chi(0;\theta)}; \theta^{(\ell_1)})$ and is estimated in the following way.

$$(3.2.10)_1 \quad |\mathfrak{y}^{,m,i+1}| \leq c_\ell \cdot d^{d_3} \cdot d^{d_3-m} \cdot \sigma \quad (c_\ell (\in \mathbb{R}^+) \text{ depends on } \ell \text{ only})$$
$|\eta_1| = \cdots = |\eta_\ell| = d$

Next, we apply the induction hypothreis (for $\ell - 1$) to the vector $\mathfrak{y}^{,m,i} = \mathfrak{y}^{m,i} - \mathbb{K}(m,i+1)\cdot\mathfrak{y}^{m,i+1}$. Then we know easily the desired facts $(3.2.8)_3$, $(3.2.8)_4$.

(ii) Next we show the assertions $(3.2.8)_3$ $(3.2.8)_4$ inductively on $\tilde{\ell}$.

For this purpose, we write the equation : $K(m, i)\cdot\mathfrak{y}^{i} = 0$: explicitly in the following form.

$$\mathbb{K}(m,i)\cdot\mathfrak{y}^{i} = \sum_{1\le\nu_1<\cdots<\nu_i\le\ell} f_{\nu_1\cdots\nu_i}\,\mathfrak{y}^{i-1}_{\nu_1\cdots\nu_i} + \sum_{\substack{1\le\nu_1<\cdots<\nu_{i-1}\le\ell,\\ \mu_1}} f_{\nu_1\cdots\nu_{i-1}\mu_1}\,\mathfrak{y}^{i-1}_{\nu_1\cdots\nu_{i-1},\ell+\tilde{\mu}_1} + \cdots + \sum_{\substack{1\le\nu_1<\cdots<\nu_a\le\ell,\\ 1\le\mu_1<\cdots<\mu_q\le\tilde{\ell}}} f_{\nu_1\cdots\nu_a,\mu_1\cdots\mu_q}\,\mathfrak{y}^{i-1}_{\nu_1\cdots\nu_a,\ell+\mu_1\cdots\ell+\mu_q} = 0$$

Corresponding to the above explicit expression of the equation, we say that a vector $\mathfrak{Y}^i$ is <u>of cotype (a)</u> if the coefficients $(f_{\nu_1 \cdots \nu_{i-a'}, \ell+\mu_1, \cdots \ell+\mu_{a'}})'s\ (a' > a)$ are zero.

Our (elementary) discussion here will be done by diminishing the cotype (a) of the given vector.

We argue by dividing the cases to a = i, or a < i. In both cases we assume : $|\mathfrak{Y}^i| \leq \tilde{b} \cdot d^{\tilde{d}_3}\ (|\mathfrak{z}_1| = \cdots = |\mathfrak{z}_\ell| = d.)$

(ii)$_1$ a = i . In this case , the equation is written in the following way .

$$\sum_{1 \leq \nu_1 < \cdots < \nu_i \leq \ell} f_{\nu_1 \cdots \nu_i} \mathfrak{Y}^{i-1}_{\nu_1 \cdots \nu_i} + \cdots\cdots + \sum_{1 \leq \mu_1 < \cdots < \mu_i \leq \tilde{\ell}} f_{\ell+\mu_1, \cdots \ell+\mu_i} \mathfrak{Y}^{i-1}_{\ell+\mu_1, \cdots \ell+\mu_i} = 0 \,;$$

Then, from (3.2.9)$_{5,6}$ the coefficients $(f_{\ell+\mu_1, \cdots, \ell+\mu_i})_{(1 \leq \mu_1 < \cdots < \mu_i \leq \tilde{\ell})}$ are expressed in the manner

$$f_{\ell+\mu_1, \cdots, \ell+\mu_i} = \sum_j f_{\ell+\mu_1, \cdots \ell+\mu_i, j} \cdot g^m_j \,, \quad \text{in } \mathcal{D}' \times \overline{\Delta_x((0);(\mathfrak{z}))}$$

where the coefficients $f_{,j}'$s of $(f_{\ell+\mu_1, \cdots \ell+\mu_i})$ are estimated in the following fashion.

$$(3.2.10)_1 \qquad |f_{\ell+\mu_1, \cdots, \ell+\mu_i, j}| \leq \tilde{c}_{\ell,1}^{\tilde{c}_{\ell,2} d_3} \cdot b \cdot d^{\tilde{\beta}_{\ell,1} d_3 - \tilde{\beta}_{\ell,2} m} \quad (|\mathfrak{z}_1| = \cdots = |\mathfrak{z}_\ell| = d)$$

In the above, $(\tilde{c}_{\ell,1}, \tilde{c}_{\ell,2})$ and $(\tilde{\beta}_{\ell,1}, \tilde{\beta}_{\ell,2})$ are depending on $\mathcal{B}$ only .

(ii) a < i : In this case the explicit

Define a vector $\mathfrak{Y}^{i+1} \in \Gamma(\mathcal{D}' \times \overline{(\Delta_x(0):(\theta))}, \mathcal{O}^{\binom{l+\tilde{\imath}}{i+1}})$

by

$$(3.2.10)_2 \qquad \mathfrak{Y}^{i+1}(\tau_1 \cdots \tau_s, \mu_{s+1}, \cdots \mu_{i+1}) = \begin{cases} f_{l+\mu_1, \cdots l+\mu_i, j} & \text{for } (j, l+\mu_1, \cdots; l+\mu_i) \\ 0, & \text{otherwise} \end{cases}$$

Then it is clear that the vector $\mathfrak{Y}'^{i} = \mathfrak{Y}^{i} - \mathbb{D}(m, i+1) \cdot \mathfrak{Y}^{i+1}$

$\in \Gamma(\mathcal{D}' \times \overline{\Delta_x((0):(\theta))}, \mathcal{O}^{\binom{l+\tilde{\imath}}{i+1}})$ is of cotype $(a-1)$. Quantita.

properties of the vector $\mathfrak{Y}'^{i}$ is easily

known from the above explict expression of $\mathfrak{Y}'^{i}$.

The above procedure provides a necessary

information in the case of $i = a$.

(ii) $a < i$: In this case

the following desired explicit.

①

expression of the equation $\mathbb{K}(m,i)(\mathfrak{f}^i)=0$ is as follows

$$(3.2.10)' \quad \mathbb{K}(m,i)\cdot(\mathfrak{f}^i) = \sum_{1\le v_1<\cdots<v_i\le \ell} f_{v_1\cdots v_i}\, \mathfrak{g}^{i-1}_{v_1\cdots v_i} + \cdots\cdots + \sum_{\substack{1\le v_1<\cdots<v_{i-a}\le \ell \\ 1\le \mu_1<\cdots<\mu_a\le \tilde{\ell}}} f_{v_1\cdots v_{i-a},\ell+\mu_1\cdots \ell+\mu_a}\, \mathfrak{g}^{i-1}_{v_1\cdots v_{i-a},\ell+\mu_1\cdots\ell+\mu_a} = 0.$$

Consider the $(\tilde{v}_1,\cdots,\tilde{v}_{i-a-1},\ell+\mu_1,\cdots\ell+\mu_a)$ - components of the above equality. Then we obtain

$$(3.2.10)'_1 \quad \sum_{v_0'<v_1} f_{v_0'\tilde{v}_1\cdots\tilde{v}_{i-a-1},\ell+\mu_1,\cdots\ell+\mu_a}\, g^m_{v_0'} - \sum_{v_1<v_1'<v_2} f_{\tilde{v}_1 v_1'\tilde{v}_2\cdots\tilde{v}_{i-a-1},\ell+\mu_1\cdots\ell+\mu_a}\, g^m_{v_1'} + \cdots + (-1)^{i-1}\sum f_{\tilde{v}_1\cdots\tilde{v}_{i-a}v_i'\ell+\mu_1\cdots\ell+\mu_a}\, g^m_{v_i'} = 0 .$$

On the otherhand, note that the equation $\mathbb{K}(m,i-a)((\tilde{f}_{\tilde{v}_1\cdots\tilde{v}_{i-a}}))$ $(1\le\tilde{v}_1<\cdots<\tilde{v}_{i-a}\le\ell) = 0$ is equivalent to the following

$$(3.2.10)'_2 \quad \sum_{v_0'<v_1} \tilde{f}_{v_0'\tilde{v}_1\cdots\tilde{v}_{i-a-1}}\, g^m_{v_0'} - \sum_{v_1<v_1'<v_2} \tilde{f}_{\tilde{v}_1 v_1'\tilde{v}_2\cdots\tilde{v}_{i-a-1}}\, g^m_{v_1'} + \cdots = 0 \quad (\text{for each } 1\le\tilde{v}_1<\cdots<\tilde{v}_{i-a+1}\le\ell).$$

Thus we obtain, for each set of indices $(\mu_1,\cdots,\mu_a)$, a vector $(f_{v_1\cdots v_{i-a+1},\ell+\mu_1\cdots\ell+\mu_a})_{1\le v_1<\cdots<v_{i-a+1}\le\ell}$ satisfying the equation: $\mathbb{K}(m,i-a+1)(\tilde{f}_{v_1\cdots v_{i-a+1};\ell+\mu_1\cdots\ell+\mu_a})_{1\le v_1<\cdots<v_{i-a+1}\le\ell} = (\tilde{f}_{v_1\cdots v_{i-a},\ell+\mu_1\cdots\ell+\mu_a})_{1\le v_1<\cdots<v_{i-a}\le\ell}$

Quantitative properties of this vector $(\tilde{f}_{v_1\cdots v_{i-a+1},\ell+\mu_1\cdots\ell+\mu_a})_{1\le v_1<\cdots<v_{i-a+1}\le\ell}$ is known from the case of $\tilde{\ell}=0$: (i.e. the case of $a=0$). Namely the vector $(\tilde{f}_{v_1\cdots v_{i-a+1},\ell+\mu_1\cdots})$ is estimated in the manner

$$(3.2.10)'_3 \quad |\tilde{f}_{v_1\cdots v_{i-a+1};\ell+\mu_1\cdots\ell+\mu_a}| \le \tilde{\beta}_1''\, {}^{\tilde{\beta}_2''\tilde{d}_3}\,\tilde{\delta}\cdot d^{\tilde{\zeta}_1(d_3)-\tilde{\zeta}_2(m)} \quad (|g_1|=\cdots=|g_\ell|=d)$$

so far as $\tilde{\zeta}_3(d_3)>m$ holds.

Here data $(\tilde{\beta}_1'',\tilde{\beta}_2'',\tilde{\zeta}_1,\tilde{\zeta}_2,\tilde{\zeta}_3)$ are those ones depending on $(\ell,\tilde{\ell},i)$ only.

We define a vector $\mathfrak{F}'^{\,i+1} \in \Gamma(\mathcal{D}\times\Delta_{r}((0),(g)), \mathcal{O}^{\binom{l+\tilde{l}}{i+1}})$ by

$$(3.2.10)'_4 \quad \mathfrak{F}'^{\,i+1}(\nu_1,\dots,\nu_i, l+\mu_1,\dots, l+\mu_{i+1-\tau}) = \begin{cases} f_{\nu_1\dots\nu_{i-a+1}, l+\mu_1,\dots, l+\mu_a} & \text{for } (\nu_1\dots\nu_{i-a+1}, l+\mu_1,\dots,l+\mu_a) \\ 0, & \text{otherwise} \end{cases}$$

It is clear that the vector $\mathfrak{F}^i - \overline{K}(m, i+1)\cdot\mathfrak{F}'^{\,i+1}$ is of cotype $(a-1)$.

(iii) Now it is easy to see that the results in (i) and (ii) yield the desired results $(3.2.8)_{3.4}$. q.e.d.

Remark. Assume that one of functions g_λ's, for example g_1, is not zero (at any points) in $\mathcal{D}\times\overline{\Delta}_r((0),(g))$. In this case, a similar consideration as in proposition 3.2.1 becomes quite simple: Namely, let $\mathfrak{F}^i$ be a vector in $\Gamma(\mathcal{D}\times\overline{\Delta}_r((0),(g)), \mathcal{O}^{\binom{l+\tilde{l}}{i}})$ which satisfies the equation

$$\overline{K}(m, i)\,\mathfrak{F}^i = 0 .$$

Define a vector $\mathfrak{F}^{i+1} \in \Gamma(\mathcal{D}\times\overline{\Delta}_r((0),(g)), \mathcal{O}^{\binom{l+\tilde{l}}{i+1}})$ by

$$(3.2.11) \quad \mathfrak{F}^{i+1}(\nu_1,\dots,\nu_{i+1}) = \begin{cases} g_1^{-m}\times f_{\nu_1\dots\nu_i} & \text{for } (1,\nu_1,\dots,\nu_i), \\ 0, & \text{otherwise}. \end{cases}$$

Then we know that the following equation

$$(3.2.11)' \qquad \mathfrak{F}^{i} - K(m, i+1)\,\mathfrak{F}^{i+1} = 0,$$

holds.

n. 2. In this numero we shall formulate our basic problems in this section. Let U be a domain in C^{N}, and let P_0 be a point in U. These data (U, P_0) are assumed to be given throughout in this section. When there is no fear of confusions, we omit these data (U, P).

A basic datum in this section will be explained as follows. By X we mean a variety ($\ni P_0$) in U. The variety X_s stands for the singular locus of X (in U). Also assume that a proper subvariety V ($\ni P_0$) of X is given. The irreducible decomposition of X_{P_0}, $X_{P_0,s}$ as wella as V_{P_0} ..... will be written in the following form.

$$X_{P_0} = \cup_\delta X_{P_0,\delta}, \quad X_{\delta P_0} = \cup_\delta X_{\delta, P_0, \delta}, \quad V_{P_0} = \cup_\delta V_{P_0,\delta}.$$

The conditions ( ) and ( ) are assumed i.e.

$V_{P_0\delta} \neq X_{P_0\delta}'$ for a pair $(V_{P_0,\delta}, X_{P_0,\delta}')$ and $V_{P_0,\delta} \not< X_\delta$ for a $V_{P_0,\delta}$) Recall that the conditions (3.1.) and (3.1.) are valid for each point P near P_0 . Such a datum (X, V) will be fixed.

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As before, our arguements will be done not only with the datum (X, V) but also with a set of functions associated with varieties in question. Namely assume that a sets $\mathfrak{G}_X = \{g'_1, \ldots g'_{\ell'}\}$ of functions defining X and $\mathfrak{G}_V = \{g_1, \ldots, g_\ell\}$ defining V on X are given. The functions $g_1, \ldots, g_\ell$ restricted on X will be denoted by $\tilde{g}_1, \ldots \tilde{g}_\ell$. We assume the following condition on $\tilde{g}_1, \ldots, \tilde{g}_\ell$.

($\tilde{\mathfrak{g}}$) $\tilde{g} \neq 0$, on each component $X_{P_0, j}$ of X_{P_0}.

Corresponding to notations given in n°. 1, define vectors $\tilde{\mathfrak{g}}^{i-1}_{\nu_1, \ldots \nu_i} \in \Gamma(X, \mathcal{O}_X^{\binom{\ell}{i-1}})$ over X by

$$(3.2.12)_1 \qquad \tilde{\mathfrak{g}}^{i-1}_{\nu_1 \cdots \nu_i}(\mu_1, \ldots \mu_{i-1}) = \begin{cases} (-1)^k \cdot g_k^m, & \text{for } (\nu_1, \ldots, \hat{\nu}_k, \ldots, \nu_i), \\ 0, & \text{otherwise} \end{cases}$$

Also define sheaf homomorphisms $\tilde{K}(m, i)$ over X by

$$(3.2.12)_2 \qquad \tilde{K}(m, i)(\tilde{f}_{\nu_1 \cdots \nu_i}) = \sum_{1 \le \nu_1 < \cdots < \nu_i \le \ell} \tilde{f}_{\nu_1 \cdots \nu_i} \tilde{\mathfrak{g}}^{i-1}_{\nu_1 \cdots \nu_i}.$$

The image $\tilde{K}(m,i) \cdot \mathcal{O}_X^{\binom{\ell}{i}}$ will be denoted by $\tilde{\mathfrak{X}}^{m, i-1}$.

Our main purpose is to give a type of a vanishing theorem for such sheaves $\tilde{\mathfrak{X}}^{m, i}$ $(i = 0, \ldots, \ell)$. An another problem, which is closely related to our basic problem, will be also studied. The basic problem will be called a vanishing theorem with algebraic division and algebraic growth condition. On the otherhand, we call the second problem a weak syzygy with quantity. These two results will be abbriviated as v.a.d.g (or v.a) and w.s.q. in the sequel. (n. a.)$_1$ We begin by formulating the first problem: in this case we consi-der a divisor D defined by $h = 0$, besides data $\{X, V, \mathcal{G}_X, \mathcal{G}_V\}$. The irreducible decomposition of the germs D_{P_0} and $\tilde{D}_{P_0}$ ($\tilde{D} = D \wedge X$) will be written as follows.

$$D_{P_0} = \bigcup_j D_{P_0, j} \quad , \quad \tilde{D}_{P_0} = \bigcup_j \tilde{D}_{P_0, j}$$

We call the collection (X, V, D, $\mathcal{G}_X$, $\mathcal{G}_V$, (u)) an underlying datum for a vanishing theorem with algebraic division and algebraic growth condition. (u.d.v.a) if the following conditions are valid.

(U. D.V.A)$_1$ $D_{P_0} \supset X_\delta$,

$(\mathrm{U.D.V.A})_2$ $\quad D_{P_0,\delta} \not\ni X_{P_0,\delta'}$ for a pair $(D_{P_0,\delta}, X_{P_0,\delta'})$.

$(\mathrm{U.D.V.A})_3$ $\quad \tilde{D}_{P_0,\delta} \not\ni V_{P_0,\delta'}$ for a pair $(\tilde{D}_{P_0,\delta}, V_{P_0,\delta'})$.

Let us consider a point P (near P_0)* in $D \cap V$. A cochain $\varphi^s \in C^s(N(\mathfrak{U}^\delta(\mathfrak{V}_X(P)-\tilde{\mathfrak{D}}), \mathfrak{X}^{m,i})$ is called <u>of algebraic growth (α_1, α_2) and algebraic division (α_3)</u> (or simply <u>of algebraic growth condition $(\alpha_1, \alpha_2, \alpha_3)$</u>) if φ^s has the following properties.

(A.D.G) For each set $\Delta_\delta(Q_0,\dots,Q_s;\tilde{\mathfrak{D}}) \ (\neq \phi) \ (Q_0,\dots,Q_s \in \bar{\mathfrak{V}}_X(P)-\tilde{\mathfrak{D}})$, the element $\varphi^s(Q_0,\dots Q_s : Q)$ has the following expansion

$$(\mathrm{A.D.G})_1 \qquad \varphi^s(Q_0,\dots Q_s;Q) = \sum_{1\le \nu_1<\dots<\nu_{i+1}\le l} \varphi^s_{\nu_1\dots\nu_{i+1}}(Q_0\dots Q_s:Q)\, \Psi^{m,i}_{\nu_1\dots\nu_{i+1}},$$

where $\varphi^s_{\nu_1\dots\nu_{i+1}} \in \Gamma(\Delta_\delta(Q_0\dots Q_s):\tilde{\mathfrak{D}}), \mathcal{O}_X)$ is estimated in the form

$$(\mathrm{A.D.G})_2 \qquad |\varphi^s_{\nu_1\dots\nu_{i+1}}(Q_0,\dots\dots,Q_s;Q)| \le \alpha_1 \cdot d(Q,\tilde{\mathfrak{D}})^{-\alpha_2} \cdot d(Q, V)^{\alpha_3}.$$

Take a neighbourhood $\mathcal{N}_0$ of P_0 and a point P in $\mathcal{N}_0$. A pair (P, φ^s) where φ^s is a cocycle in $C^s(N(\mathfrak{U}^\delta(\mathfrak{V}_X(P)-\tilde{\mathfrak{D}}), \mathfrak{X}^{m,i})$ with

We use the phrase ' a point P near P_0 ' as the synonym of (a point P in a neigh-bourhood of P_0). Here the existence of a neighbourhood of P_0, in which assertions in question are valid, is important. But informations (such as measures of the neighbourhood etc.......) are not asked. Because detailed informations concerning neighbourhoods of P are not used.

algebraic growth condition $(\partial_1, \partial_2, \partial_3)$ (for a suitable $(\partial) = (\partial_1, \partial_2, \partial_3)$) is called a testifying datum for v.a. (with quantities r, (δ), (∂)). (t. d. v. a.)

We mean by $\mathcal{T}_{v.a}((\mathrm{U.D}), \mathcal{N})$, the set $\{P, \varphi_i^s\} : P \in \mathcal{N}$ of all the t.d. v. a.'s. In the sequel informations about $\mathcal{N}$ are not so important and we shall omit a neighbourhood $\mathcal{N}$ in our formulation. Let us consider the following datum $Q^i_{v.a}((\mathrm{U.D})_{v.a}) = (N(\delta^\circ), M_1, M_2, P_1, P, E_2, E_3, E_{31}, E_{32}, \mathcal{L})$ $(i = 1, \cdots, \ell)$ depending on the u.d.v.a. $(\mathrm{UD})_{v.a}$ only. Such a datum will be called an expressing quantitative datum for v.a (ex. q. d v.a.) if the following conditions are valid.

(Ex. Q.V.A) For a t.d. v. a. (P, φ_i^s) with quantities (r, (δ), (∂)) there exists a $(s-1)$ — cochain $\varphi_i^{s-1} \in ((\mathrm{N}(\mathfrak{N}(\tilde{\tilde{\mathfrak{T}}}_X) - \tilde{\mathfrak{D}}), \mathfrak{X}^{m,i}_{\delta'}))^{p'}$ of algebraic growth $(\partial_1', \partial_2', \partial_3')$ so that the following are valid.

$$(\mathrm{Ex.Q.V.A})_1 \quad \delta_{\check{C}ech}(\varphi_i^{s-1}) = (\varphi_i^s),$$

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$(\text{Ex. Q. V. A})_2$ $\quad x' = \mathbb{M}_1(x)$, $(\delta') = \mathcal{L}(\delta)$, $\alpha_1' = \mathcal{M}_2(x)^{-1} \cdot \mathbb{P}(\alpha_1, \delta_1) e^{\mathbb{P}_2(\alpha_2, \delta_2)} e^{\mathbb{P}_3(\alpha_3, \delta_3)}$,
$\alpha_2' = \mathbb{L}_2(\alpha_2 + \delta_2)$, $\alpha_3' = \mathbb{L}_{31}(\alpha_3) - \mathbb{L}_{32}(m)$.

So far as $x < x^0$, $(\delta) < (\delta^0)$, $\alpha_3 (\alpha_3)$ are valid

If there exists an ex. q. d. v.a. for a given datum $\mathcal{T}_{v.a}((U, D))$, we say that v.a. (vanishing theorem with algebraic division and algebraic growth condition) is valid for $(U, D)_{v.a}$.

Our basic assertion in this section is formulated in the following manner.

Theorem 3.2.1. (A vanishing theorem with algebraic division and algebraic growth condition)

For any u.d. v.a. $(U, D) = \{X, V, D, \mathcal{F}_X, \mathcal{F}_V, (k)\}$ the vanishing theorem with algebraic division and algebraic growth condition is valid.

We shall make some remarks on the above theorem.

(C) A basic fact is that data (δ_2'), $((\alpha_{v_2}))$ are independent of the integer m in question.

This fact will be made an essential use of in the sequel ((c.f) § 4. Theorem 4.1). Concerning the above comment, we shall add one another remark.

As we remked previously, it is not true in general that the following sequance is valid.

$$(3.2.1)^{\sim} \quad 0 \to \tilde{\mathfrak{X}}^{m,\ell} \xrightarrow{\mathbb{X}(m,\ell)} \cdots \xrightarrow{\mathbb{X}(m,2)} \tilde{\mathfrak{X}}^{m,1} \xrightarrow{\mathbb{X}(m,1)} \tilde{\mathfrak{X}}^{m,0} \to 0$$

If we work with a resolution of $\tilde{\mathfrak{X}}^{m,0}$ (without entering into dependences of a chosen resolution on integers m carefully) it seems not possible to assure the independenceness of (δ_2), (∂_2) of $\mathcal{M}$.

This is the essential reaaon why we work with series of sheaves $\tilde{\mathfrak{X}}^{m,i}$ to which dependencess of quantities on the integers m are rather easily examined. The author believes that our discussion of series of sheaves $\tilde{\mathfrak{X}}^{m,i}$ in this section ows its own interst besides the above explained fact.

(n°. 2)$_2$ In addition to the above theorem 3.2.1 we shall make one anotuer arguement which we call a weak syzygy with quantity (w.sy. q) Both theorems v.a and w.sy. q. are , as will be shown in later , related mutually. We formulate our second problem as follows : in this case we do not consider the divisor D . Instead we consider a subvariety V^* of V so that the condition

$$V^* \supset V \cap X_\Delta .$$

holds. The cases $V^* = V \cap X_\Delta$ or $V^* = V$ are included. Also the possibility $V^* = \phi$ is admitted (Of course , in this case , $X_\Delta = \phi$). When the above condition is satisfied , we call the collection (X, V, V^*, q_X, q_{V},) an <u>underlying datum for a weak syzygy with quantities</u> (u.d.w, sy. q.) and we will write the above collection as $(U.D)_{w.s.q}$. Functions related to the variety V^* do not appear in our formulation of problems. Here our arguements will be done in a neighbourhood of P_c (Informations of neighbourhoods are not important in the later arguements and we shall omit to write a neigubourhood explicitly .) Now our consideration will be divided to the following two cases (1) $V^* \neq V$, (2) $V^* = V$.

In a little while we are concerned with a nontrivial case (1). A pair of $P \in V - V^*$ and of an element $\tilde{f}_{i,Q} \in \Gamma(\Delta_\varepsilon(P) : \mathcal{O}_X^{(\ell)})$ will be called a testifying datum for w. sy. q.. (t. d. w. sy. q) (w. sy. q. stands for the abbriviation of weak syzygy with quantity) if the following condition is true.

$$(W.\ SY.\ Q)_1 \qquad \tilde{K}(m,\ i) \cdot (\tilde{f}_{i,Q}) = 0 .$$

For the case of $i = 0$, we do not assume any algebraic conditions for $f_{0,Q}$.

We say that $\mathfrak{F}^{m,i}_{Q}$ is of q.p. (quantitative property)

(r, b , d_3) if the following conditions are valid.

$(\mathbb{W}.sy.Q)_2$ There exists an element $\mathfrak{F}^{m,i} \in \Gamma(\tilde{\Delta}_x(Q), \theta_X^{(i)})$

in such a manner that

$(\mathbb{W}.sy.Q)_{2,1}$ the stalk $\mathfrak{F}^{m,i}_{Q}$ of $\mathfrak{F}^{m,i}$ at Q is $\mathfrak{F}^{m,i}_{Q}$.

and moreover,

$(\mathbb{W}.sy.Q)_{2.2}$ In $\tilde{\Delta}_x(Q)$, the element $\tilde{\mathfrak{F}}^{m,i} = \left(\tilde{f}^{m,i}_{\mu}\right)_{\mu=1,\cdots,(i)}$ is

estimated in the following way

$(\mathbb{W}.sy.Q)_{2.3}$ $\left|\tilde{f}^{m,i}_{\mu}(Q')\right| \leq b \cdot d(Q', V)^{d_3}$.

In the sequal the estimations $(W.sy.Q)_{2.3}$ will be

abbriviated as ; $\left|\tilde{\mathfrak{F}}^{m,i}\right| \leq b \cdot d(Q', V)^{d_3}$.

By $(\mathbb{T}.\mathfrak{D}.)^{i}_{w.sy.q}$, we mean the set of all the t.d.w. sy:

$(Q, \mathfrak{F}^{m,i}_{Q})$; $Q \in (V - V^*)$, where $\tilde{\mathfrak{F}}^{m,i}_{Q}$ has a q. p.

(r, b , d_3) with suitable quantities (r , b , d_3) .

A datum $Q^{*,i}_{w.sy.q} = \{(\delta^i)(\tilde{\delta}^i_1, \tilde{\delta}^i_2), \tilde{M}^i_1, \tilde{M}^i_2, \tilde{M}^i_3, \tilde{P}^i, \tilde{L}^{*,i}_{31}, L^{*,i}_{32}\}$ $(i \geq 0)$

will be called an <u>expressing quantitative datum for (W. Sy q)</u> (an <u>ex. q. d. w. sy. q.</u>) if the following stateme-

-nts are valid with such a datum $Q^{i}_{w.sy}$.

$(Ex.Q.\mathbb{W}.sy.Q)^{i}$ <u>For any pair $(Q, \mathfrak{F}^{m,i}_{Q}) \in (\mathbb{T}.\mathfrak{D})_{w.sy.q}$ $(i \geq 0)$</u>

<u>of a quantitative property (r, b, d_3) , there exists</u>

an element $\mathfrak{Y}'^{m,i+1}_{Q} \in \mathcal{O}^{\binom{m}{i+1}}_{X,Q}$ of a q.p. (r' , b' , d_3') so that the equation

$$(\mathrm{Ex.Q.W.Sy.Q})^i_1 \quad \tilde{K}^{m,i+1}(\mathfrak{Y}'^{m,i+1}_{Q}) = \mathfrak{Y}^{m,i}_{Q} \qquad (i=0,\cdots\cdots, l-1)$$

with the data

$$(\mathrm{Ex.Q.W.Sy.Q})^i_2 \quad r' = \mathbb{M}^i_1(r) \ , \quad b' = \mathbb{M}^{-1}_2(b)\,\mathbb{K}^i_3(b) \cdot (\mathcal{C})^{P(d_3,m)}$$

$$d_3' = \mathbb{L}^i_{3,1}(d_3) - \mathbb{L}^i_{3,2}(m)$$

is valid, so far as the inequalities

$$(\mathrm{Ex.Q.W.Sy.Q})^i_3 \quad \mathbb{L}^i_{3,1}(d_3) - \mathbb{L}^i_{3,2}(m) > 0 \ , \qquad r \leq \delta^i_i d (Q , V^*)^{\tilde{\delta}^i_3}$$

hold.

(Remark) If the variety V^* is empty then the above assertion is understood as follows . In the ex. q. d. w. sy q. we replace (δ) by $\tilde{r}_i$, and the other materials are unchanged. In the statement , the part $r < \delta_i d (Q , V^*)^{\delta}$ is changed to $r < \tilde{r}$. Remaining parts are unchangesd.

We say that a weak syzgy with quantities holds for an u. d. w. sy. q. $(U. D)_{w.sy.q}$ if we can find an ex. q.d. w. sy. q $Q^i_{w.sy.q}$ for $(\mathfrak{U}, \mathfrak{D})_{w.sy.q}$.

(Remark) Arguements hitherto are done for the case : $V \neq V^*$. If $V = V^*$, we regard that w. sy. q. holds.

Now our second assertion is formulated in the following manner.

Theorem 3.2.2 (A weak syzgy with quantities)

For any u.d.w.sy.q. $(U.D)_{w.sy.q} = \{X, V, V^*, \mathfrak{g}_X, \mathfrak{g}_V\}$ the w.sy. q. holds.

We mean by Theorem $3.^{n}_{v.a}$ and Theorem $3.^{n}_{w.sy.q}$ statesments which assert the valididy of assertions in Theorem $3.^{n}_{v.a}$ and Theorem $3._{w.sy.q}$ for the cases where the dimension of X is at most n .

Remaining parts of this sectiom will be devoted to investigations of theorems $3.2.1_{v.a}$ and $3.2.2_{w.sy.q}$.

n. 3 There exist certain relations between theorem $3.2.1_{v.a}$ and theorem $3.2.2_{w.sy.q}$. We shall begin by clarifining them and by reducing theorems $_{v.a}$, $_{w.sy.q}$ and Theorem$_{w.sy.q}$ to more weaker problems . Reductions will be done in several ways.

(1) Assume that an u.d. v.a. $(U.D)_{v.a.} = \{X, V, D, \mathfrak{g}_X, \mathfrak{g}_V, h\}$ is given. Instead of the original assertion (v.a), Theorem $3.2.1_{v.a}$, we shall consider the following weaker assertion.

A datum $Q^{i}_{pre\,v.a}((U.D.)_{v.a}) = \{(\tilde{\delta}_0) M^{i}_{1}, M^{i}_{2}, P^{i}_{1}, P^{i}_{2}, P^{i}_{3}, L^{i}_{2}, L^{i}_{3}, \mathcal{L}^{i}\}$

is said to be an ex. q. d. pre v.a (an expressing quantitative datum for pre-vanishing theorem with algebraic growth condition) if the following statement, weaker than v. a , is valid.

(EX. Q. P.V.A) For a pair (Q, $\tilde{\mathcal{Y}}^{s}$) $\in (\mathcal{T}\mathcal{D})_{v.a}$ with quantitative conditions (Υ, (δ), $(\tilde{\delta})=(\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3)$) $((\delta)<(\tilde{\delta}))$, there exists a (s-1) - cochain $C^{s-1}(N(\hat{\mathfrak{U}}^{\delta'}(U_{r'}(P)-\mathfrak{D}))\tilde{\mathcal{X}}^{m'_i})$ with algebraic growth (d'_1, d'_2, d'_3) : $d'_3 = 0$ in such a manner that the following relations

$$(EX.Q.P.V.A)_1 \qquad \tilde{\mathcal{Y}}^{s} = \delta_{\check{C}ech}(\tilde{\mathcal{Y}}^{s-1}) ,$$

$$(EX.Q.P.V.A)_2 \qquad r' = M^{i}_{1}(r) , \quad (\delta') = \mathcal{L}^{i}(\delta) , \quad d'_1 = M^{i}_{2}(r)^{-1} P^{i}(\delta_1, d_1) \cdot e^{\ldots}$$

$$d'_2 = L^{i}_{2}(d_2, \delta_2) ,$$

holds so far as $d_3 > Z(m)$ is true.

In a similar way to the case of v .a. , we say that the pre v.a. holds for an u.d. v. a . $(U, D)_{v.a}$ if there exists an ex q d . pre. v. a. for $(U. D)_{v.a}$.

A weaker assertion than the original theorem theorem $3_{v.a}$ is formulated in the following way.

Lemma 3.21 $_{pre.v.a}$ (Pre vanishing theorem with algebraic growth condition)

For a given u.d.v.a. (U.D) , pre v.a. is valid.

By lemma $3.^{n}_{pre.v.a}$, we mean the validity of the above assertion for the cases where the dimension of X is at most n. Our first assertion of the theorem $3.2.1_{.v.a}$ will be done in the following two steps.

Lemma 3. 2. 2 Assume that an u.d.v.a. $(U.D)_{v.a} = \{ X, V, D, \mathfrak{F}_X, \mathfrak{F}_V, \mathfrak{h} \}$ is given and that the w.sy. q. is valid for the u.d. w.sy. q. $(U.D)_{w.sy.q} = \{ X, V, V^*, \mathfrak{F}_X, \mathfrak{F}_V : V^* = V \cap D$. Then we know that

(3. 2. 13)$_1$ pre. v. a. holds for (U. D) v. a.

Moreover, we know that

(3 . 2. 13)$_2$ the assumptions that for $(U.D)_{v.a}$, the w.sy. q. holds and the assumption that pre v. a. is valid for $(U.D)_{v.a}$ imply the theorem $3.2.1_{v.a}$ (v.a)

Concerning the second assertion of the w. sy. q. , our reduction will be done as follows : Assume that an u. d wy. sy. q. $(U.D)^*_{w.sy.q} = \{X, V, V^*, \mathfrak{F}_X, \mathfrak{F}_V\}$ as well as $(U.D)'^*_{w.sy.q} = \{X, V, V'^*, \mathfrak{F}_X, \mathfrak{F}_V\}$ are given. Moreover , we assume the following condition.

$$(3.2.14) \quad V_{\mathfrak{P}_0} \supset V'^*_{\mathfrak{P}_1} \supsetneqq V^*_{\mathfrak{P}_0} .$$

Here we also admit the possiblity of $V = V^*$.

Then our reduction step for theorem$_{w.sy.q}$ is stated in the following ways.

Lemma $3.2.3^n_{w.sy.q}$ Assume that an u.d. w. sy. q $(U.D.)^*_{w.sy.q}$ and $(U.D)'^*_{w.sy.q}$ are given so that the condition ( ) is valid. Moreover, assume the validity of theorem $3.^{n-1}_{w.sy.q}$ as well as the w. sy. q. for $(U.D)'^*_{w.sy.q}$. Then there exists a subvariety V''^* of V which satisfies the following conditions.

$$(3.2.15)_1 \quad V'^* \supsetneqq V''^* \supset V^*$$

$(3.2.15)_2$ The w. sy. q. holds for $(U.D)''^* = \{X, V, V''^*, \mathfrak{F}_X, \mathfrak{F}_V\}$

Proof of reduction steps : Lemma 3.2.1, 3.2.2 and Lemma $3.2.3^n_{w.sy.q}$ will be given in the later numeros.

Here we shall see that the above lemmas imply basic as assertions Theorems $3.1_{V.Z}$ and $3.2.1_{w.sy.q}$ .if we know the validity of Theorem $3^{n-1}_{w.sy.g}$. At first it is quite clear that the lemma $^{n}_{w.sy.g}$, applied inductively to V'^{*} , leads to the following implication.

$(3.2.16)_1$ Theorem $3^{n-1}_{w.sy.g} \Longrightarrow$ Theorem $3.^{n}_{w.sy.g}$.

On the otherhand , it is also obvious that

$(3.2.16)_2$ Theorem $3.^{n}_{w.sy.g}$ implies theorem $3.^{n}_{v.a}$.

Thus we know that lemmas and Theorem is sufficient to show Theorem and Theorem .

Remaining parts of this section will be devoted to verifications of lemmas 3.2.1, 3.2.2, 3.2.3 and theorem 3. . We quickly indicate our divices ; (I) Lemmas 3.2_1, 3.2_2 have many similarities , and will be discussed paralelly (in one place) . On the otherhand, Our verification of lemma $3._{w.sy.g}$ is reduced to further weaker assertions :

n. 4 . In order to discuss Lemma 3.2 $_{w.sy.g}$, we shall make further reductions of problems. We begin by fixing certain notation used here and by pointing out simple facts used in our discussion of Lemma 3. $_{w.sy.g}$. In the first place, let $(U.D)_{w.sy.g} = \{X, V, V^*, g_X, g_V\}$ be an underlying datum for a weak syzygy. Let us consider the following condition on V :

(3.2.17) At the origin P_0, the relation ( ) is valid.

(3.2.19)' $V^*_{P_0,j} \neq V_{P_0,j'}$,

for any pair ($V^*_{P_0,j}$, $V_{P_0,j'}$) of irreducible components $V^*_{P_0,j}$ of $V^*_{P_0}$ and $V_{P_0,j'}$ of V_{P_0} at P_0.

We quickly show the following

<u>Proposition 3.2.2 The validity of w.s.q. for all the u.d.sy.q. $(U.D)_{w.sy.g}$ satisfying the condition (3.2.17) leads to the validity of w.sy.q. as in Theorem 3.2.2.w.sy.g.</u>

<u>Proof.</u> Let us write the irreducible decomposition of V and V^* at P_0 in the following manner.

$$V = \bigcup_j V_{P_0,j} \quad , \quad V^* = \bigcup_j V^*_{P_0,j} \quad .$$

Divide irreducible components of V^* in two types : (i) Irreducible components $V^*_{P_0,\delta_1}$'s which do not coincide with any irreducible component $V_{P_0,\delta}$ of V_{P_0}. (ii) irreducible components which coincide with an irreducible component $V_{P_0,\delta}$ of V_{P_0}. Define germs $V^*_{P_0,i}$ (i = 1, 2) by $V^*_{P_0,1} = \bigcup_{\delta_1} V^*_{P_0,\delta_1}$ and $V^*_{P_0,2} = \bigcup_{\delta_2} V^*_{P_0,\delta_2}$, where $V^*_{P_0,\delta_1}$'s as well as $V^*_{P_0,\delta_2}$'s exhaust components of the first and of the second types respectively. Let $\tilde{V}^*_{P_0,1}$ be the germ defined by $\tilde{V}^*_{P_0,1} = V^*_{P_0,1} \cup (V^*_{P_0} \cap X_{P_0,\lambda})$. Take a variety $\tilde{V}^*_1$ in a neighbourhood of P_0 where the stalk (of $\tilde{V}^*_1$ at P_0 is $\tilde{V}^*_{P_0,1}$) The condition (3.2.16) imposed on X, V shows the validity of w. sy. q. for the u.d. w. sy. q. $(U. D)^{\sim}_{w.sy.q} = \{ X, V, \tilde{V}^*_1, \mathfrak{F}_X, \mathfrak{F}_V \}$.

Because the germ $\tilde{V}^*_{P_0,i}$ is contained in $V^*_{P_0}$, the validity of w. sy. q. for the u.d. w. sy. q. $(U. D)^{\sim}_{w.sy.q}$ implies the validity of the w. sy. q. for the $(U. D)_{w.sy.q} = \{X, V, V^*, \mathfrak{F}_X, \mathfrak{F}_V \}$. q.e.d.

Here we remark that, for an u.d. v. a. $(U.D)_{v.a} = \{X, V, D, \mathfrak{F}_X, \mathfrak{F}_V, (h) ,\}$ the intersection $V^* = V \cap D$ satisfies the conditions ( ) and ( ). We restrict, henceforth our attension to those u.d. w.sy.q.'s (U.D)$_{w.sy.q}$ = { X, V, V^*, $\mathfrak{T}_X$, $\mathfrak{T}_V$ } in which the variety V^* satisfies the condition (3.2.16) besides ( ). More precisely, we shall mean by Theorem 3.2.2$'_{w.sy.q}$ the one which says that the w.sy. q. is valid for any u.d. w.sy.q. (U.D)$_{w.sy.q}$ in which the condition (3.2.17) on V^* is true (except the case $V^* = V$). Also by Lemma 3.2.$^{n'}_{w.sy.q}$, we understand the assertion which is obtained from Lemma 3.2$^{n}_{w.sy.q}$ in the following manner : (1) In the statement of Lemma 3.2$^{n}_{w.sy.q}$, we impose the further condition (3.2.17) on V^*, V'^*. (2) Also we replace the induction hypothesis of the validity of theorem 3.2.$^{n-1}_{w.sy.q}$ by theorem 3.2.$^{n-1'}_{w.sy.q}$. Finally we impose the condition (3.2.16) on the variety V'^* to be found out. Then the following implications are preserved.

$(3.2.16)'_1$ Theorem 3.2.$'_{w.sy.q}$ $\Longrightarrow$ Theorem 3.2.v.a,

$(3.2.16)'_2$ Theorem 3.2.$^{n-1'}_{w.sy.q}$, Lemma 3.2$^{n'}_{w.sy.q}$ $\Longrightarrow$ Theorem 3.$^{n'}_{w.sy.q}$

The first implication is valid because , for u.d. v.a

$(\overline{U}. D)_{v.a} = (X, V, D, \mathfrak{F}_X, \mathfrak{F}_V, (h))$, the condition (3.2.17) is valid for $V^* = V \cap D$ because of (3.1.). On the otherhand, the implication $(3.2.16)'_2$ is seen immeadiately.

In later numeros, we work with Theorem $3.2 2'_{w.sy.q}$ and lemma $3.2.3'_{w.sy.q}$ rather than Theorem $3.2.'_{w.sy.q}$ and Lemmma $^{m}_{w.sy.q}$. This restriction is not of an essential nature. But our ideas will be fixed in several places by this restriction.

Now, for our purposes, we shall need further reductions:

Let $(\overline{U}. D)_{v.a} = \{ X, V, D, \mathfrak{F}_X, \mathfrak{F}_V, (h) \}$ be an u.d. v.a. We assume that the w. sy. q. is valid for the w. sy. q. $(U. D) \underset{w.sy.q}{\supset} \{ X, V, V^* = D \cap V, \mathfrak{F}_X, \mathfrak{F}_V \}$. A pair $(P, \tilde{\varphi})$ of a point $P \in V \cap D$ and of an element $\tilde{\varphi} \in \Gamma(\mathfrak{F}_{(b)} - \tilde{D}\mathcal{O})^{(m)}$ is called <u>a meromorphic testifying datum for the weak syzgy with quantities</u> (with $(r, b, \alpha_2, \alpha_3,)$) if the following conditions are satisfied.

$(Me.W.SY.Q)_1$ $\tilde{\kappa}(m, i) \cdot \tilde{\varphi} = 0$ (if $i \neq \ell, 0$)

and is estimated in the way

$(Me.W.SY.Q)_2$ $|\tilde{\varphi}^{m_i^j}| \leq b \cdot d(Q', \tilde{D})^{-\alpha_2} d(Q', V)^{\alpha_3}$; $Q' \in \overline{U}_r(Q) - \tilde{D}$.

In a similar way to w. sy. q, we mean by $\mathcal{T}^{i}_{mero.w.sy.q}$ the set of all the t.d. mero . w. sy. q. (with suitable quan--tities ((r, b, d_2 , d_3)). For this set $T^{i}_{mero.w.sy.q}$, we consider the following datum $Q^{i}_{mero.w.sy.q} = \{\tilde{\delta}^{i}_{me}, M^{i}_{me}, M^{i}_{me}, P^{i}_{me}, P^{i}_{me}, P^{i}_{me}, L^{i}_{me}, L^{i}_{me}, L^{i}_{me}\}$ associated with (U.D)$_{v.a}$ v.a . Such a datum $Q^{i}_{mero.w.sy.q}$ is called an weak quantitative datum for mero--morphic weak syzygy with quantity (ex. q. d. mero. w.sy. q.) if the following statements, which is weaker and is regarded as a meromorphic form of w. sy. q , are valid.

(EX.Q.D.ME.W.SY)$_1$ For a t.d. mero. w.sy. q. (P, $\mathfrak{Y}^{m,i}$) with (r, b, d_2, d_3) we find an element $\mathfrak{Y}^{m,i+1} \in \Gamma(\overline{U_r(P)} - D, \mathcal{O}_X^{(\tilde{m})})$ so that the equation

(EX.Q.D.ME.W.SY)$_1'$ $\quad \tilde{K}(m, i+1)\, \mathfrak{Y}^{m,i+1} = \mathfrak{Y}^{m,i}$,

and the estimation

(EX.Q.D.ME.W.SY)$_2'$ $\quad |\mathfrak{Y}^{m,i+1}| \leqq b' \cdot d(Q, \tilde{D})^{-d_2'}\, d(Q', \nabla)^{d_3'}$; $Q \in \overline{U_{r'}(P)} - D$;

is valid with quantities (r', b', d_2', d_3') determined by

$$r' = M^{i}_{me,1}(r), \qquad b' = M^{i}_{me,2}(r)^{-1}\, M_{me,2}(b),$$

$$d_2' = L_{me,2}(d_2), \qquad d_3' = L^{i}_{me,3,1}(d_3) - L^{i}_{me,3,2}(m).$$

If there exists an ext. mero. w. sy. q. for the datum then we say that the mero. w. sy. q. is valid for $(U.D)_{v.2}$.

As two steps to Lemma 3.2 $_{w.sy.q}'$, we show the following two assertions. in later.

Lemma 3.2.4 $^{mero,n}_{w.sy.q}$. For an u.d.v .a. $(U.D)_{v.2} = (X, V, D, \mathfrak{A}_X, \mathfrak{A}_V, (h))$, where the w. sy. q. is valid for $(U.D)_{w.sy.q} = \{X, V, V^* = V \cap D, \mathfrak{A}_X, \mathfrak{A}_V\}$, the meromorphic weak syzygy with quantities is valid for $(U.D)_{v.2}$.

Lemma 3.2. Lemma 3.2 $^{mero,n}_{w.sy.q}$ and Lemma 3.2 $^{n-1'}_{w.sy.q}$ implies Lemma 3.2 $^{n'}_{w.sy.q}$.

Remaining parts of this section will be devoted to verifications of the aforementioned lemmas : lemma 3. 2 1. lemma 3.2.2, lemma 3.2.3 and lemma 3.2.4.

We first discuss the lemma 3.2.1 and after that we enter into lemmas 3.2.2, 3 and lemmas 3.2.4...

n. 5 Inorder to discuss lemma 3.2.1, we begin with a simple remark. Assume that an u.d. w.sy. q. $(U.D)_{w.sy.q} = \{X, V, V^*, \mathcal{Y}_X, \mathcal{Y}_V\}$ is given. We assume that the w. sy. q. is valid for $(U.D)_{w.sy.q}$.

Furthermore, we assume that a variety V'^* satisfying the condition $V^* \supset V'^* \supset V^*$ is given. (We include the case $V^* = V'^*$) We generalize the notion of w. sy. q given in n° 3 in the following manner : Consider a point $Q \in (X \leftarrow X_j) - V'^*$ and an element $\mathfrak{Y}^i_Q \in \mathcal{O}^{(\mathfrak{m})}_{X,Q}$. A pair (Q, $\tilde{\mathfrak{Y}}^i_Q$) is called a (t. d. w.sy. q)' with quantities (r, b , α_3) if the following condition, similar to , is valid.

(w. sy. Q)' There exists an element $\mathfrak{f}^i = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}_{\mu \in (\mathfrak{m})} \in \Gamma(\overline{\Delta_r(Q)}, \mathcal{O}^{(\mathfrak{m})}_X)^*$, whose stalk at Q is $\check{\mathfrak{Y}}^i_Q$, so that the equation

(w .sy. Q)$'_1$ $\tilde{K}$ (m, i) ($\mathfrak{Y}^i$) = 0 (i = 1, ..., ℓ)

and the estimation

(w. sy. Q)$'_2$ $|\mathfrak{Y}^i| \leq b \cdot d(i : V)^{\alpha_3}$; $Q' \in \Delta_r(Q)$

are valid.

By $T_{w.sy.q}((U, D), V'^*_{w.sy.t})$, we mean the set of all the (t. d. w. sy. q)' with quantities (r , b , α_3) (for suitable quantities (r , b , α_3) . A datum

$\mathcal{L}'^i_{w.sy.q}((U, D), V'^*)$ = $\{ \tilde{\delta}^{*i}, \mathcal{L}^i M_1, M_2^{*i}, M_3^{i}, E_{11}^{*i}, E_{31}^{*i}, E_{32}^{*i} \}$ is said to be an (ex.d.w. sy. q)' (for $\tilde{\mathfrak{X}}^{m,i}$) if the following

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is valid.

$(EX.W.SY.Q)'_{i}$ If $d($ $Q,$ $\Upsilon')$ $<$ $\tilde{\delta}^{i}_{1} d($ $Q, X^{\vee}_{\lambda} V'^{*})^{\tilde{\delta}^{i}_{2}}$ ($Q \in (X - X_{s})$ $- V'^{*}$) is true, then there exists an element

$\eta^{m,i+1} \in \Gamma(\Delta_{r'}(Q) : \mathcal{O}_{X}^{\binom{m}{i+1}})$ $(i=0,\cdots,k-1)$ so that the equation

$(EX.W.SY.Q)'_{1}$ $\tilde{K}(\eta^{m,i+1}) = \eta^{m,i}$ in $\Delta_{r'}(Q)$

as well as the estimation

$(EX.W.SY.Q)'_{2}$ $|\eta^{m,i+1}(Q')| \leq b' \cdot d(Q', V)^{d_3'}$; $Q' \in \Delta_{r'}(Q)$

hold. where quantities r', b', and d_3' are determined by

$$(3.2.18) \qquad r' = M_1'(\Upsilon), \quad b' = M_2'(\Upsilon)^{-1} \cdot M_3'(b) \cdot P'_{11}(c)^{P'_{12}(d)}$$

$$d_3' = L'_{31}(d_3) - L'_{32}(m),$$

so far as $\Upsilon < \tilde{}$

Now, the following elementary proposition will be made use of in the later arguements.

Proposition 3.2.3 . If the w. sy. q . is valid for $(U.D)_{w.sy.q} = (X, V, V^{*}, \mathcal{G}_{X}, \mathcal{G}_{V})$ and if a subvariety V'^{*} ; $V \supset V'^{*} \supset V^{*}$; is given, then there exists an $\{ex.d.w.\ sy..q\}'$. $Q'^{i}_{w.sy.q}((U,D), V'^{*}) = \{\tilde{\delta}'^{i}, \hat{c}'^{k}, \tilde{M}'^{i}_{1}, \tilde{M}'^{i}_{2}, \tilde{M}'^{i}_{3}, \tilde{P}'^{i}_{11}, \tilde{P}'^{i}_{12}, \tilde{L}'^{i}_{31,32}\}$ for $\{(U.D)_{w.sy.q}, V'^{*}\}$.

Proof. It is obvious that, for a point $Q \in V - V'^{*}$, assertions $(EX.W.SY.Q)'_{1}$, $(EX.W.SY.Q)'_{2}$ are valid with an ex. w. sy . d . q . $Q^{i}_{w.sy.g}((U,D)_{w.sy.g})$.

(i i) Next we consider the case where a point Q is not in V . Choose a sufficiently small couple ($\tilde{\delta}'_0$) . Our arguement will be done separetedly according to whether Q is in $N_{\tilde{\delta}'_0}(V, V'^{*})$ or Q is not in $N_{\tilde{\delta}'_0}(V, V'^{*})$. Recall that if Q is not in $N_{\tilde{\delta}'_0}(V, V'^{*})$ then the following inequality

$$(3.2.19)^{1}_{1} \quad d(Q, V'^{*}) \leqq \tilde{c}'_1 \cdot d(Q, V)^{\tilde{c}'_2}$$

holds with a suitable ($\tilde{c}'_1$, $\tilde{c}'_2$) (c.f. §1,)

Thus , for a sufficiently small ($\tilde{\delta}''_0$), the following facts are valid in view of (1) and the Lojasiewicz's inequality

$(3.2.19)^{1}_{2}$ There exists a function $\tilde{g}$ (for example $\tilde{g}_1$) so that $|\tilde{g}_1(, Q')|^{-1} \leqq \tilde{h}'_1 d(Q : V)^{-\tilde{h}_2}$ with suitable constants ($\tilde{h}'_1$, $\tilde{h}'_2$) so far as Q' is in $\tilde{\Delta}_{\delta''_0}(Q : V'^{*}_{\Lambda})$.

Then, from ( ) , our assertion of the existence of a datum $Q'' = \{ \tilde{\delta}''_1, \iota'' \tilde{M}''_1, \tilde{M}''_2 \tilde{M}''_3 \tilde{P}''_{11}, \tilde{P}''_{12}, \tilde{L}''_{31}, \tilde{L}''_{32} \}$, with which

$(\mathcal{E}(\text{w.sy.}Q))'$ holds for any points $Q \notin N_{\gamma'}(V : V'^{*})$, follows easily.

(iii) Consider a point $Q \in N_{\gamma'_0}(V : V^{*})$ and take a point $Q_V \in V$ so that $d(Q, Q_V) = d(Q, V)$ holds.

From the existence of an ex. t.w.sy. / . $Q^{\lambda}_{w.sy.\delta}((U, D)_{w.sy.\delta})$ for $(U, D)_{w.sy.\delta}$, we know the existence of a datum $Q''' = \{ (\tilde{\delta}''), (\tilde{c}'''^{i}_1, \tilde{c}'''^{i}_2,) \; M''' , M , P''' , P''' \; P, L''' \}$, depending the datum $(U. D)_{v.a}$ only , in such a manner that the following fact is valid.

(3.2.19) If $r \geqq \tilde{c}'''^{i}_1 d(Q, Q_V)^{\tilde{c}'''^{i}_2}$, then , for a (t.d.w.sy.)$'$ $(Q, \mathfrak{f}^{m,i}_a)$, there exists an element $\mathfrak{f}'''^{m,i+1} \in \Gamma(\Delta_{\gamma'}(Q_V), \mathcal{O}^{(m)})$ so that the relations below are valid.

$$(3.2.19)^2_1 \quad \tilde{K}(m, i+1) . \mathfrak{f}^{m,i+1} = \mathfrak{f}^{m,i},$$

$$(3.2.19)^2_2 \quad |\tilde{\mathfrak{f}}^{m,i+1}| \leqq b'' . d(Q' : V)^{\delta'''_3} \quad (Q' \in \Delta_{\gamma'}(Q_V))$$

with quantities

$$(3.2.19)^2_3 \; r''' = M'''_1(r), \; b''' \leqq M'''_2(T)^{-1} M'''_3(\delta) \cdot P_{11}(\iota)^{P_{12}(Q)}.$$

Moreover, we can assume the inequality

$$r''' \geqq 4 \cdot d(Q, Q_V)$$

(Remark) For the above purpose , take $(\tilde{c}'''^{i}_1, \tilde{c}'''^{i}_2)$

so that $\tilde{m}''_{12}\tilde{c}_2 < 1$, $\tilde{m}''_{11}\tilde{c}^{\tilde{m}''_{12}} > 4$ where $M_1(r) = \tilde{m}'''_{11} r^{\tilde{m}'''_{12}}$.

Thus we know that, so far as $r \geq \tilde{c}_1 d(Q, Q_V)^{\tilde{c}_2}$ holds, there exists an element $\mathfrak{y}^{m,i+1} \in \Gamma(\tilde{\Delta}_{r''}(Q), \vartheta^{(m)}_{(i+1)}) : \tilde{K}(\mathfrak{y}^{m,i+1})$ $\tilde{\mathfrak{y}}^{m,i}$ whose quantities (r'', b'', d_3''') are determined by $(3.2.19)^2_3$. Next consider the case in which the inequality $r \leq \tilde{c}_1^{i} d(Q, Q_V)^{\tilde{c}_2'''^{i}}$ holds. For an arbitrary (small enough) couple $(\tilde{\delta}'''')$, there exists a monomial M'''', depending on $(\tilde{\delta}'''')$, $(U.D)_{w.sy.q}$ only, so that $r'' = M''''(r) < \tilde{\delta}_1'''' \cdot d(Q, V)^{\tilde{\delta}_2''''}$ holds.

In such a situation we know, from ( ), the existence of a datum $\mathfrak{Q} = \{\tilde{\iota}''' \tilde{M}_1''', \tilde{M}_2''', \tilde{M}_3''' \tilde{P}_{11}''', \tilde{P}_{12}''', \tilde{L}_{31}''' L_{32}\}$ depending on M''' and $(U,D)_{w.sy.q}$ only, in such a manner that, for a pair $(Q, \mathfrak{y}^{m,i})$: $\mathfrak{y}^{m,i} \in \Gamma(\Delta_r(Q), \vartheta^{(\eta)}_X)$ the assertions $(\mathrm{EX.w.sy.Q})'_1$, $(\mathrm{EX.w.sy.Q})'_2$ are true with quantities $r'' = \tilde{M}_1'''(r)$, $b'' = \tilde{M}_2(r)^{-1} \cdot \tilde{M}_3'''(b) \cdot \tilde{P}_{11}'''(\tilde{\iota}''')$, $d_3''' = L_{31}'''(d_3) - L_{32}'''(m)$. [$\tilde{P}_{12}'''(d_3)$ written above]

It is clear that (;), $(3.2.19)'$ and $(3.2.19)^2$ lead to our assertion of the existence of an ex w. sy. q. d. $Q'^{i}_{w.sy.q}$ for $\{(U.D), V^*\}$. q. e. d.

Now we prove lemma $3.2.1_1$ and lemma $3.2.1_2$ in 3-2-40.

a pararell way .

Proof of lemma 3.2.1 (i) . Let $(\mathcal{V}.\mathcal{D})_{v.a} = ($ X, V, D , $\mathcal{J}_X$, $\mathcal{J}_V$ (h)) be an u. d. v. a. so that the w. sy. q. holds for the w.sy. q. $(U.D)^*_{w.sy.q} = ($ X , V , $V^*=V\cap D$, $\mathcal{J}_X$, $\mathcal{J}_V$). By lemma3.2.1$^i_1$, lemma3.2.1$^i_2$ we mean the validity of lemma3.2.1$_1$ and lemma3.2.1$_2$ for the sheaf $\mathcal{X}^{m_i'}(i' \leq i)$. Both assertions lemma 3.2.1$_1$ and lemma 3.2.1$_2$ are proven in an inductive way on i . We discuss problems in the following manner.

$(3.2.20)_{l,0}$ To show that

$(3.2.20)_1$ the lemma3.1. implies the assertion lemma 3.2.1$_1$,

and that

$(3.2.20)_2$ the assertion lemma$)^{0}_1$ implies lemma$)^{l}_2$.

$(3.2.20)'$ To show that

$(3.2.20)'_1$ the assertions lemma$)^{i}_{1,2}$ are reduced to the assertions lemma3.2.1$^{i+1}_{1,2}$($0 \leq i \leq l$) .

Our inductive devices lemma3.2.1$^{i}_{1,2}$($0 \leq i \leq l$) are done pararelly in both cases lemma3.2.1$^i_1$ and lemma 3.2.1$^i_2$ .On the otherhand the assertions lemma$^{l}_1$ and lemma$^{l}_2$ separetedly

(ii) We fix a datum $Q^{0} = \{\gamma^{0}, M_1^{0}, M_2^{0}, P_{11}^{0}, P_{12}^{0}, L^{0}, \mathcal{L}^{0}\}$ with which lemma 3.1 is valid for the structure sheaf $\mathcal{O}_X$ and an ex. w. sy. q. d. $Q^{i}_{w.sy.q} = \{\gamma, M_1^{i*}, \tilde{M}_2^{i*}, \tilde{M}_3^{i*}, \tilde{P}_{21}^{i*}, \tilde{P}_{22}^{i*}, \tilde{L}_{31}^{i*}, \tilde{L}_{32}^{i*}\}$ for $\mathfrak{X}^{m,i}$.

$(ii)_1$ First we consider the inductive steps $(\quad)_{\ell,1}$ and $(\quad)_{\ell,2}$. In both cases, we fix a cocycle $\tilde{\varphi}^{s} \in Z^{s}(N(\hat{\mathfrak{A}}^{\delta}(U_r(P)-D)), \tilde{\mathfrak{X}}^{m,\ell})$ with algebraic growth condition $(\tilde{\mathfrak{d}}_1, \tilde{\mathfrak{d}}_2, \tilde{\mathfrak{d}}_3) : P \in D \cap V$ *. Also we fix an expression of a cocycle $\tilde{\varphi}^{s}$ in the following manner

$$(3.2.21)_1 \qquad \tilde{\varphi}^{s}(Q_0, \ldots, Q_s) = \varphi^{s}_{1,\ldots,\ell}(Q_0 \ldots Q_s) \cdot \mathfrak{F}^{m,\ell}_{1,\ldots,\ell},$$

where $\tilde{\Delta}_{\delta}(Q_0 \ldots Q_s) = \bigcap_{\nu=0}^{s} \tilde{\Delta}_{\delta}(Q_\nu) \neq \phi$ and $\varphi^{s}_{1,\ldots,\ell}(Q_0, \ldots, Q_s)$ is in $\Gamma(\tilde{\Delta}_{\delta}(Q_0, \ldots, Q_s), \hat{\mathcal{O}}_X)$.

Moreover, the following estimation of $\varphi^{s}_{1,\ldots,\ell}(Q_0 \ldots Q_s : Q')$ is assumed.

$$(3.2.21)_2 \qquad |\varphi^{s}_{1,\ldots,\ell}(Q_0, \ldots Q_s : Q')| \leq \tilde{\mathfrak{d}}_1 \cdot d(Q', \hat{D})^{-\tilde{\mathfrak{d}}_2} \cdot d(Q', V)^{\tilde{\mathfrak{d}}_3}.$$

Remark that the fact $\tilde{\varphi}^{s}$ is a s-cocycle is equivalent to say that $\{\varphi^{s}_{1,\ldots,\ell}(Q_0, \ldots, Q_s)\}_{Q_0,\ldots Q_s} : \Delta_\delta(Q_0, \ldots, Q_s) \neq \phi;$ is in $Z^{s}(N(\pi \Delta^{\delta}_{r}(\overline{U}_P - D)), \mathcal{O}_X)$.

$(ii)_{1,1}$ In the first case the assertion $(\quad)^{\ell}$ is obviously an immeadiate consequence of Lemma 3.1.

* We assume that P is in a neighbourhood N_0 of P_0 (fixed once and for all).

(ii)$_{1,2}$ In the second case, our situation is more subtle. We use the fact that $\tilde{\mathfrak{X}}^{m',0}$ is a subsheaf of $\mathfrak{X}^{m,0}$ ($m' \geq m$). Note that the existence of an ex.d.pre. v.a. q. $Q^{a}_{pre,v,a} = \{\tilde{\delta}^{a}_{1}, \tilde{M}^{a}_{1}, \tilde{M}^{a}_{2}, \tilde{\mathbb{P}}^{a}_{11}, \mathbb{P}^{a}_{12}, \tilde{L}^{a}_{2}, \tilde{L}^{a}_{31}, \tilde{L}^{a}_{32}, \mathcal{L}^{a}\}$ for the sheaf $\mathfrak{X}^{m,1}$ leads to the following.

(3.2.25)$_1$ There exists a datum $Q''^{0} = \{\delta''^{0}, M''^{0}_{1}, M''^{0}_{2}, \mathbb{P}''^{0}_{11}, \mathbb{P}''^{0}_{12}, \mathbb{P}''^{0}_{21}, \mathbb{P}''^{0}_{22}, L''^{0}_{2}, L''^{0}_{3}, \mathcal{L}''^{0}\}$, depending (U. D)$_{v.a}$ only, so that the following modified form of Theorem 2.2.1. is valid with the datum Q''^{0}.

(3.2.25)$_2$ For a given cocycle $\tilde{\varphi}^{s} \in Z^{s}(N(\tilde{\alpha}^{\delta}(\tilde{U}_{\gamma}(\varrho)-\tilde{D})), \tilde{\mathfrak{X}}^{m,0}_{\rho})$ with algebraic growth condition $(\tilde{d}_1, \tilde{d}_2, 0)$, there exists a $(s-1)$ — cochain $\psi^{s-1} \in C^{s-1}(N(\hat{\alpha}^{\delta}(\tilde{U}_{\gamma}(\varrho)-\tilde{D})), \tilde{\mathfrak{X}}^{m,0})$ of algebraic growth $(\tilde{d}''_1, \tilde{d}''_2, 0)$ satisfying the condition

$$\delta_{\check{C}ech}(\psi^{s-1}) = \varphi^{s} \quad \text{in } C^{*}(N(\tilde{\alpha}^{\delta''}(\tilde{U}_{\gamma}(\varrho)-\tilde{D})), \tilde{\mathfrak{X}}^{m'',0})$$

where quantities (r''), (δ'') m'', $(\tilde{d}''_1, \tilde{d}''_2)$ are given in the following way.

(3.2.25)$'_a$ $r'' = M''^{0}_{1}(r)$, $(\delta'') = \mathcal{L}''^{0}(\delta)$, $m'' = [L(m)]$

$\tilde{d}''_1 = M''^{0}_{2}(r)^{-1} \cdot \mathbb{P}''^{0}_{11}(d_1, \delta_1)t \cdot t^{\mathbb{P}''^{0}_{12}(d_2,\delta_2)}$, $d''_2 = L''^{0}_{2}(d_2, \delta_2)$.

The above fact is deduced from ( ) ( ) in a quite elementary way and we shall omit its details.

Next we remark the following : Assume that a cocycle

$\tilde{\tilde{\varphi}}^{s} \in Z^{s}(N(\mathfrak{U}(\tilde{\tilde{\mathfrak{U}}}_{r}(\varepsilon)-\tilde{D}) , \tilde{\mathfrak{X}}^{m,r})$ of algebraic growth $(\tilde{\tilde{d}}) = (\tilde{\tilde{d}}_1, \tilde{\tilde{d}}_2, \tilde{\tilde{d}}_3,)$ is given . Then the validity of the w. sy. q. for the u.d. w. sy.q. $(U. D)_{w.sy.q} = \{X, V, V_{\cap} D, \mathcal{Y}_V, \mathcal{Y}_X\}$ leads to the following statement.

(3.2.22) <u>There exists a s-cocycle $\tilde{\tilde{\varphi}}^{\prime s} \in Z^{s}(N(\mathfrak{U}(\tilde{\tilde{\mathfrak{U}}}^{\prime}_{r}(\varepsilon)-\tilde{D}), \tilde{\mathfrak{X}}^{m',0})$ of algebraic growth $(\tilde{\tilde{d}}_1'', \tilde{\tilde{d}}_2'', 0)$ so that</u>

<u>$(3.2.22)_1$ $\tilde{\tilde{\varphi}}^{\prime s} = \tilde{\tilde{\varphi}}^{s}$ (as elements in $Z^{s}(N(\mathfrak{U}(\tilde{\tilde{\mathfrak{U}}}^{\prime}_{r}(\varepsilon)-\tilde{D}), \mathfrak{X}^{m',0})$, so far as $\tilde{\tilde{d}}_3 > \tilde{\tilde{L}}_3^{0'}(m)$.</u>

<u>where quantities $(\tilde{\tilde{\delta}}')$, $\tilde{\tilde{m}}'$, $(\tilde{\tilde{d}}_1', \tilde{\tilde{d}}_2')$ are given by</u>

<u>$(3.2.22)_2$ $\tilde{\tilde{\delta}}' = \tilde{\tilde{\mathcal{A}}}^{0}(\tilde{\tilde{\delta}})$, $\tilde{\tilde{m}}' = [\tilde{\tilde{L}}_3^{0}(\tilde{\tilde{d}}_3)] + m$,</u>

<u>$\tilde{\tilde{d}}_1' = \tilde{\tilde{P}}_{11}(\tilde{\tilde{d}}_1, \tilde{\tilde{\delta}}_1)$, $\tilde{\tilde{d}}_2' = \tilde{\tilde{L}}_2^{0}(\tilde{\tilde{d}}_2, \tilde{\tilde{\delta}}_2)$ $\tilde{\tilde{P}}_{22}(\tilde{\tilde{d}}_3, m)$, where $\tilde{\tilde{\mathcal{A}}}^{0}$, $\tilde{\tilde{L}}$'s and $\tilde{\tilde{P}}$'s depend on $(U.D)_{v.a}$ only.</u>

Now it is <u>easy</u> to see that $(3.2.22)_1$ and $(3.2.22)_2$ imply the following conclusion.

<u>(3.2.22) Under the condition of the validity of w. sy. q. for (X, V, $V_{\cap}D$, $\mathcal{Y}_X$, $\mathcal{Y}_V$) and the pre v.a. for $(U.D)_{v.a}$, the sheaf $\tilde{\mathfrak{X}}^{m,0}$ implies the v.a. for the sheaf $\tilde{\mathfrak{X}}^{m,0}$.</u>

(iii) Next we shall consider implications ($)_{i}$. In this step, both cases of the pre. v. a. and v.a. are argued in a pararell way. Remark the following.

(3.2.23)$_1$ If the relation ; $2^{-1}\cdot\delta_1 > \max(\delta_1'\cdot\ell^{\delta_2'},\ \delta_1''\cdot\ell^{\delta_2''})$, $\delta_2 < \min(\delta_2',\delta_2'')$ are valid for couples (δ), (δ') and (δ''), then, for any points $Q_0,\ldots,Q_s$, Q satisfying the condition

$$\Delta_{\delta'}(Q_0,\ldots\ldots,Q_s\ ;\ \tilde{D}) \ni Q\ ,$$

the inclusion relation

$$\Delta_{\delta''}(Q;\tilde{D}) \subset \Delta_{\delta}(Q_j;\tilde{D}) \quad (j=0,\ldots\ldots,s)$$

is true.

(3.2.23)$_2$ Assume that the following conditions are valid for couples ($\tilde{\delta}'$), ($\tilde{\delta}''$) and ($\tilde{\delta}'''$)

(2.2.23)$_2'$ $\quad 2^{-1}\cdot\tilde{\delta}_1''' > (\tilde{\delta}_1'\cdot\ell^{\tilde{\delta}_2'},\ \tilde{\delta}_1''\cdot\ell^{\tilde{\delta}_2''}) \quad \tilde{\delta}_2''' < (\tilde{\delta}_2',\ \tilde{\delta}_2'')$.

Then, for any points $Q_0,\ldots\ldots,Q_s, Q \in$ ( ) satisfying the condition

$$\Delta_{\tilde{\delta}'}(Q_0,\ \ldots.\ Q_s) \ni Q.$$

the inclusion relation

(3.2.23)$_3$ $\quad \Delta_{\tilde{\delta}'''}(Q) \supset \Delta_{\tilde{\delta}''}(Q_j)\ (j=0,\ldots\ldots,s)$

holds.

Now let us take up a s-cocycle $\Psi^s \in Z^s(N(\hat{\mathfrak{N}}^s(\mathfrak{U}_r(\varepsilon\cdot\vartheta), \tilde{\mathfrak{X}}^{m,i}) \ (i=0,\dots,\ell-1)$ with algebraic growth condition $(\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3)$. For each non empty intersection $\tilde{\Delta}_s(Q_0 \dots\dots Q_s, \tilde{\mathfrak{D}}) (= \bigcap_{\nu=0}^{s} \tilde{\Delta}_s(Q_\nu : \tilde{\mathfrak{D}}))$, we express $\Psi^s(Q_0, \dots Q_s)$ explicitly in the following form.

$$\Psi^s(Q_0, \dots, Q_s) = \sum_{1\le \tau_1<\dots<\tau_i \le \ell} \psi^s_{\tau_1\dots\tau_i}(Q_0, \dots\dots, Q_s)\, \tilde{\mathfrak{f}}^{m,i}_{\tau_1\dots\tau_i},$$

where $\psi^s_{\tau_1\dots\tau_i}(Q_0,\dots,Q_s)$ is in $\Gamma(\tilde{\Delta}_s(Q_0, \dots Q_s, \tilde{\mathfrak{D}}), \mathcal{O}_X)$.

We assume the following estimation.

$$|\psi^s_{\tau_1\dots\tau_i}(Q_0, \dots Q_s, Q)| \le d_1 \cdot d(Q, \mathfrak{D})^{-\delta_2} \cdot d(Q, V)^{\delta_3}.$$

Then one can choose maps $\{\mu\}$ of e-$\mathcal{L}$-types depending on the fixed ~~e.x.~~ d.w.s.f. ξ $\mathcal{O}^i(\dots)$ only with which we know the following.

(3.2.24) If the intersection $\tilde{\Delta}_{s'}(Q_0, \dots\dots, Q_{s+1}, \tilde{\mathfrak{D}}')$ then, for any point $Q \in \tilde{\Delta}_{s'}(Q_0, \dots\dots Q_{s+1}, \mathfrak{D}')$, there exists a neighbourhood $\tilde{\Delta}_{s''}(Q : \tilde{\mathfrak{D}}) \supset \tilde{\Delta}_{s'}(Q_0, \dots\dots, Q_{s+1}, \tilde{\mathfrak{D}})$ and a vector $\tilde{\psi}^{s+1}_{Q_0\dots Q_{s+1}} \in \Gamma(\tilde{\Delta}_s(Q, \tilde{\mathfrak{D}}), \mathcal{O}_X^{\binom{\ell}{i+1}})$ so that the equation

$$(3.2.24)_1 \quad \left\{\sum_{\mu} (-1)^{\mu-1} \psi^s_{\tau_1\dots\tau_i}(Q_0, \dots \hat{Q}_\mu \dots Q_{s+1})\right\}_{1\le\tau_1<\dots<\tau_i\le\ell} = K(\tilde{\psi}^{s+1}_{Q_0\dots Q_{s+1}});\ (\tilde{\psi}^{s+1}_{Q_0\dots Q_{s+1}}) = (\tilde{\psi}^{s+1}_{\tau_1\dots\tau_{i+1}}(Q_0\dots))$$

holds.

(Remark) As maps $\mathcal{L}_i (i=1,2)$ of e-ℓ type , we can choose such ones explicitly : Let $\mathcal{L}_1$ be a map $\mathcal{L}_1 : (\delta_1, \delta_2) \longrightarrow (\delta') = (e^{-2\delta_1}, 2\delta_2)$, and let $\mathcal{L}_M : (\delta') \to (\delta'')$ be an (e-ℓ)-map satisfying the condition $\delta_1'' d(Q,\mathcal{D})^{\delta_2''} < M(\delta_1' d(Q,\mathcal{D}))^{\delta_2'}$ for any $Q \in I - \mathcal{D}$. Then the composition

$$\mathcal{L}_2 = \mathcal{L}_1 \circ \mathcal{L}_M \circ \mathcal{L}_1$$ suffices for our purpose.

Now , by (3.2.) , the following estimation

$$(3.2.26) \qquad \left| \tilde{\mathcal{Y}}^{s+1}_{V_1 \cdots V_{s+1}}(Q_0, \ldots, Q_{s+1} : Q) \right| \leq \tilde{\alpha}_1' \cdot d(Q; \mathcal{D})^{-\tilde{\alpha}_2^{*\prime}} \cdot d(Q, \mathbf{V})^{\tilde{\alpha}_3}$$

is valid , where quantities $(\tilde{\alpha}_1^{\prime *}, \tilde{\alpha}_2^{\prime *}, \tilde{\alpha}_3^{\prime *})$ are determined by

$$\tilde{\alpha}_1^{\prime *} = \mathbb{P}_1^{\prime *, i}(\tilde{\alpha}_1, \tilde{\delta}_1) \cdot e^{\mathbb{P}_2^{\prime *, i}(\tilde{\alpha}_2, \tilde{\delta}_2)} \cdot e^{\mathbb{P}_3^{\prime *, i}(\tilde{\alpha}_3, n)} \quad \tilde{\alpha}_2^{\prime *} = \mathbb{L}_2^{\prime *, i}(\delta_2 + \tilde{\alpha}_2),$$

$$\tilde{\alpha}_3^{*\prime} = \mathbb{L}_{31}^{\prime *, i}(\tilde{\alpha}_3) - \mathbb{L}_{32}^{\prime *, i}(\mathcal{M}).$$

In the above the datum $(\mathbb{P}_1^{\prime *, i}, \ldots, \mathbb{L}_{32}^{\prime *, i})$ is of course depending on $(U, D)_{w, \mathcal{Y}, \mathcal{G}}$ only. We write the right side of $(3.2.24)_1$ (i.e. the term $\tilde{K}(n, i+1) \cdot \tilde{\mathcal{Y}}^{s+1}_{Q_0 \cdots Q_{s+1}}$) as $\tilde{\mathfrak{Y}}^{s+1}_{Q_0 \cdots Q_{s+1}}$.

This ~~element~~ $\{\tilde{\mathfrak{Y}}^{s+1}_{Q_0 \cdots Q_{s+1}}\}$ is in $\Gamma(\triangle_\delta(Q_0 \cdots Q_M, \mathcal{D}), \tilde{\mathfrak{X}}^{m, i+1})$

The collection $\{\tilde{\mathfrak{Y}}^{s+1}_{Q_0 \cdots Q_{s+1}}\}_{Q_0 \cdots Q_{s+1}}$ defines a (s + 1)-cocycle : $\tilde{\mathfrak{Y}}^{s+1} \in Z^{s+1}(N(\hat{\pi}^{\delta}(\bar{U}_i(P)) - \mathcal{D}), \tilde{\mathfrak{X}}^{m, i+1})$ (i = 0, 1 ,..., ℓ-1). Apply the induction hypothesis for (i + 1) (of the both assertions pre. v. a. and v. a.) to $\tilde{\mathfrak{Y}}^{s+1}$. Then we know the existence of data $Q^{i+1}_{pce.v.a} = \{\tilde{\delta}, M_1^{i+1}, M_2^{i+1}, \mathbb{P}_1^{i+1}, \mathbb{P}_2^{i+1}, \mathbb{P}_3^{i+1}, \mathbb{L}_2^{i+1}, \mathbb{L}_{3,1}^{i+1}, \mathbb{L}_{3,2}^{i+1}, \mathcal{L}^{i+1}\}$ as well as $Q^{i+1}_{v.a} = \{\delta, M_1^{i+1}, M_2^{i+1}, \mathbb{P}_1^{i+1}, \mathbb{P}_2^{i+1}, \mathbb{P}_3^{i+1}, \mathbb{L}_2^{i+1}, \mathbb{L}_{3,1}^{i+1}, \mathbb{L}_{3,2}^{i+1}, \mathcal{L}^{i+1}\}$ depending on $(\mho.D)_{v.a}$ only, so that the following facts are valid with these data.

(3.2. 27) There exists a s − cochain $\tilde{\mathfrak{F}}'^{s}_{v.a} \in C^{s}(N(\hat{\mathcal{R}}^{s'}(\overline{\mho}_{r}-\tilde{\mathfrak{D}}), \tilde{\mathfrak{X}}^{m,i+1})$ (resp. $\tilde{\mathfrak{F}}''^{s}_{pce.v.a} \in C^{s}(\mathcal{N}(\hat{\mathcal{R}}^{s'}(\overline{\mho}_r\text{-}\mathfrak{D})), \tilde{\mathfrak{X}}^{m,i+1})$ of algebraic growth $(\alpha_1', \alpha_2', \alpha_3')$ (resp. $(\alpha_1'', \alpha_2'', 0)$) so that the equation

$$(3.2.27)_1 \quad \delta_{Céch}(\tilde{\mathfrak{F}}'^{s}) = \mathfrak{F}^{s+1} \quad \left(\text{resp. } \delta_{Céch}(\tilde{\mathfrak{F}}''^{s+1}) = \mathfrak{F}^{s+1}\right)$$

holds. Here quantities $\{r', (s'), (\alpha') = (\alpha_1', \alpha_2', \alpha_3')\}$ (resp. $\{r'', (s''), (\alpha'') = (\alpha_1'', \alpha_2'', \alpha_3'')\}$) are determined by the following formulas.

$$(3.2.27)_1^{v.a} \quad r' = M_1^{i+1}(r), \qquad \alpha_1' = M_2^{i+1}(r)^{-1} \cdot \mathbb{P}_1^{i+1}(\alpha_1^*, s_1^*) \cdot \mathcal{C}^{\mathbb{P}_2^{i+1}(\alpha_2^*, s_2^*)} \cdot \mathcal{C}^{\mathbb{P}_3^{i+1}(\alpha_3^*, s_3^*)}$$

$$\alpha_2' = \mathbb{L}_2^{i+1}(s^*, \alpha^*), \qquad \alpha_3' = \mathbb{L}_{3,1}^{i+1}(\alpha_3^*) - \mathbb{L}_{3,2}^{i+1}(m).$$

$$(3.2.27)_2^{pce.v.a} \quad r'' = M_1^{i+1}(r), \qquad \alpha_1'' = M_2^{i+1}(r)^{-1} \mathbb{P}_1^{i+1}(\alpha_1^*, s_1^*) \cdot \mathcal{C}^{\mathbb{P}_2^{i+1}(\alpha_2^*, s_2^*)} \cdot \mathcal{C}^{\mathbb{P}_3^{i+1}(\alpha_3^*, s_3^*)}$$

$$\alpha_2'' = \mathbb{L}_2^{i+1}(\alpha_2^*, s_2^*), \qquad \alpha_3'' = 0$$

Moreover, define a s − cocycle $\{\mathcal{Y}'^{s}_{v_0\cdots v_s}\}_{1\le v_0<\cdots<v_s\le \ell} \in Z^{s}(N(\mathcal{R}^{s'}(\overline{\mho}_r(\mathbb{P})-\mathfrak{D}), \mathcal{O}^{(\xi)})$ (resp. $\{\mathcal{Y}''_{v_0\cdots v_{s}}\}_{1\le v_0<\cdots<v_{s}\le\ell} \in Z^{s}(N(\mathcal{R}^{s'}(\overline{\mho}_r(\mathbb{P})-\mathfrak{D}), \mathcal{O}^{(\xi)})$) by

$$(3.2.27)' \quad \{\mathcal{Y}^{s}_{v_0\cdots v_s}(Q_0, \ldots, Q_s)\}_{1\le v_0<\cdots<v_s\le\ell} = \Big\{\mathfrak{F}'^{s}(Q_0, Q_1, \ldots, Q_s) \cdots\cdots + \{\mathcal{Y}^{s}_{v_0\cdots v_s}(Q, \ldots\ldots Q)\}_{1\le v_0<\cdots<v_s\le\ell}\Big\}.$$

$$(3.2.27)'' \quad \{\varphi''^{s}_{\nu_1\cdots\nu_i}(Q_0, \ldots\ldots, Q_s)\}_{1\le\nu_1<\cdots<\nu_i\le\ell} = \mathfrak{F}''^{s}(Q_0, \ldots, Q_s)$$

$$+ \{\varphi^{s}_{\nu_1\cdots\nu_i}(Q_0, \ldots\ldots, Q_s)\}_{1\le\nu_1<\cdots<\nu_i\le\ell}.$$

it is clear that the following equations are valid.

$$(3.2.28)_1 \quad K(m, i)\{\varphi^{s}_{\nu_1\cdots\nu_i}(Q_0,\cdots Q_s)\}_{1\le\nu_1<\cdots<\nu_i\le\ell} = K(m, i)\{\varphi'^{s}_{\nu_1\cdots\nu_i}(Q_0,\cdots Q_s)\}_{1\le\nu_1<\cdots<\nu_i\le\ell}$$

$$(3.2.28)_2 \quad K(m, i)\cdot\{\varphi^{s}_{\nu_1\cdots\nu_i}(Q_0,\cdots Q_s)\}_{1\le\nu_1<\cdots<\nu_i\le\ell} = K(m, i)\{\varphi''^{s}_{\nu_1\cdots\nu_i}(Q_0,\cdots Q_s)\}_{1\le\nu_1<\cdots<\nu_i\le\ell}$$

Of course the left side of the above equations $(3.2.28)_1$ and $(3.2.28)_2$ are nothing else than the given $\mathfrak{F}^{s}$ after taking a refinement map $\tilde{r}': \hat{\mathfrak{N}}^{\delta'}(\overline{U}_{\ell}(\varepsilon)-D) \hookrightarrow \hat{\mathfrak{N}}^{\delta}(\overline{U}_{\ell}(\varepsilon)-D)$, $(\tilde{r}': \hat{\mathfrak{N}}^{\delta''}(\overline{U}_{\ell}(\varepsilon)-D) \hookrightarrow \hat{\mathfrak{N}}^{\delta}(\overline{U}_{\ell}(\varepsilon)-D))$

Because $\{\varphi'^{s}_{\nu_1\cdots\nu_i}\}_{1\le\nu_1<\cdots<\nu_i\le\ell}$ (resp. $\{\varphi''^{s}_{\nu_1<\cdots<\nu_i}\}_{1\le\nu_1<\cdots<\nu_i\le\ell}$) are cocycles (with coefficient in $\mathcal{O}^{(\ell)}_{i}$) , we apply Lemma 3.2. to $\{\varphi''^{s}_{\nu_1\cdots\nu_i}\}_{1\le\nu_1<\cdots<\nu_i\le\ell}$ and ( ) to $\{\varphi'^{s}_{\nu_1\cdots\nu_i}\}_{1\le\nu_1<\cdots<\nu_i\le\ell}$.

Then we obtain the following decomposition of $\{\varphi''^{s}\}$ (resp. $\{\varphi'^{s}\}$)

$$(3.2.29) \quad \delta_{\check{c}ech}(\varphi'^{s-1}) = \varphi'^{s} \quad \text{in } C^{*}(N(\hat{\mathfrak{N}}^{\delta}(\overline{U}_{\ell}(\varepsilon)-D)), \mathcal{O}^{(\ell)}_{i})$$

$$(\text{ resp. } \delta_{\check{c}ech}(\varphi''^{s-1}) = \varphi''^{s} \text{ in } C^{*}(N(\hat{\mathfrak{N}}^{\delta}(\overline{U}_{\ell}(\varepsilon))-D), \mathcal{O}^{(\ell)}_{i}))$$

in which cochains φ'^{s-1} (resp. φ''^{s-1}) are estimated in the following ways,

$$(3.2.30)_1 \quad |\varphi'^{s-1}(Q)| \leqq \bar{\bar{\partial}}'_1 \cdot d(Q:D)^{-\bar{\bar{\partial}}'_2} \cdot d(Q:\mathfrak{V})^{\bar{\bar{\partial}}'_3}$$

$$(3.2.30)_2 \quad |\varphi''^{s-1}(Q)| \leqq \bar{\bar{\partial}}''_1 \cdot d(Q, D)^{-\bar{\bar{\partial}}''_2}$$

$(3.2.30)'_1$ $\tilde{\bar{\mathfrak{z}}}'_1 = \bar{\tilde{M}}'_2(x)^{-1} \cdot \tilde{\bar{P}}'_{11}(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}) \cdot \iota^{\bar{\tilde{P}}'(\mathfrak{z}_2, \tilde{\mathfrak{z}}_2)} \cdot \iota^{\bar{\tilde{P}}'(\mathfrak{z}_3, m)}$, $\tilde{\bar{\mathfrak{z}}}'_2 = \bar{\tilde{L}}'^{i}_2(\mathfrak{z}_2, \delta_2)$, $\tilde{\bar{\mathfrak{z}}}'_3 = \bar{\tilde{L}}^{i}_{31}(\mathfrak{z}_3) \cdot \bar{\tilde{L}}^{i}_{32}(m)$

$\bar{\tilde{\mathfrak{X}}}'^{i} = \bar{\tilde{M}}^{i}_1(x)$, $\bar{\tilde{\delta}}' = \bar{\tilde{L}}'(\delta)$

$(3.2.30)'_2$ $\tilde{\bar{\mathfrak{z}}}''_2 = \bar{\tilde{M}}''_2(x)^{-1} \cdot \tilde{\bar{P}}''_{11}(\tilde{\mathfrak{z}}_1, \tilde{\delta}_1) \cdot \iota^{\bar{\tilde{P}}''(\tilde{\mathfrak{z}}_2, \tilde{\mathfrak{z}}_2)} \cdot \iota^{\bar{\tilde{P}}''(\mathfrak{z}_3, m)}$ $\tilde{\bar{\mathfrak{z}}}''_2 = \bar{\tilde{L}}''^{i}_2(\mathfrak{z}_2, \delta_2)$, $\bar{\tilde{\mathfrak{X}}}''^{i} = \bar{\tilde{M}}''^{i}_1(x)$, $\bar{\tilde{\delta}}'' = \bar{\tilde{\mathfrak{z}}}''(\delta)$.

in the above data $\bar{\tilde{M}}'$s, $\bar{\tilde{L}}'$s, $\bar{\tilde{P}}'$s and $\bar{\tilde{\mathfrak{z}}}'$s as well as $\bar{\tilde{M}}''$s, $\bar{\tilde{L}}''$s, $\bar{\tilde{P}}''$s and $\bar{\tilde{\mathfrak{z}}}'$s are depending on $(U.D)_{v.a}$ only.

It is clear that the equations $(3.2.28)$, $(3.2.29)_2$, $(3.2.29)$, $(3.2.30)_1$, $(3.2.30)_2$ and the estimations $(3.2.30)'_1$, $(3.2.30)'_2$ leads to the desired assertion of ( ). q.e.d.

n. In this numero, we shall be concerned with lemmas $3.2.^{mero}_{w.sy.\mathfrak{g}}{}'$ and lemma$3.2.^{n'}_{w.sy.\mathfrak{g}}$. We start with verifying lemma$3.2.^{mero}_{w.sy.\mathfrak{g}}{}'$.

<u>Proof of lemma $3.2.^{mero}_{w.sy.\mathfrak{g}}{}'$.</u> Let $(U.D)_{v.a} = \{X, V, D. \mathfrak{Y}_X, \mathfrak{Y}_V, (\ldots$ be an u.d. v. a. so that the w. sy. q. is valid for $(U.D)_{w.sy.\mathfrak{g}} = \{X, V, V^* = V \cap D, \mathfrak{Y}_X, \mathfrak{Y}_V\}$. By lemma $3.2._{v.a}$, we can assume that v.a. is valid for $(U.D)_{v.a}$. We fix an <u>ex. l.d. v.a.</u> $\mathcal{Q}^{i}_{v.a} = \{\mathfrak{Z}^{i}, M^{i}_1, M^{i}_2, P^{i}_1, P^{i}_2, P^{i}_3, L^{i}_2, L^{i}_{3,1}, L^{i}_{3,2}, \mathcal{L}^{i}\}$ once and for all. We also fix an <u>ex. q. d. w.sy. $\mathfrak{g}$.</u> $\mathcal{Q}^{i}_{w.sy.\mathfrak{g}} = (\mathfrak{Z}^{i}, M^{i}_1, M^{i}_2, M^{i}_3, P^{i}, L^{i}_{3,1}, L^{i}_{3,2})$. Take a point P in T

and take a positive number r . Moreover, assume that a vector $\tilde{\varphi}^{m,i} \in \Gamma(\tilde{\Delta}_r(P)-\tilde{\mathcal{D}}, \mathcal{O}^{(\ell)})$ so that the equation

$$(3.2.31)_1 \quad \tilde{K}(m, i)\tilde{\varphi}^{m,i} = 0 \quad (i \neq 0, \ell)$$

and the estimation

$$(3.2.31)_2 \quad |\tilde{\varphi}^{m,i}| \leq b\cdot d(Q', \tilde{\mathcal{D}})^{-d_2} d(Q', T)^{d_3} \; ; \; Q \in \tilde{\Delta}_r(P) - \tilde{\mathcal{D}} \; ;$$

hold.

Fix a couple $(\tilde{\delta})$ of positive numbers in such a way that the couple $(\tilde{\delta})$ is chosen in an independent manner from the choice of P . Then the w.sy. q. for $(U.D)_{w.sy.q}$ is applied for any point P (near P_0) and for $\tilde{\varphi}^{m,i} \in \Gamma(\tilde{\Delta}_r(P)-\tilde{\mathcal{D}}, \mathcal{O}^{(\ell)})$, and so we find a vector $(i = 0, 1, \ldots l-1)$ $\tilde{\varphi}^{m,i+1}_Q$ in $\Gamma_{\tilde{\delta}}(\Delta_{\tilde{\delta}}(Q;\tilde{\mathcal{D}}), \mathcal{O}^{(\ell)})$; $Q \in \tilde{U}_r(P)-\tilde{\mathcal{D}}$; so that the equation

$$(3.2.32)_1 \quad \tilde{K}(m, i+1)\cdot \tilde{\varphi}^{m,i+1}_Q = \tilde{\varphi}^{m,i} \quad \text{in } \Gamma(\Delta_{\tilde{\delta}}(Q;\tilde{\mathcal{D}}), \mathcal{O}^{(\ell)})$$

and the estimation

$$(3.2.32)_2 \quad |\tilde{\varphi}^{m,i+1}_{Q}(Q')| \leq b'\cdot d(Q', \tilde{\mathcal{D}})^{-d_2'} \cdot d(Q', T)^{d_3'} \; ; \; Q' \in \Delta_{\tilde{\delta}}(Q;\tilde{\mathcal{D}}) \, ,$$

are valid, where b', d_2', d_3', are given by

$$(3.2.32)_3 \quad b' = M_1(b)\cdot P(d_3,\delta_2)\cdot e^{P(d_3,\delta_2)}\cdot e^{P(d_3,m)}, \; d_2' = L_2(d_2+\delta_2), \; d_3' = L_{31}(d_3) - L_{32}(m),$$

3-2-51 with data $\{M, P, P, P, L_2, L_{31}, L_{32}\}$ depending on $(U.D)_{v.a}$ only.

If $\Delta_\delta(Q, \tilde{\mathfrak{D}}) \cap \Delta_\delta(Q', \mathfrak{D}) \neq \phi$, then the equation

$$(3.2.32)_4 \quad \tilde{K}(m, i+1)(\mathcal{Y}^{m,i+1}_Q - \mathcal{Y}^{m,i+1}_{Q'}) = 0,$$

holds.

If $i + 1 = \ell$ the equation ( ) shows that the equation

$$(3.2.32)_5 \quad \mathcal{Y}^{m,i+1}_Q - \mathcal{Y}^{m,i+1}_{Q'} = 0,$$

is true.

In view of the equation ( ), the assertion ( ) of the present lemma is verified in the case of $i+1 = \ell$.

We assume that $i + 1 \neq \ell$. Apply the w. sy. q. for the equation ( ): namely there exists a couple(γ) depending on $Q_{w.sy.\ell}$ only with which we find an element $\tilde{\mathcal{Y}}^{m,i+2}_{Q,Q'} \in \Gamma(\Delta_\gamma(Q,Q';\mathfrak{D}), \mathcal{O}^{(\ell_{i+2})})$ satisfying the following.

$$(3.2.33)_1 \quad \tilde{K}(m, i+2)\cdot\tilde{\mathcal{Y}}^{m,i+2}_{Q,Q'} = \tilde{\mathcal{Y}}^{m,i+1}_Q - \tilde{\mathcal{Y}}^{m,i+1}_{Q'},$$

$$(3.2.33)_2 \quad |\tilde{\mathcal{Y}}^{m,i+2}_{Q,Q'}| \leqq b'' d(Q', \tilde{\mathfrak{D}})^{-\partial_2''} d(Q', T)^{\partial_3''} \quad (Q' \in \Delta_\delta(Q,Q';\mathfrak{D}))$$

In the above $\{b'', \partial_2'', \partial_3''\}$ are given by

$$(3.2.33)_3 \quad b'' = M(b)\cdot P(\delta_1)\cdot e^{P(\partial_3,\delta)} \cdot e^{P(\partial_3, m)}$$

$$\partial_2'' = L(\partial_2 + \delta_2), \qquad \partial_3'' = L_{31}(\partial_3) - L_{32}(m).$$

(Of course, polynomials M_b, P's and linear functions L' are determined by $\mathcal{Q}_{w,s,\ell}((\upsilon, \mathcal{D})_{u,s,\ell})$ only.)

For any pair $(Q, Q') \in \overline{\Delta}_r(\mathcal{D})\text{-}\mathcal{O}$, we fix solutions $\mathcal{Y}^{m,i+2}_{Q,Q'}$ (i = 0 ... $\ell-2$). Denoting the right side of (by $\mathcal{Y}^{(m,i+2),1}_{Q,Q'} \in \Gamma(\mathcal{A}_\delta(\mathcal{O}\mathcal{D}), \mathfrak{X}^{m,i2})$ we obtain a 1- cocycle $\{\mathcal{Y}^{(m,i+2),1}_{Q,Q'}\}_{Q,Q'}$ by the equation

$$(3.2.34) \quad \mathcal{Y}^{(m,i+2),1}_{(Q,Q')} = \mathcal{Y}^{(m,i+2),1}_{Q,Q'}$$

Now our position is to apply the v. a. to the above cocycle $\mathcal{Y}^{(m,i+2),1}$. Then we find a 0- cocycle $\mathcal{Y}'^{(m,i+2),0} \in C^0(\mathcal{N}(\mathcal{U}(\overline{\mathcal{U}}_r(\mathcal{D})\text{-}\mathcal{O}), \mathfrak{X}^{m,i2}_{\delta''}))$ with algebraic growth condition $(\partial_1, \partial_2, \partial_3)$, so that the equation

$$(3.2.35) \quad \delta_{\text{Cech}}(\mathcal{Y}'^{(m,i+2),0}) = \mathcal{Y}^{(m,i+2),1}$$

Here quantities $\{\Upsilon', \partial_1', \partial_2', \partial_3'\}$ are expressed as follows.

$$(3.2.35)' \quad b = M^{-1}(\Upsilon)\, P(\quad), \qquad = L(\quad)$$

$$= L(\quad) - L(\quad).$$

3-2-53

As in ( ), we define an element (( ).

(Q (P)) by

(3.2.36) = ,

It is clear that we obtain the following equations.

(3.2.36)' , for Q , Q

(P) if (,) is not empty.

(3.2.36)'' K(m, i + 1) = .

Finally, we obtain the following estimation of the element .

(3.2.36)''' bd (, V) d (, v)

with (, , : = =

= .

In the above data are depending on the datum only.

The equations ( ), ( ) and the estimation ( ) gives an answer for our problem. q.e.d.

n. 6 . In this numero we shall consider the final stage of proving lemma 5.2 : Our arguements here are divided into three steps. Because each step requires certain detailed arguements, we shall outline our method here : We start with a pair $\{(U.D)^{*}_{w.sy.q}, (U.D)^{-1}_{w.sy.q}\}$ of two u.d. w. sy. q . as in lemma . Such a pair will be called an admissible pair of u. d. w. sy. q. Our first step is to find a suitable divisor $D \supset V'^{*}$ defined by (h) so that $(X, V, D, \mathfrak{g}_X, \mathfrak{g}_V, (h))$ turns out to be an underlying datum for v. a . Then we can apply the mero. v.a for a point $P \in D \cap V'^{*}$ (except a proper subvariety V''^{*} of V'^{*})... Secondly we reduce our result , which is obtained in a meromorphic mean by making use of mero. v.a .. to a problem in $\tilde{\mathfrak{D}} = X \cap D$. This is a key step in reducing the dimension of the ' ambient space X ' and the theorem of Artin — Ree (c.f. M.Nagata[ ]) with a care about quantitative properties) enters into . Finally we combine results done in the first and the second steps .

3-2-55

(i) Let $(U.D)'^{*}_{w.sy.g}$, $(U.D)^{*}_{w.sy.g}$ be an <u>admissible pair of u.d. w. sy. q.</u> . We express the irreducible decomposition of $V'^{*}_{P_0}$ in the following form:

$$V'^{*}_{P_0} = \bigcup_{\delta_1} V'^{*}_{P_0,\delta_1} \bigcup_{\delta_2} V'^{*}_{P_0,\delta_2}$$

, where the first components $V'^{*}_{P_0,\delta_1}$'s exhaust all the irreducible components of $V'^{*}_{P_0}$ which do not coincide with any irreducible components of V^{*}_{0} while the second components $V'^{*}_{P_0,\delta_2}$'s are irreducible components of $V'^{*}_{P_0}$ which are at the same time irreducible components of $V^{*}_{P_0}$. Germs $V'^{*}_{P_0,\delta_1}$'s and $V'^{*}_{P_0,\delta_2}$ are called of first and second type respectively. Let D^{*} be a divisor defined by h^{*} (in a small neighbourhood of P_0). The irreducible decomposition of $D^{*}_{P_0}$ is written in the following form.

$$D^{*}_{P_0} = \bigcup_{\delta} D^{*}_{P_0,\delta} .$$

Moreover, we mean by $\widetilde{\mathfrak{D}}^{*}$ the intersection $\widetilde{\mathfrak{D}}^{*} = D^{*} \wedge \Sigma$ and by $\widetilde{\mathfrak{D}}^{*}_{P_0}$ the germ of $\widetilde{\mathfrak{D}}^{*}$ at P_0 , in the following form.

$$\tilde{D}^{*}_{P_0} = \bigcup_{\gamma} \tilde{D}^{*}_{P_0,\gamma} .$$

Let us consider the following conditions on $D^{*}_{P_0}$ and $\tilde{D}^{*}_{P_0}$.

(3.2.)$_{1.1}$ There exists no relations : $D^{*}_{P_0,\gamma} \supset X_{P_0,\gamma'}$ between irreducible components $D^{*}_{P_0,\gamma}$'s of $D^{*}_{P_0,\gamma}$ and $X_{P_0,\gamma'}$'s of X_{P_0}.

Note that the above condition implies the following

(3.2.37)$'_{1.1}$ There is no identities : $D'^{*}_{P_0,\gamma} = X_{P_0,\delta'}$ between irreducible components $D^{*}_{P_0,\delta}$'s of $D^{*}_{P_0}$ and $X_{P_0,\delta'}$'s of X_{P_0}.

Moreover, we assume the following conditions.

(3.2.37)$_{1.2}$ $\tilde{D}^{*}_{P_0} \supset X_{s,P_0}$.

(3.2.37)$_{2.1}$ There is no inclusion relations ; $\tilde{D}^{*}_{P_0,\delta} \supset V_{P_0,\delta'}$, $V_{P_0,\delta} \supset \tilde{D}^{*}_{P_0,\delta}$ between irreducible components $\tilde{D}^{*}_{P_0,\delta}$ of $\tilde{D}^{*}_{P_0}$ and $V_{P_0,\delta'}$ of V_{P_0}.

(3.2.37)$_{2.2}$ $\tilde{D}^{*}_{P_0} \supset V'^{*}_{P_0}$

(3.2.37)$_{2.3}$ Functions $\tilde{g}_i$ (i = 1,......, ℓ) do not vanish on components $\tilde{D}^{*}_{P_0,\delta}$ of $\tilde{D}^{*}_{P_0}$, where $\tilde{D}^{*}_{P_0,\delta}$ satisfy the condition

$\tilde{D}^*_{\varrho,\gamma} \not\subset X_{\Delta,\varrho_0}$.

(3.2.37)$_3$ For each irreducible component V'^*_{ϱ,δ_1} of first type, there exists a proper subgerm $V''^*_{\varrho_0,\delta_1}$ of $V'^*_{\varrho_0,\delta_1}$ in such a manner that the conditions below are true.

(3.2.37)$'_3$ Germs of varieties X_{ϱ_0} and $\tilde{D}^*_{\varrho_0}$ are smooth* in $V'^*_{0,\delta_1} - V''^*_{0,\delta_1}$ and the ideal sheaf $\tilde{\mathcal{J}}(\tilde{D}^*_0) \subset \mathcal{O}_X$ is generated by $\tilde{h}^*$ in $V'^*_{0,\delta_1} - V''^*_{0,\delta_1}$; $\tilde{h}^*$ is the restriction of h on X .

(In the above , varieties $V''^*_{\varrho_0,\delta_1}$'s are not determined uniquely by $(D^*, V'^*_{\varrho_0,\delta_1})$. In the actual arguements , done in later, varieties $V''^*_{\varrho_0,\delta_1}$'s will be fixed for purposes of arguements.) It is not difficult to see the existence of a divisor D^* satisfying the above conditions: ( )$_{1.1}$~( )$_{3}$.
But, for completeness we shall show a method to assure the existence of a divisor D^* quickly .
Let us write the irreducible decomposition of X_{Δ,ϱ_0} as follows.

$$X_{\Delta,\varrho_0} = \bigcup_{\gamma} X_{\Delta,\varrho_0,\gamma} .$$

---

* By this we mean that X, $\tilde{D}^*$ is smooth at each point in $V'^*_{0,\delta_1} - V''^*_{0,\delta_1}$

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Fix sets of generators $\{\tilde{f}_{s,j,\tau(j)}\}$ of the ideals $\mathcal{J}(X_{s,P_0,j})$'s and $\{f'_{j_1,\tau_1(j_1)}\}$ of $\mathcal{J}(V'^{*}_{P_0 j_1})$. Also fix generators $\{f''_{j_2,\tau_2(j_2)}\}$ of $(V''^{*}_{P_0 j_2})$, where germs $V'^{*}_{P_0,j_1}$'s and $V'^{*}_{P_0,j_2}$'s are of first and of second types respectively. By $F_{\tau,\tau_1\tau_2}$, we mean the product $F_{\tau,\tau_1,\tau_2} = \prod_{j} \tilde{f}_{s,j,\tau(j)} \cdot \prod_{j_1} f'_{j_1\tau_1(j_1)} \cdot \prod_{j_2} f_{j_2,\tau(j_2)}$. It is clear that the germ determined by $\{F_{\tau,\tau_1\tau_2}\}_{\tau,\tau_1,\tau_2}$ is nothing else than $X_{s,P_0} \vee V'^{*}_{P_0}$. we shall define a divisor D^{*}_{c} to be the zero locus of the holomorphic function h^{*}_{c} of the form

$$(3.2.38) \qquad h^{*}_{c} = \sum_{\tau,\tau_1,\tau_2} c_{\tau,\tau_1,\tau_2} \cdot F_{\tau,\tau_1,\tau_2}$$

In a little while we consider a divisor D^{*}_{c} parametrized by $(c) = (c_{\tau,\tau_1,\tau_2})$: It is clear that the zero locus of h^{*}_{c} contains X_{s,P_0} and $V'^{*}_{P_0}$. Take an irreducible component $V'^{*}_{P_0 j_1}$ of $V'^{*}_{P_0}$ of first type. Let the codimension of $V^{*}_{P_0,j_1}$ in U be d. Then, from the fact that $V'^{*}_{P_0,j_1}$ is of first type, the following is obvious.

(3.2.39) There exist functions $F^{1}_{\tau'\tau'_1\tau'_2}, \ldots, F^{d}_{\tau^d\tau^d_1\tau^d_2}$ so that $(F'_{\tau'\tau'_1\tau'_2}, \ldots F^{d}_{\tau^d\tau^d_1,\tau^d_2})$ generate the ideal sheaf of $V^{*}_{j_1}$

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(except a proper subvariety $V'^{*}_{\mathfrak{z}_1}$ of $V'^{*}_{\mathfrak{z}_1}$.)

Now , in a little while, $\Sigma^* \cong \{(c)\}$ denotes the parameter space of the function h^*_L . We consider parameters (c) near the origin 0 of Σ^* . To show the existence of a divisor D^*_L satisfying the conditions $(3.2.37)_{1.1}$ ~ $(3.2.37)_3$, it is of course enough to show the following.

(3.2.40) For each condition $(3.2.37)_{1.1}$, $(3.2.37)_{2.1}$, $(3.2.37)_{2.2}$, $(3.2.37)_{2.3}$, $(3.2.37)_3$, there exists a prpper subgerm Σ'^*_0 of Σ^*_0 (= the germ of the whole space Σ^* at 0), so that ,for $\Sigma^*_0 - \Sigma'^*_0$, each condition $(3.2.37)_{1.1}$, $(3.2.37)_{2.1}$, ..., is true.

For conditions $(3.2.37)_{1.1}$, $(3.2.37)_{2.1}$, the above is clear from the conditions (3.1.), (3.1.) .

On the other hand , it is clear that ( ) leads easily ( ) for the condition $( )_3$. Finally consider the condition ( ). This follows from the following simple observation : Let $\tilde{\mathfrak{D}}^*_{g_\mu, P_0}$ be the germ of the intersection $\tilde{\mathfrak{D}}^*_{g_\mu} \cap X$, and let $\tilde{\mathfrak{D}}^*_{P_0 g_\mu, \dot{\sigma}}$ be an irreducible component of $\tilde{\mathfrak{D}}^*_{g_\mu, P_0}$ (satisfying the condition $\tilde{\mathfrak{D}}^*_{g_\mu, P_0} \not\subset X_{\Delta, P_0}$) . Two possibilities occur : (i) $\tilde{\mathfrak{D}}^*_{g_\mu, P_0} \not\supset V^*_{P_0 \dot{\sigma}}$ for any irreducible component $V_{P_0, \dot{\sigma}}$ of V_{P_0} (i i) For a component $V_{P_0, \dot{\sigma}}$ of V_{P_0} the relation $V_{P_0, \dot{\sigma}} \subset \tilde{\mathfrak{D}}^*_{g_\mu, P_0}$ holds. In the second case , the consideration for $(3.2.37)_{2.1}$ assures ( ) for $\tilde{\mathfrak{D}}^*_{g_\mu, P_0}$. On the otherhand , in the first case there is no relation of the form $V'^*_{P_0 \dot{\sigma}} \supset \mathfrak{D}^*_{P_0, \hat{g}_\mu, \dot{\sigma}}$ (with an irreducible component $V'^*_{P_0, \dot{\sigma}}$, of $V'^*_{P_0}$) This is an immeadiate consequence of the altitude theorem of Krull ((c. f)) M. Nagata [ ]) ..... $\tilde{\mathfrak{D}}^*_{P_0, \hat{g}_\mu, \dot{\sigma}}$ is of codimension one in X_{P_0} , while the condition ( ) shows that V'^* are at least two dimensional codimension in X . It is quite clear that ( ) follows for the case of (ii) from the above consideration.

3-12-61

we say that a divisor D^* defined by $h^* = 0$, is called adequate (for an admissible pair $(U.D)^*_{w.sy.\xi}$ and $(U.D)'^*_{w.sy.\xi}$). Also a set of germs $\{V''^*_{P,\delta_1}, V''^*_{P,\delta_2}\}_{\delta_1,\delta_2}$ where $V^*_{P_{\delta_1}}$ ($V^*_{P_{\delta_2}}$) is associated with each irreducible component $V'^*_{P_0,\delta_1}$ ($V'^*_{P_0,\delta_2}$) of $V'^*_{P_0}$ will be called to be attached to $\{(U.D)^*_{w.sy.\xi}$, $(U.D)'^*_{w.sy.\xi}$, $D^*\}$ if the following conditions are satisfied.

(3.2.41) For each component $V'^*_{P_0,\delta}$ of $V'^*_{P_0}$ (either of first or of second type)

$$V''^*_{\delta} \subset V'^*_{\delta}$$

holds. Moreover, if $V'^*_{\delta_1}$ is of first type then $V''^*_{\delta_1}$ is a proper subgerm of $V'^*_{\delta_1}$.

(3.2.41') For each $V'^*_{\delta_1}$ of first type, the condition $(3.2.37)_3$ is satisfied by $V''^*_{\delta_1}$.

The arguements done here is nothing else than to say that there exists an adequate divisor D^* to $\{(U.D)^*_{w.sy.\xi}$, $(U.D)'^*_{w.sy.\xi}\}$ and a set of subgerms $\{V''^*_{\delta_1}, V''^*_{\delta_2}\}$ attached to $\{(U.D)^*_{w.sy.\xi}$, $(U.D)'^*_{w.sy.\xi}\}$ and D^*.

Henceforth, we fix such data (D^*, $\{V''^*_{\delta_1}, V''^*_{\delta_2}\}$).

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(ii) Take an irreducible component $V'^{*}_{P_0,\delta_1}$ of firsr type. Then there are uniquely determined irreducible components $\widetilde{D}^*_{P_0,\tilde{\delta}_1}$ of $\widetilde{D}^*_{P_0}$ and X_{P_0,δ_1} of X_{P_0} so that the relations $\widetilde{D}^*_{P_0,\tilde{\delta}_1} \supset V'^{*}_{P_0,\delta_1}$ and $X_{P_0,\tilde{\delta}_1} \supset V'^{*}_{P_0,\delta_1}$ hold. The germ $\widetilde{D}^*_{P_0,\tilde{\delta}_1}$ is not contained in X_{δ,P_0} and the relation $X_{P_0,\tilde{\delta}_1} \supset \widetilde{D}^*_{P_0,\tilde{\delta}_1}$ holds. Moreover, we note that <u>the dimen-sion of $\widetilde{D}^*_{P_0,\tilde{\delta}_1}$ is at least 2</u>. Remark that there exists a monomial $M^*_{\delta_1}$ and a couple ($\delta'^{*}_{\delta_1}$), which are depending on (U. D)$_{w.sy.g}$ and ($V'^{*}_{P_0,\delta_1}$, $V''^{*}_{P_0,\delta_1}$) only, in such a manner that the following is valid.

(3.2.42) For a point $P^*_{\delta_1} \in V'^{*}_{P_0,\delta_1} - V''^{*}_{P_0,\delta_1}$ and for a positive number $r < \delta'^{*}_{\delta_1,0}\, d(P^*_{\delta_1} , V''^{*}_{\delta_1})$, there exists an analytic map $\pi_{P^*_{\delta_1}} : \Delta_r(P^*_{\delta_1}) \longrightarrow \Delta_r(P^*_{\delta_1})_\wedge \widetilde{D}^*$ so that $\pi_{P^*_{\delta_1}}$ is identity on $\Delta_{\tilde{r}}(P^*_{\delta_1})_\wedge \widetilde{D}^* : \tilde{r} = M^*_{\delta_1}(x)$.

This fact is shown quickly as follows : From the cind condition ( ), we can choose functions ($\tilde{h}^{\delta_1}_1$, ..., ..., $\tilde{h}^{\delta_1}_{d-1}$, h^*) , where $\tilde{h}^{\delta_1}_1$, ... , $\tilde{h}^{\delta_1}_{d-1}$ are in $\mathcal{I}(X_{P_0,\delta_1})$ and a set of coordinates (x_1, ...$x_?$, x_d) so that the following Jacobian conditions are true.

$(J)_1$ $\quad J(h_1^{\tilde{\delta}_1}, \ldots, h_{\tilde{d}}^{\tilde{\delta}_1}) = \det \dfrac{\partial(h_1^{\tilde{\delta}_1}, \ldots, h_{\tilde{d}}^{\tilde{\delta}_1})}{\partial(x_1, \ldots, x_{\tilde{d}})} \neq 0$ in $V'^{*\tilde{\delta}}_{\tilde{\delta}_1} - V''^{*}_{\tilde{\delta}_1}$.

$(J)_2$ $\quad J(h_1^{\tilde{\delta}_1}, \ldots, h_{\tilde{d}}^{\tilde{\delta}_1}\, h) = \det \dfrac{\partial(h_1^{\tilde{\delta}_1}, \ldots, h_{\tilde{d}}^{\tilde{\delta}_1}, h)}{\partial(x_1, \ldots, x_{\tilde{d}}, x_{\tilde{d}+1})} \neq 0$, in $V'^{*}_{\tilde{\delta}_1} - V''^{*}_{\tilde{\delta}_1}$

In the above statements, $\tilde{d}$ = codim $(X_{\ell_0, \tilde{\delta}_1})$ in U. From the above conditions $(J)_1$ and $(J)_2$, the assertion ( ) follows immeadiately. in view of prop. 12. Here we choose $(\tilde{x}_\chi) = (x) - (x_1, \ldots, x_{\tilde{d}}) = (x_{\tilde{d}+1}, \ldots, x_r)$ as coordinates of $X_{\ell_0, \tilde{\delta}}$ around a neighbourhood of $V'^{*}_{\ell_0, \tilde{\delta}_1} - V''^{*}_{\ell_0, \tilde{\delta}_1}$. Also from ( ), we may assume that (x_α) are coordinates of $X_{\tilde{\delta}}$ in $\Delta_{\tilde{\delta}}(P^{*\tilde{\delta}}, V^{\alpha *}_{\tilde{\delta}_1})$ for each point $P^{*\tilde{\delta}} \in V'^{*}_{\tilde{\delta}_1} - V''^{*}_{\tilde{\delta}_1}$. In $\Delta_{\tilde{\delta}}(P^{*\tilde{\delta}}, V''^{*}_{\tilde{\delta}_1})$, $\tilde{h}$ is regarded as holomorphic function of coordinates $(\tilde{x}_\chi)$. Assume that an element $f_{P^{*\tilde{\delta}_1}} \in (\Gamma(\tilde{\Delta}_r(P^*), \tilde{\mathfrak{X}}^{m,0})$ is given $(r < \tilde{\delta}_1(P^{*\tilde{\delta}_1}, V''^{*}_{\tilde{\delta}_1})^{\tilde{\delta}_2})$. We assume the following explict expansion of $f_{P^{*\tilde{\delta}_1}}$.

(3.2.43) $\quad f_{P^{*\tilde{\delta}_1}} = \sum_{\mu=0}^{\ell} \tilde{h}_\mu \cdot \tilde{g}_\mu^{\,m}$, where $\tilde{h}_\mu$ is in $\Gamma(\tilde{\Delta}_r(P^{*\tilde{\delta}_1}), \tilde{\mathcal{O}}_X)$ and is estimated in the following way.

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(3.2.44) $|k_\mu| \leq \tilde{b} \cdot d(Q, V)^{\tilde{d}_3}$.

Moreover, we assume that $\tilde{f}_{P^{*\delta_1}}$ is divisible by $\tilde{h}$ in $\Delta_r(P^{*\delta_1})$. Now we show the following fact.

(3.2.45) There exist $\{\tilde{M}^{*\delta_1}, \tilde{M}_2^{*\delta_1}, \tilde{M}_3^{*\delta_1}, \tilde{L}^{*\delta_1}, \tilde{P}^{\delta_1}\}$ ( ) with which two conditions below are true.

(3.2.45)$_1$ So far as $\tilde{d}_3 \geq \tilde{L}^{*\delta_1}(m)$, the function $\tilde{f}_{P^{*\delta_1}}$ is written in the following form in $\tilde{\Delta}_{r'}(P^{*\delta_1})$ ~~sr~~$M_1(r)$.

(3.2.45)$_{1,1}$ $\tilde{f}_{P^{*\delta_1}} = \tilde{h} \cdot \{\sum_\mu \tilde{k}_\mu \cdot g_\mu^m\}$, where $\tilde{k}_\mu$ is holomorphic in $\tilde{\Delta}_{r'}(P^{*\delta_1})$ and is estimated as

(3.2.45)$_{1,2}$ $|k'_\mu| \leq \{\tilde{M}_2^{*\delta_1}(r)\}^{-1} \tilde{M}_3^{*\delta_1}(b) \cdot e^{\tilde{P}_2^{\delta_1}(d_3, m)}$.

(Remark) The above fact is, in its algebraic form, both nothing else than a special case of the Artin– Ree's theorem ((c.f.) M.Nagata [ ]). A quantative examination, which is quite simple, will be made use of in later.

We quickly show the above statement. The restricted functions of $\tilde{f}$, $\tilde{g}_\mu$, $\tilde{k}_\mu$, ..., on $\Delta_r(P^{*\delta_1}) \cap \mathfrak{D}^*$ are

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denoted by $\tilde{f}^*$, $\tilde{g}^*_\mu$, $\tilde{k}^*_\mu$, ... Then the conditions ( ) and ( ) imply the following equation.

$$(3.2.46) \quad \sum_{\mu=0}^{l} \tilde{k}^*_\mu \cdot \tilde{g}^m_\mu = 0.$$

From the Lojaswiecz 's inequality, we obtain the following estimation of $\tilde{k}^*_\mu$.

$$|k^*_\mu| \leqq b \cdot c_1^{\tilde{\delta}_3} \cdot d(Q, V_\cap D^*)^{c_2 \tilde{\delta}_3}; \quad Q \in \tilde{\Delta}_r(P^{*,\lambda})_\cap \tilde{\mathfrak{D}}_{P,\delta_1}.$$

where (c_1, c_2) are positive numbers determined by (, D^*, h^*) only.

Apply the induction hypothesis : Lemmas $^{n-1}_{w,sy,f}$ for $(U.D)^{*,n-1}_\delta$ $= (X_\delta, V_\cap X_\delta, V^{**}_\cap X_\delta, \mathcal{O}_X, \mathcal{O}_V)$. Then , so far as $\tilde{\delta}_3 \geqq \tilde{\tilde{L}}^{*,\delta_1}(m)$, we find a vector $\tilde{\mathcal{f}}^*_{P^{*\delta_1}} \in \Gamma(\tilde{\Delta}_r(P^{*,\delta_1})_\cap \mathfrak{D}^*, \mathcal{O}^{(l)}_{\mathfrak{D}^*})$ $r'' = \tilde{\tilde{M}}^{*,\delta}_1(r)$ so that the equation

$$\tilde{K}^*(m, 2) \cdot \tilde{\mathcal{f}}^*_{P^{*\delta}} = (\tilde{k}^*_\mu)_{\mu=1,\dots,l},$$

as well as the estimation

$$|\tilde{\mathcal{f}}^*_{P^{*\delta}}| \leqq \tilde{\tilde{M}}^{*,\delta}_2(r^{-1}) \cdot \tilde{\tilde{M}}^{*,\delta}_3(b) \cdot e^{\tilde{\tilde{P}}^{*,\delta}(\delta_3, m)} \cdot d(Q^*; V_\cap \tilde{\mathfrak{D}}^*)^{\tilde{\tilde{L}}^*_{3,1}(\delta_3) - \tilde{\tilde{L}}^*_{3,2}(m)} \; : \; Q^* \in \tilde{\mathfrak{D}}^*_\cap \tilde{\Delta}_r(P^*_{\delta_1}),$$

holds, Here $\tilde{\tilde{M}}^{*,\delta_1}_1, \tilde{\tilde{M}}^{*,\delta_1}_2, \dots \tilde{\tilde{L}}^{*,\delta_1}_2, \tilde{\tilde{P}}^{*,\delta_1}$ are determined by $((U,D)^*, (U,D)'^*, D^*, V^*,$ $h^*)$ only.

Combining ( ) and ( ), we know the ass

assertion ( ).

The statement a(b) was given in such a manner that the exponent $\tilde{d}_3$, originally given, is decreased to 0 . Now it is easily deduced that the assertion ( ) leads to the following statement. (This step is completely similar to the steo : pre. v. a ⇒ v. a,), and so we shall omit any details .)

(3. 2. 49) There exists a datum ( ) depending on (U.D), (U. D) D, only with which the following is valid:

(3. 2. 49) For a given element $\tilde{f}_{P^*_{\delta_1}} \in [\tilde{\Delta}_r(P^*_{\delta_1}), \mathfrak{X}^{m_0}]$ with the following estimation :

$$\tilde{f}_{P^*_{\delta_1}} = \Sigma_\mu \tilde{k}_\mu g^m_\mu, \quad k_\mu \in F(\tilde{\Delta}_r(P^*_{\delta_1}), \mathcal{O}_r) \; ; \; |k_\mu| \leq b ,$$

as well as the following divisiblity condition

$$\tilde{f}_{P^*_{\delta_1}} \equiv 0 \; (\tilde{h}^*) \quad in \quad \tilde{\Delta}_r(P^*_{\delta_1})$$

we find an another expression of $\tilde{f}_{P^*_{\delta_1}}$,

$$\tilde{f}_{P^*_{\delta_1}} = \tilde{h}^* \cdot (\Sigma^{\ell}_{\mu=0} \tilde{k}'_\mu \cdot g^{m'}_\mu) , \quad \tilde{k}'_\mu \in \tilde{\Delta}_r(P^*_{\delta_1})$$

where $\tilde{k}_\mu$, m' are estimated in the manner

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$(3.2.49)_2 \quad |\tilde{k}'_\mu| \leq M(\ r\ )^{-i} M(\ b\ )\epsilon^{i}, \quad m' = [L(m)]$

The above estimation is a direct consequence of ( ) and will be used in (i i i) .

(i i i) Our position here is to combine arguements in (i) and (i i) with the mero. w. sy. q. Start with a given admissible pair $(U.D)^*_{w.sy.q}$ $(U.D)'^*_{w.sy.q}$. Also we start with a fixed adequate divisor and a set of germs $V''^*_{P_0}$ attached to $(U.D)^*_{w.sy.q}$ $(U.D)'^*_{w.sy.q}$ and D^*. Furthermore, we fix an ex. mero. w. sy. q. once and for all. Arguements here will be divided into sevsral steps.

$(iii)_1$ We first show the existence of
= { M , M ,M , P L, L L ,} with which
the assertion ( ) is valid for any pair $(P^*, \psi^i_{P^*})$, where P^* is in $V'^*_{P_0} - V''^*_{P_0}$. For this,

it is easily seen that (it is enough) to show the existence of data $Q^{i}_{w.sy.q}((U, D),(U, D),\mathcal{D}^{*}, V^{\prime\prime *}, V^{\prime *}_{j}) = (\tilde{\gamma}_0^{*,\delta_i}, M_1^{*,\delta_i}, M_2^{*,\delta_i}, M_3^{*,\delta_i}, P_1^{*,\delta_i}, P_2^{*,\delta_i}, L^{*,\delta_i}, L_1^{*,\delta_i})$ for any components $V^{\prime *}_{j_1}$ of first type, with which the assertions ( ) and ( ) are true for any elements $(P^{\prime *}_{j}, \gamma^{i}_{P}) \in$ satisfying the further condition : $P^{*}_{j} \in V^{\prime *}_{j} - V^{\prime\prime *}_{j}$.

(i i i)$_{1.1}$ First we consider the case of $i = 0$.

Take a positive number $r < \tilde{\xi}_1 \cdot d(P^{\prime *}_{j}, V^{\prime\prime *}_{j})^{\gamma_2}$ and also take an element $\tilde{f} \in \Gamma(\Delta_r(P^{\prime *}_{j}), \mathcal{O}_X)$ which is estimated in the following way.

$$|\tilde{f}(Q)| \leq b \cdot d(Q, V^{\prime *}_{j_1})^{d_3} : Q \in \Delta_r(P^{\prime *}_{j}).$$

By the mero. w. sy. q., $f(Q)$ is expressed in the following manner.

$$(3.2.50) \qquad f(Q) = h^{-\lambda}(\textstyle\sum_{\mu} \tilde{f}_{\mu} \cdot \tilde{g}_{\mu}^{m}) \quad \text{in } \Delta_{r'}(P^{\prime *}_{j})$$

where $r' = M(r)$, $m = [L(d_3)]$ and $\tilde{f}_{\mu} \in \Gamma(\Delta_{r'}(P^{\prime *}_{j}), \mathcal{O}_X)$ is estimated in the following way.

$$(3.2.50)' \qquad |f_{\mu}| \leq M(r)^{-1} m(b) e^{P(d_3)}$$

Here $M, \ldots, P$, are depending on $(U, D)_{V.d.} =$

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$\{X, V, \mathcal{D}^{*}, \mathcal{Y}_X, \mathcal{Y}_V\ (h)\}$ only.

Moreover, we point out that the order of the pole $\hat{d}$ is independent of the choice of P'^{*}_{j} and of f. ( ).

Next we apply the assertion ( ) to the function $\{\sum_\mu \tilde{f}_\mu \cdot \tilde{g}_\mu^m\} = \tilde{h}^{\lambda} \cdot \tilde{f}$. Then the function f is expressed in the following manner.

$$(3.2.5/) \quad f = \tilde{h}^{-(d-1)} \left(\sum_\mu \tilde{f}_\mu^{(d-1)} \cdot \tilde{g}_\mu^{m'} \right) \text{ in } \tilde{\Delta}_{r''}(P).$$

In the above, r'', m'' are determined as follows.

$$(3.2.5/)_1 \quad r'' = M(r), \quad m'' = [L(m')].$$

Furthermore, $\tilde{f}_\mu^{(d-1)}$'s are estimated in the manner

$$(3.2.5/)_2 \quad |f_\mu^{(d-1)}| \leq M(r')^{-1} \cdot M(b') \, e^{P(m)}( )$$

Repeat the above procedure $\tilde{d}$ times. Because the order $\tilde{d}$ of poles is independent of P^{*}_{j} and of f, we know that the assertion ( ) holds. This gives a proof of the case of $i = 0$.

$(iii)_{1.2}$ We discuss the case $i \neq 0$. Start with a positive number $r < \tilde{\delta}_1(P^{*}_{\delta}, V''^{*}_{\delta_1})$ and a vector

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Of course , in the above, $M_{\Delta}^{3}M, \mathfrak{M}$, $\mathfrak{P}$, $\mathfrak{L}$, are determined by $(\mathfrak{U}, \mathfrak{D})_{v,2}$ only.

Extend the vector $\mathfrak{y}^{*,m,i+2}$ by the following equation

$$(3.2.52) \quad \tilde{\mathfrak{y}}^{i+2} = \sum_{\mu} \mathfrak{U}^{*}(\tilde{\mathfrak{y}}^{m,i+2}) \cdot \tilde{g}_{\mu}^{\alpha_4},$$

to the set $\tilde{\Delta}_{r}(P^{*})$ with a projection $\pi_{P^{*}}$.

(c. f. (i) in) .Furthermore, define a vector $\tilde{\mathfrak{y}}^{i+1} \in \Gamma(\Delta_{r}(P), \mathfrak{D})$ by

$$(3.2.53)_1 \quad \tilde{\mathfrak{y}}^{i+1} = \mathfrak{y}^{m,i+1} + \tilde{K}(m, i+2) \cdot \tilde{\mathfrak{y}}^{m,i+2}.$$

Of course

Of course $\tilde{K}(m, i+1) \mathfrak{y}^{i+1} = \tilde{K}(m, i+1) \cdot \mathfrak{y}^{i+1} = \tilde{h}^{d_i} \cdot \mathfrak{y}^{i}$

Furthermore , the estimations ( ) and ( ) shows that the vector $\tilde{\mathfrak{y}}^{i+1}$ is expanded in the following way.

$$(3.2.53)_2 \quad \tilde{\mathfrak{y}}^{i+1} = \sum \mathfrak{y}_{\mu}^{i+1} \cdot g_{\mu}^{m''}.$$

Here m'' is expressed in the fashion

$$m'' = [L_{3,1}(\alpha_4') - L_{3,2}(m)]$$

and $\mathfrak{y}_{\mu}^{i+1}$'s are estimated in the manner

$$(3.2.53)_3 \quad |\mathfrak{y}_{\mu}^{i+1}| \leq M_{\cdot}(r)^{-1} M(b) \, e^{P(\alpha_4, m)}.$$

moreover, the equations ( ), ( ) show that

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by

( ) $r'' = M(r)$, $\alpha_3'' = [L(\alpha_3')]$.

here M, L depend on $(\bar{U}, D)_{\nu, \check{a}}$ only.

Moreover $\mathfrak{y}_{\mu}^{i+1} \in \Gamma(\tilde{\Delta}(P_{+}^{*}), \mathcal{O}_X^{(k)})$ is estimated, with M, P depending on (, $\bar{U}$, D $)_{\nu a}$ only, as follows.

( ) $|\mathfrak{y}_{\mu}^{i+1}| \leq M(r)^{-1} M(b) \cdot e^{P(\alpha_3, x)}$.

If $i = \ell - 1$, then the expressions ( ), ( ) and the equation ( ) reduce the expressions ( ), ( ) and assertion to the cse of $i = 0$. Therefore we assume that $i \neq 0, \ell - 1$. As in the case of $i = 0$, we deminish the order $\tilde{d}_i$ of the pole $\tilde{h}^{*}_{=0}$ in the right side of ( ). Our arguement is completely similar to the arguement of (ii)) and will be done in the following divices. Restrict a vector $\tilde{\mathfrak{y}}^{i+1}$, a mapping $\tilde{K}(m, i+1)$ .... on $\tilde{D}^{*}$. Then the equation

$$(\quad) \quad \tilde{K}^{*}(m, i+1)(\tilde{\mathfrak{y}}^{*, i+1}) = 0,$$

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$\psi^{i} \in (\Delta_{r}(P_{0}^{*}) , \mathcal{O}^{(l)})$ with quantities

(r, b, α_3), Assume the following equation.

( ) $\tilde{K}(m, i)\cdot\psi^{i} = 0 \quad (i = 1, 2, \cdots, b-1)$

The mero, w. sy. q. claims the existence of a vector

$\tilde{\psi}^{i+1} \in \Gamma(\Delta_{r'}(P), \mathcal{O}_{X}^{(l+1)})$ so that the following equation

( ) $\hat{K}(m, i+1)\cdot(\psi^{i+1}\cdot h^{-\gamma_i}) = \psi^{i}$,

and the estimation

( ) $|\psi^{i+1}| \leqq M(r)^{-1} M(b)\, e^{P(\alpha_3, m)} \cdot d(Q, V)^{\alpha_3'}$

hold.

In the above quantities r', α_3' are given by

( ) $r' = M(r)$, $\alpha_3' = L(\alpha_3) - L(m)$.

Here , of course, M, M , P, L's are determined by $(U, D)_{V,a}$ only.

From the arguement in ( ), ψ^{i+1} is furtuer expressed (in $\Delta_{r''}(P^{*})$) in the following manner

(ψ^{i+1}) $= \sum_{\mu} \cdot f_{\mu}^{i+1} g_{\mu}^{\alpha_3''}$,

with quantities r'' and α_3'' are , which are given

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holds. We estimate the vector $\mathfrak{F}^{*,m+1}$ in the form

( ) $|\mathfrak{F}^{*,m+1}| \leq c_1^{\delta_3'} M(r)^{-1} \cdot M(b) \cdot e \cdot d^{P(d_2,m)}(\tilde{\mathfrak{O}}^*, V)^{c_2 \delta_3'} : \mathfrak{O}^* \in \Delta(P_{\delta_1}^*) \cap \tilde{\mathfrak{O}}^*$,

with constants (c_1, c_2) depending on $(U, D)_{V,a}$ only.

Apply the theorem$^{n-1}_{w,sy,\xi}$ to obtain a vector $\mathfrak{F}^{*,m+2} \in \Gamma(\Delta_r(P), \tilde{\mathfrak{O}}^{(l_{i+2})}_{\mathfrak{O}^*})$,

satisfying the equation

( ) $\tilde{K}^*(m, i+2) \cdot \mathfrak{F}^{*,i+2} = \mathfrak{F}^{*,i+1}$.

From theorem$^{n-1}_{w,sy,\xi}$ we know that $\mathfrak{F}^{*,i+2}$ is estimated in the following fashion,

( ) $|\tilde{\mathfrak{F}}^{*,i+2}_{i+2}| \leq M(r)^{-1} \cdot M(b) e d^{P(d_3)}(\mathfrak{O}, V \cap \tilde{\mathfrak{O}}^*)^{\delta_4'}, \mathfrak{O} \in \Delta_r(P_{\delta_1}^*) \tilde{\mathfrak{O}}^*$

$\delta_4' = l_{3}(d_3) - l_{j_2}(m)$, where M, P, l_r are depending on $(U, D)_{V,a}$ only.

Moreover, we apply the theorem$^{n-1}_{w,sy,\xi}$ (for $i=0$) to the vector $\tilde{\mathfrak{F}}^{*,i+2}$. Then the vectors $\mathfrak{F}^{*,i+2}$ are expressed in the following way.

( ) $\tilde{\mathfrak{F}}^{*,i+2} = \Sigma_\mu \mathfrak{F}^{*,i+2}_\mu \cdot \tilde{\mathfrak{F}}^{*\,d_4''}_\mu$, in $\tilde{\Delta}_r(P_{\delta_1}^*) \cap \tilde{\mathfrak{O}}^*_{\delta_1}$.

In the above, r, $\mathfrak{F}^{*,i+2}_\mu$ are estimated as

( ) $r = M(r)$, $|\mathfrak{F}^{*,i+2}_\mu| \leq M(r) \cdot M(b) \cdot e^{P(d_4)}$

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( )$_4$ the vector $\tilde{f}'^{i+1}$ is zero on $\tilde{\mathcal{D}}^*$.

Needless to say that , the above $\mathbb{M}$'s, $\mathbb{P}$ $\mathbb{L}$'s depend on $(\mathrm{U}.\mathrm{D})_{v,\alpha}$ only. Apply the assertion ( ) to the vector $\tilde{f}'^{i+1}$. Then the equation ( ) can be written, inductively on the order of in the following way .

( ) $\tilde{K}(m, i+1)(\tilde{f}'^{i+1}_{d-1}) = h^{\tilde{d}_i - 1} \cdot \tilde{f}^{i}$ in $\Delta_{r}(P^*_{\delta})$; $r = M(\alpha)$,

where $\tilde{f}'^{i+1}_{d-1}$ is expressed as follows.

( ) $\tilde{f}'^{i+1}_{d-1} = \Sigma_{\mu} \tilde{f}^{i+1}_{\tilde{d}_i - 1, \mu} \tilde{g}^{m''}_{\mu}$,

with $m'' = \mathbb{L}_{3,1}(\alpha_3) - \mathbb{L}_{3,2}(m)$.

Moreover, $\tilde{f}'^{i+1}_{\tilde{d}_i - 1, \mu}$'s are estimated in the fashion

( ) $|\tilde{f}'^{i+1}_{\tilde{d}_i - 1, \mu}| \leq \{\mathbb{M}(r)\}^{-1} \mathbb{M}(f) \, e^{\mathbb{P}(\alpha_3)}(\quad)$.

Because $\mathbb{M}$'s $\mathbb{P}$, $\mathbb{L}$'s are depending on $(\mathrm{U}, \mathrm{D})_{v, \alpha}$ and the <u>degree of the pole $\tilde{d}_i$ is independent of P^*_{δ}</u> and $\tilde{f}^{i}$, we conclude (by repeating procedures $\tilde{d}_i$ times) arguements of this part.

(iii)$_2$ Now , it is easy to derive our assertion in this Lemma ( ) , from the results in ( ).

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What is remainded here is to show the existence of a datum with which the assertion ( ) is valid for any point in . - V . But deducing the desired results from the conclusion in (i i) is done in an entirely same way as in the proof of proposition . We, therefore, leave this part untouched. q.e.d.

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