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On some estimations of the trigonometrical sums in an algebraic number field

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§ 1 Introduction

We have discussed the Waring Problem in an algebraic number field[1] as an application of Vinogradov's mean value theorem [2]. This is another application of our previous paper [2]. Using the Korobov's method we shall prove the estimations of some trigonometrical sums in minor arcs in an algebraic number field.[3] Our results will be given as Theorem 1 and Theorem 2 in §3. In the rational number field the estimations of this type are very interesting and important in the analytic theory of numbers. We use a letter c to denote a positive constant depending on K alone. It is not necessarily the same one each time it occurs. The constant c may well depend on another parameter $*$. In this case we write as $c(*)$ to explain the meaning. We use the same terminology and notations as before.[1,2]

§ 2 Lemmata

Let $\underline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ and $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ be k+2 dimensional complex row vectors and let $\underline{\mu}^* = (\mu_0, \mu_1, \ldots, \mu_{k+1})^*$ be a k+2 dimensional complex colum vector (we call $\underline{\mu}^*$ the transposed of $\underline{\mu}$). Further let A be a k+2 dimensional quadratic (symmetric) matrix such that

$$A = \begin{bmatrix} \binom{0}{0}\alpha_0, & \binom{1}{1}\alpha_1, & \dots, & \binom{k}{k}\alpha_k, & \binom{k+1}{k+1}\alpha_{k+1} \\ \binom{1}{0}\alpha_1, & \binom{2}{1}\alpha_2, & \dots, & \binom{k+1}{k}\alpha_{k+1}, & 0 \\ & & \dots\dots\dots\dots & & \\ \binom{k}{0}\alpha_k, & \binom{k+1}{1}\alpha_{k+1}, & \dots\dots & & \\ \binom{k+1}{0}\alpha_{k+1}, & 0, & \dots\dots\dots\dots & & , 0 \end{bmatrix}$$

where $\binom{i}{j}$ are binomial coefficients and $\binom{0}{0} = 0$. We define two $k+2$ dimensional complex vectors $\underline{\beta} = (\beta_0, \beta_1, \dots, \beta_{k+1})$ and $\underline{\gamma}^* = (\gamma_0, \gamma_1, \dots, \gamma_{k+1})^*$ by the following relations:

$$\underline{\lambda} A \underline{\mu} = \underline{\beta}\,\underline{\mu}^* = \underline{\lambda}\,\underline{\gamma}^* .$$

Then we have

$$\beta_i = \binom{i}{i}\alpha_i\lambda_0 + \binom{i+1}{i}\alpha_{i+1}\lambda_1 + \dots + \binom{i+1}{i}\alpha_{k+1}\lambda_{k+1-i} \quad (0 \leq i \leq k+1)$$

and

$$\gamma_i = \binom{i}{0}\alpha_i\mu_0 + \binom{i+1}{1}\alpha_{i+1}\mu_1 + \dots + \binom{k+1}{k+1-i}\alpha_{k+1}\mu_{k+1-i} \quad (0 \leq i \leq k+1).$$

Let $N_s(T; \nu_1, \dots, \nu_k)$ be the number of solutions of the system of k ($k \geq 2$) diophantine equations

$$\begin{cases} \lambda_1 + \lambda_2 + \dots + \lambda_s = \mu_1 + \mu_2 + \dots + \mu_s + \nu_1 \\ \lambda_1^2 + \lambda_2^2 + \dots + \lambda_s^2 = \mu_1^2 + \mu_2^2 + \dots + \mu_s^2 + \nu_2 \\ \dots\dots\dots\dots \\ \lambda_1^k + \lambda_2^k + \dots + \lambda_s^k = \mu_1^k + \mu_2^k + \dots + \mu_s^k + \nu_k, \end{cases}$$

where $\mathfrak{O} \ni \lambda_j$, $\lambda_j \prec T$, $(1 \leq j \leq s)$ and $\nu_1, \ldots, \nu_k$ are arbitrarily fixed integers in $\mathfrak{O}$. Let $N_s(T;0,\ldots,0) = N_s(T) = N_{s,k}(T)$.

We have now to describe two lemmas. They are essential to prove the following theorems.

Lemma 1 $k \geq 2$ denotes a positive integer. R is a sufficiently large positive number. T is a positive real number with the condition that

$$(2R)^{k(1+1/(k-1))^{\tau-1}} < T$$

with a positive integer $\tau \geq 1$. Further let s be a positive integer with

$$s \geq \frac{n}{2(n-1)} k(k+1) + \tau k - 1.$$

Then we have the following estimate:

$$N_s(T) \leq c^{(k^2+2s+\frac{(\tau-1)}{k}s-\frac{1}{4}(\tau-1)(k+1))\tau} s^{2k} T^{2sn-\frac{1}{2}k(k+1)n+\delta n},$$

where

$$\delta = \delta(k,\tau) = \tfrac{1}{2}k(k+1)(1-1/k)^{\tau}.$$

This is the Vinogradov's mean value theorem in an algebraic number field.

Lemma 2 Let m, m_1 be two natural numbers and let T_1, T be real numbers such that $0 < T_1 \leq T$. We define a trigonometrical sum S as follows:

$$S = S(\alpha_1,\ldots,\alpha_{k+1}) = \sum_{\lambda \prec T} E(f(\lambda))$$

where $f(\lambda) = \alpha_1\lambda + \ldots + \alpha_{k+1}\lambda^{k+1}$ and the summation means that λ runs through all $\lambda \in \mathfrak{O}$ such that $\lambda \prec T$. Let

$$\tilde{\beta}_\nu = \binom{\nu+1}{\nu}\alpha_{\nu+1}\tilde{\lambda}_1 + \ldots + \binom{k+1}{\nu}\alpha_{k+1}\tilde{\lambda}_{k+1-\nu} \quad (1 \leq \nu \leq k).$$

Further, we write

$$V_{m,m_1} = \sum_{\substack{\tilde\lambda_1,\dots,\tilde\lambda_k \\ \tilde\mu_1,\dots,\tilde\mu_k}} N_m(T;\tilde\lambda_1,\dots,\tilde\lambda_k)N_{s1}(T_1;\tilde\mu_1,\dots,\tilde\mu_k)E(\tilde\beta_1\tilde\mu_1+\dots+\tilde\beta_k\tilde\mu_k)$$

where the summation means that $\tilde\lambda_i, \tilde\mu_i$ ($1 \leqq i \leqq k$) run through all $\tilde\lambda_i, \tilde\mu_i \in \mathfrak{P}$ such that

$$|\lambda_\nu^{(p)}| < mT^\nu, \quad |\mu_\nu^{(p)}| \leqq m_1T_1^\nu \quad (1 \leqq p \leqq n_1),$$

$$|\lambda_\nu^{(q)}| \quad 2mT^\nu, \quad |\mu_\nu^{(q)}| \leqq 2m_1T_1^\nu \quad (n_1+1 \leqq q \leqq n_1+n_2)(1 \leqq \nu \leqq k).$$

Then we have

$$\left|\tfrac{1}{2}S\right|^{4mm_1} \leqq T^{4m_1mn}\, T^{-2mn}\, T_1^{-2m_1n}\, V_{mm_1} + (c\,T_1T^{n-1})^{4mm_1}.$$

This is fundamental lemma of Korobov in an algebraic number field.

§3 Theorems

__Theorem 1.__ Let T be a sufficiently large number and $k \geqq 2$ be a positive integer. We may suppose that $k^{\frac{1}{2}}\log k < r < k - k^{\frac{1}{2}}\log k$ and we define $\bar{s}$ and $s = \underline{s}$ by $\bar{s} = \mathrm{Max}(k-r, r)$ and $s = \underline{s} = \mathrm{Min}(k-r, r)$. Let $f(u) = \alpha_1 u + \alpha_2 u^2 + \dots + \alpha_{k+1}u^{k+1}$, $\alpha_i = x_{i1}\rho_1 + \dots + x_{in}\rho_n$, $(x_{i1},\dots,x_{in}) \in U$, ($1 \leqq i \leqq k+1$). We put a number α_{k+1} of $E(h,t)$ which is defined by the Farey division with respect to (h,t), where $h = c^n mk^s D^2 T^{\bar{s}}$, $c \geqq 32$, $m = [s^2\log 2e^2 n]$ and $t = T^{\underline{s}}$. Then we have

$$|S| \leqq c\,T^{\,n - \frac{1}{96(\log n+3)^2 k^2 \log k}},$$

where c is a suitable positive constant depending only on K.

Theorem 2 Let ε be a positive number and assume that $\varepsilon n < r < (1-\varepsilon)n$ $(0 < \varepsilon < 1)$ in theorem 1. Then we have

$$|S| \leq c(\varepsilon, n)\, T^{n - \frac{c(\varepsilon,n)}{k^2}}$$

where

$$c(\varepsilon, n) = 1/(32\log(2e^2 n)\log(2e^2 n/\varepsilon^2)).$$

To obtain these results we must use some results of Siegel and Mitsui of the estimations of trigonometrical sums in an algebraic number field (4,5,6).

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