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Monotonic spaces

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0. Introduction. In this paper all spaces are assumed to be regular(T_1 + T_3), all mappings to be continuous onto, and all images and preimages are those by mappings, unless otherwise specified. The index i always runs through the positive integers. As for undefined terminologies refer to Arhangel'skiĭ[1 and 2] and Michael[13].

Let Ⓒ be a class of spaces. A mapping which is the composition of open mappings and perfect mappings is said to be an OP-mapping. An OCP-mapping is defined similarly if open mappings in this definition are replaced by open compact mappings. The operations O, OP, OCP, etc., are defined as the operations taking images under the corresponding type of mappings. The operation P^{-1} is the one of taking preimages under perfect mappings. Then the operations OPP^{-1} and $OCPP^{-1}$ are defined naturally. By these operations we can define the classes of spaces OP(Ⓒ), $OCPP^{-1}$(Ⓒ), etc., which are the minimal invariant classes of spaces containing Ⓒ under the operations OP, $OCPP^{-1}$, etc..

If Ⓒ is a class of good spaces, some of good properties for Ⓒ will be still inherited to OP(Ⓒ), etc.. This idea stems from Arhangel'skiĭ[2]. Along this line a great progress has been made by Wicke and Worrell[19-26, 28, 29]. Nagami[16] is also devoted to the study in this field. Chaber-Čoban-Nagami[5], introducing the concept of monotonic spaces defined extrinsically, gives another view point for the theory by Wicke and Worrell. The main purpose of this paper is to give self-contained proofs for the results in [5] with the usage of intrinsic definition of monotonic spaces. Many results essentially due to Wicke and Worrell will be proved without any specifications, while in the final section we acknowledge the correspondence between them and the equivalent original theorems due to Wicke and Worrell. It should be noted that our present view point, transferring extrinsic definitions in [5] to intrinsic ones, is essentially obtained by Čoban independently.

Section 1 gives basic definitions of monotonic spaces from the present view point. Section 2 gives basic properties of monotonic spaces. In Section 3 characterization theorems for monotonically complete spaces are given. In Section 4 further properties of monotonically non-complete spaces will be studied. In Section 5 a role of the concept of complete mappings will be shown. Examples which illustrate the position of each kind of monotonic spaces are given in Section 6. The final Section 7 is devoted to exhibit the relation between the concepts and propositions in this paper and the original ones by Wicke and Worrell.

1. Basic definitions and lemmas.

1.1. DEFINITION. Let X be a space and $\{U_i\}$ a monotonically decreasing sequence of subsets of X. $\{U_i\}$ is said to be a k-sequence (by Michael [13, Definition 1.2]) if the following two conditions are satisfied.

(1) $\cap U_i$ is compact and non-empty.

(2) For each open neighborhood U of $\cap U_i$ there exists an m with $U_m \subset U$.

In this case $\{U_i\}$ is said to be converging to $\cap U_i$.

1.2. DEFINITION. Let X be a space.

$$\{\mathfrak{U}_i = \{U(\alpha_i) \neq \emptyset : \alpha_i \in A_i\} ; \varphi^i{}_j : A_i \to A_j (i > j)\}$$

is said to be a directed structure or simply a structure of X if it satisfies the following three conditions.

(1) For each i, $\mathfrak{U}_i$ is an open covering of X.

(2) $\{A_i ; \varphi^i{}_j\}$ forms an inverse system.

(3) $U(\alpha_i) = \cup\{U(\alpha_{i+1}) : \varphi^{i+1}{}_i(\alpha_{i+1}) = \alpha_i\}$, $\alpha_i \in A_i$, $i=1,2,\cdots$.

In [5] we introduced the concept of sieve which was defined extrinsically. A structure can be considered as an intrinsic sieve.

A structure is said to be strict or closure-refining if it satisfies the following condition.

(4) If $\varphi^{i+1}{}_{i}(\alpha_{i+1}) = \alpha_i$, then $\mathrm{Cl}\, U(\alpha_{i+1}) \subset U(\alpha_i)$.

A sequence $\{U(\alpha_i)\}$ with $(\alpha_i) \in \mathrm{inv} \lim A_i$ can be considered as an <u>intrinsic thread</u>, while we defined it in [5] extrinsically. Consider the following four conditions for a structure, where (α_i) is a generic element of inv lim A_i.

(5) $\{U(\alpha_i)\}$ is a k-sequence.

(6) $\{U(\alpha_i)\}$ is a k-sequence or $\cap U(\alpha_i) = \emptyset$.

(7) $\{U(\alpha_i)\}$ is a k-sequence such that $\cap U(\alpha_i)$ is a singleton.

(8) $\{U(\alpha_i)\}$ is a k-sequence such that $\cap U(\alpha_i)$ is a singleton, or $\cap U(\alpha_i) = \emptyset$.

A structure satisfying (5)/(6)/(7)/(8) is said respectively to be an mcc-/mp-/mcm-/mm-<u>structure</u>, which can be considered as an <u>intrinsic</u> mc-/mp-/λ-/md-<u>sieve</u>(cf.[5, Definition 2.1]). A space having an mcc-/mp-/mcm-/mm-structure is said respectively to be an mcc-/mp-/mcm-/mm-<u>space</u>. The empty space is any one of these spaces. In the sequel we always treat non-empty spaces to avoid this trivial case. An mcc-/mp-/mcm-/mm-space is respectively an abbreviation for a <u>monotonically Čech complete/monotonic p-/monotonically complete metric/monotonically metric space</u>. A monotonically complete metric/monotonically metric space can be considered as an intrinsic equivalent to a space with a λ-base/monotonically developable space in [5].

The following two lemmas are easy exercises.

1.3. LEMMA. *Let* $\{H_i\}$ *be a* k-*sequence of subsets of a space* X. *If, for each* i, $H_i \supset G_i \neq \emptyset$ *and* $G_i \supset \overline{G_{i+1}}$, *then* $\{G_i\}$ *is a* k-*sequence*.

1.4. LEMMA. *Let* $f: X \to Y$ *be a mapping. If* $\{H_i\}$ *is a* k-*sequence of subsets of* X, *then* $\{f(H_i)\}$ *is a* k-*sequence in* Y.

1.5. LEMMA. *Let* X *be a space and* $\{\mathcal{W}_i\}$ *a sequence of open coverings of* X. *If* $\{\mathcal{U}_i = \{U(\alpha_i): \alpha_i \in A_i\};\ \varphi^i_j\}$ *is an* mcc-/mp-/mcm-/mm-*structure of* X, *then there exists respectively a closure-refining* mcc-/mp-/mcm-/mm-*structure* $\{\mathcal{V}_i = \{V(\beta_i): \beta_i \in B_i\};\ \psi^i_j\}$ *and transformations* $g_i: B_i \to A_i$ *satisfying the following three conditions*.

(1) $\mathcal{V}_i < $ (*refines*) $\mathcal{W}_i$.

(2) g_i *is a refine-transformation from* $\mathcal{V}_i$ *to* $\mathcal{U}_i$.

(3) $g_i \psi^{i+1}_i = \varphi^{i+1}_i g_{i+1}$.

Proof. For each $\alpha_1 \in A_1$ let $B(\alpha_1)$ be the set such that $\{V(\beta_1): \beta_1 \in B(\alpha_1)\}$ is the collection of all non-empty open sets of X refining $\mathcal{W}_1 | U(\alpha_1)$. Let B_1 be the disjoint sum of $B(\alpha_1)$, $\alpha_1 \in A_1$. Define $g_1 : B_1 \to A_1$ by:

$$g_1(\beta_1) = \alpha_1, \ \beta_1 \in B(\alpha_1).$$

Set

$$C = \{(\alpha_2, \beta_1) \in A_2 \times B_1 : \ \beta_1 \in B(\varphi^2{}_1(\alpha_2)), \ U(\alpha_2) \cap V(\beta_1) \neq \emptyset\}.$$

For each $(\alpha_2, \beta_1) \in C$ let $B(\alpha_2, \beta_1)$ be the set such that

$$\{ V(\beta_2) : \ \beta_2 \in B(\alpha_2, \beta_1) \}$$

is the collection of all non-empty open sets of X whose closures refine $\mathcal{W}_2 | U(\alpha_2) \cap V(\beta_1)$. Let B_2 be the disjoint sum of $B(\alpha_2, \beta_1)$, $(\alpha_2, \beta_1) \in C$. Define $g_2 : B_2 \to A_2$ and $\varphi^2{}_1 : B_2 \to B_1$ by:

$$g_2(\beta_2) = \alpha_2, \ \varphi^2{}_1(\beta_2) = \beta_1, \ \beta_2 \in B(\alpha_2, \beta_1),$$

where $(\alpha_2, \beta_1) \in C$. Continuing in this fashion we obtain the desired closure-refining monotonic structure $\{ \mathcal{V}_i = \{ V(\beta_i) : \beta_i \in B_i \} ; \varphi^i{}_j \}$ of the corresponding kind by virtue of Lemma 1.3. That completes the proof.

1.6. DEFINITION. Let $\{H_i\}$ a decreasing sequence of non-empty subsets of a space X. $\{H_i\}$ is said to be a <u>weak</u> <u>k-sequence</u> if it satisfies the following condition.

If, for each i, $H_i \supset G_i \neq \emptyset$ and $G_i \supset \overline{G_{i+1}}$, then $\{G_i\}$ is a k-sequence.

If 'k-sequence' in the definition of monotonic structures is weakened to 'weak k-sequence' we obtain the concept of a <u>weak</u> mcc-/mp-/mcm-/mm-<u>structure</u>.

Since we need only the fact that $\{ \mathcal{U}_i \}$ is a weak monotonic structure in the proof of Lemma 1.5, we get at once the following.

1.7. LEMMA. A space has an mcc-/mp-/mcm-/mm-structure if and only if it has a weak mcc-/mp-/mcm-/mm-structure.

2. Basic properties of monotonic spaces.

(cc) represents the class of Čech complete spaces. This type of notion is used for other classes of spaces. Since, by Lemma 1.4, a monotonically complete structure is mapped to a monotonically complete structure of the same kind under a mapping, the following is obvious.

2.1. PROPOSITION. O(mcc) = (mcc). O(mcm) = (mcm).

2.2. PROPOSITION. OC(mp) = (mp). OC(mm) = (mm).

Proof. Let $f: X \to Y$ be an open compact mapping of an mp-/mm-space X and $\{ \mathfrak{U}_i = \{ U(\alpha_i): \alpha_i \in A_i \}; \varphi^i_j \}$ an mp-/mm-structure of X. By Lemma 1.5 we can assume without loss of generality that $\{ \mathfrak{U}_i; \varphi^i_j \}$ is closure-refining. To see that $\{ f(\mathfrak{U}_i); \varphi^i_j \}$ is an mp-/mm-structure of Y let (α_i) be a generic element of inv lim $\{ A_i; \varphi^i_j \}$.

When $\{ U(\alpha_i) \}$ is a k-sequence, $\{ f(U(\alpha_i)) \}$ is also a k-sequence by Lemma 1.4. Conversely, consider the case when $\cap f(U(\alpha_i)) \neq \emptyset$. Pick a point y from this intersection. Since $f^{-1}(y)$ is compact and Cl $U(\alpha_i) \cap f^{-1}(y) \neq \emptyset$ for any i, $\cap$ Cl $U(\alpha_i) = \cap U(\alpha_i) \neq \emptyset$. Thus $\{ U(\alpha_i) \}$ is a k-sequence and hence $\{ f(U(\alpha_i)) \}$ is a k-sequence. The rest to be proved is trivial. That completes the proof.

Since an mcc-/mp-structure of the image space is transferred respectively an mcc-/mp-structure by the operation P^{-1}, we get at once the following.

2.3. PROPOSITION. P^{-1}(mcc/mp) = (mcc/mp).

2.4. DEFINITION. A structure $\{\mathcal{U}_i;\ \varphi^i_j\}$ of a space X is said to be *saturated* if the following condition is satisfied.

For each i and each element $U(\alpha_i) \in \mathcal{U}_i$, $\{U(\alpha_{i+1}) \in \mathcal{U}_{i+1}:\ \varphi^{i+1}_i(\alpha_{i+1}) = \alpha_i\}$ is a base of $U(\alpha_i)$.

The structure $\{\mathcal{V}_i\}$ constructed in Lemma 1.5 is moreover assumed to be saturated without any change of its proof.

2.5. LEMMA. *Let* X *be a space. Let for each* i, $\{U(\alpha_i):\ \alpha_i \in A_i\}$ *be a finite open collection of* X *and* $\varphi^{i+1}_i : A_{i+1} \to A_i$ *a transformation satisfying the following two conditions.*

(1) $\varphi^{i+1}_i(\alpha_{i+1}) = \alpha_i$ *implies* $U(\alpha_{i+1}) \subset U(\alpha_i)$.

(2) $(\alpha_i) \in \text{inv lim } A_i$ *implies* $\{U(\alpha_i)\}$ *is a* k-*sequence*.

Set $U_i = \cup\{U(\alpha_i):\ \alpha_i \in A_i\}$, $\cap U_i = K$. *Then* K *is non-empty and compact and* $\{U_i\}$ *is converging to* K.

This is essentially proved in Nagami[16, Lemma 8.2].

2.6. PROPOSITION. P(mcc/mp/mcm/mm) = (mcc/mp/mcm/mm).

Proof. Let $f: X \to Y$ be a perfect mapping and X a space with a saturated closure-refining mcc-/mp-/mcm-/mm-structure $\{\mathcal{U}_i = \{U(\alpha_i):\ \alpha_i \in A_i\};\ \varphi^{i+1}_i\}$. Let B_i be the set of all finite subsets of A_i. Set

$$V(\beta_i) = \cup\{U(\alpha_i):\ \alpha_i \in \beta_i\},\quad \beta_i \in B_i,$$

$$W(\beta_i) = Y - f(X - V(\beta_i)),\quad \beta_i \in B_i.$$

Let C_i be the set of all $\beta_i \in B_i$ such that for a point $y(\beta_i) \in W(\beta_i)$ and for each $\alpha_i \in \beta_i$, $U(\alpha_i) \cap f^{-1}(y(\beta_i)) \neq \emptyset$. Set $D_1 = C_1$. For $n \geqq 2$ let D_n be the subset of $C_1 \times \cdots \times C_n$ consisting of all sequences $(\beta_1, \cdots, \beta_n)$ satisfying the following two conditions.

(1) $\varphi^{i+1}{}_i(\beta_{i+1}) \subset \beta_i$, $i=1,\cdots,n-1$.

(2) $\mathrm{Cl}\, V(\beta_{i+1}) \subset f^{-1}(W(\beta_i))$, $i=1,\cdots,n-1$.

Set

$$H(\beta_1, \cdots, \beta_n) = W(\beta_n),$$

$$\textcircled{H}_n = \{ H(\delta_n) : \delta_n \in D_n \}.$$

Define $\psi^{n+1}{}_n : D_{n+1} \rightarrow D_n$ by:

$$\psi^{n+1}{}_n(\beta_1, \cdots, \beta_n, \beta_{n+1}) = (\beta_1, \cdots, \beta_n).$$

Then $\{ \textcircled{H}_n ; \psi^{n+1}{}_n \}$ is a saturated structure of Y. Assume that for $(\delta_n) \in \operatorname{inv\,lim} D_n$, $\bigcap H(\delta_n) = K \neq \emptyset$. Set

$$\delta_n = (\beta_1, \cdots, \beta_n),\ n=1,2,\cdots.$$

Then by (1), $\{ \beta_i ; \varphi^{i+1}{}_i \}$ forms an inverse system. Let y be an arbitrary point of K. Set

$$\gamma_i = \{ \alpha_i \in \beta_i : f^{-1}(y) \cap U(\alpha_i) \neq \emptyset \}.$$

Then $\gamma_i \neq \emptyset$ and $\varphi^{i+1}{}_i(\gamma_{i+1}) \subset \gamma_i$. Thus $\{ \gamma_i \}$ is an inverse subsystem of $\{ \beta_i \}$. Pick an element $(\alpha_i) \in \operatorname{inv\,lim} \gamma_i$. Then

(3) $\bigcap U(\alpha_i) \neq \emptyset$.

To prove that

(4) $\bigcap U(\alpha'_i) \neq \emptyset$, $(\alpha'_i) \in \operatorname{inv\,lim} \beta_i$,

assume that there exists an $(\alpha'_i) \in \operatorname{inv\,lim} \beta_i$ with $\bigcap U(\alpha'_i) = \emptyset$. Pick for each i points x_i and x'_i with

$$x_i \in f^{-1}(y(\beta_i)) \cap U(\alpha_i),$$

$$x'_i \in f^{-1}(y(\beta_i)) \cap U(\alpha'_i).$$

Since $\{U(\alpha_i)\}$ is a k-sequence by (3), the sequence $T = \{x_i\}$ has a cluster point x in $\cap U(\alpha_i)$. Since $\cap U(\alpha'_i) = \emptyset$ and $U(\alpha'_i) \supset \mathrm{Cl}\, U(\alpha'_{i+1})$, the sequence $T' = \{x'_i\}$ is discrete and infinite. Moreover the equality $\cap U(\alpha'_i) = \emptyset$ implies that $\{y(\beta_i)\}$ is infinite. Therefore by the perfectness of f, f(T') is discrete and infinite. On the other hand f(T) has a cluster point f(x). But $f(x_i) = f(x'_i) = y(\beta_i)$ for each i. This contradiction shows that (4) is true.

Since $\cap V(\beta_n) = \cap f^{-1}(W(\beta_n)) = f^{-1}(K)$ by (1), then $\{f^{-1}(W(\beta_n))\}$ is converging to a compact set $f^{-1}(K)$ by Lemma 2.5. Thus $\{H(\delta_n) = W(\beta_n)\}$ is a k-sequence.

Consider the case when $\cap U(\alpha_i)$ is a singleton or empty for each $(\alpha_i) \in \mathrm{inv\,lim}\, A_i$. Assume that $\cap H(\gamma_i) = \cap W(\beta_i)$ contains two points y and y' for an element $(\gamma_i) \in \mathrm{inv\,lim}\, D_i$, where $\gamma_i = (\beta_1, \cdots, \beta_i)$. It is easy to find elements (α'_i) and (α''_i) of $\mathrm{inv\,lim}\{\beta_i;\ \varphi^{i+1}_i\}$ such that

$$\cap U(\alpha'_i) = \{x\},\ f(x) = y,$$

$$\cap U(\alpha''_i) = \{x'\},\ f(x') = y'.$$

Pick for each i points x_i and x'_i with

$$x_i \in U(\alpha'_i) \cap f^{-1}(y(\beta_i)),$$

$$x'_i \in U(\alpha''_i) \cap f^{-1}(y(\beta_i)).$$

Since $\lim x_i = x$, $\lim f(x_i) = f(x) = y$. Since $\lim x'_i = x'$, $\lim f(x'_i) = f(x') = y'$. Since for each i, $f(x_i) = f(x'_i) = y(\beta_i)$, then $y = y'$, which is a contradiction.

All of the essential part to be proved has been verified and the proof is finished.

The following three theorems are almost evident from the definition.

2.7. THEOREM. A locally mcc-/mp-/mcm-/mm-space is respectively an mcc-/mp-/mcm-/mm-space.

2.8. THEOREM. The countable product of mcc-/mp-/mcm-/mm-spaces is respectively an mcc-/mp-/mcm-/mm-space.

2.9. THEOREM. A closed subset of an mcc-/mp-/mcm-space is respectively an mcc-/mp-/mcm-space. A subset of an mm-space is an mm-space.

2.10. THEOREM. A G_δ set S of an mcc-/mp-/mcm-space X is respectively an mcc-/mp-/mcm-space.

Proof. Let $\{ \mathfrak{U}_i = \{U(\alpha_i): \alpha_i \in A_i\}; \varphi^i{}_j \}$ be a monotonic structure of any kind of X. We only consider the case when S is not empty. Set $S = \cap G_i$ with G_i open and with $G_i \supset G_{i+1}$. By an analogous argument to that for Lemma 1.5 there exist, for each i, an open collection $\mathfrak{V}_i = \{V(\beta_i): \beta_i \in B_i\}$, a transformation $\psi^i{}_j : B_i \to B_j (i > j)$ and a transformation $g_i : B_i \to A_i$ satisfying the following three conditions, where $\mathfrak{V}_i{}^\#$ denotes the sum of all elements of $\mathfrak{V}_i$.

(1) $\mathfrak{V}_i{}^\# = G_i$.

(2) $\psi^{i+1}{}_i$ is a closure-refining transformation from $\mathfrak{V}_{i+1}$ to $\mathfrak{V}_i$.

(3) $\varphi^{i+1}{}_i g_{i+1} = g_i \psi^{i+1}{}_i$.

Then $\{\mathfrak{V}_i | S; \psi^i{}_j\}$ is a monotonic structure of S of the corresponding kind. That completes the proof.

2.11. THEOREM. (mcc) $\cap$ (mm) = (mcm).

Proof. Let X be an mcc- and mm-space. Let $\{ \mathfrak{U}_i = \{ U(\alpha_i): \alpha_i \in A_i \} ; \varphi^i_j \}$ and $\{ \mathfrak{V}_i = \{ V(\beta_i): \beta_i \in B_i \} ; \psi^i_j \}$ be respectively an mcc-structure and an mm-structure of X. Let C_i be the set of all elements $(\alpha_i, \beta_i) \in A_i \times B_i$ such that $U(\alpha_i) \cap V(\beta_i) \neq \emptyset$. Define $\sigma^{i+1}_i : C_{i+1} \to C_i$ by:

$$\sigma^{i+1}_i(\alpha_{i+1}, \beta_{i+1}) = (\varphi^{i+1}_i(\alpha_{i+1}), \psi^{i+1}_i(\beta_{i+1})).$$

Then $\{ \{ U(\alpha_i) \cap V(\beta_i): (\alpha_i, \beta_i) \in C_i \} ; \sigma^{i+1}_i \}$ is a weak mcm-structure of X and hence by Lemma 1.7 X is an mcm-space. That completes the proof.

3. *Monotonic complete spaces.*

3.1. LEMMA(Engelking[8, Theorem 2, p.143]). *A completely regular space X is Čech complete if and only if there exists a sequence $\mathfrak{U}_i$, i=1,2,···, of open coverings of X satisfying the following condition.*

If $\mathfrak{F}$ is a non-empty collection of closed sets with the finite intersection property such that, for each i, $\mathfrak{F}$ has an element refining $\mathfrak{U}_i$, then $\cap \{ F: F \in \mathfrak{F} \} \neq \emptyset$.

3.2. PROPOSITION. *A Čech complete space X is an mcc-space.*

Proof. Let $\{ \mathfrak{U}_i \}$ be a sequence of open coverings of X as in Lemma 3.1. Let $\{ \mathfrak{V}_i \}$ be a closure-refining structure of X such that $\mathfrak{V}_i < \mathfrak{U}_i$ for each i. Then $\{ \mathfrak{V}_i \}$ is as can easily be seen an mcc-structure. That completes the proof.

3.3. THEOREM. *A space is an mcc-space if and only if it is the image of a (paracompact) Čech complete space under an open mapping.*

Proof. Sifficiency is obvious by Propositions 2.1 and 3.2.

Necessity. Let $\{ \mathfrak{U}_i = \{ U(\alpha_i): \alpha_i \in A_i \} ; \varphi^i_j \}$ be a saturated closure-refining mcc-structure of a space X. Set

$$Y = \{ ((\alpha_i),x) \in (\text{inv lim } A_i) \times X: x \in \cap U(\alpha_i) \} .$$

The topology of inv lim A_i is the one induced by the discrete topology of A_i's. Y enjoys the relative topology of the product topology. Let $f:Y \to X$ and $g:Y \to \text{inv lim } A_i = M$ be the restrictions of the corresponding projections.

f is evidently onto. To prove f is open let (α,x) be an arbitrary point of Y, where $\alpha = (\alpha_i)$. Let $(\alpha|n)$ be the cubic neighborhood of α determined by the first n coordinates $(\alpha_1,\cdots,\alpha_n)$. Let V be an open neighborhood of x. Set $W = (\alpha|n) \times V$. It suffices to prove $f(W \cap Y) \supset U(\alpha_n) \cap V$, since $U(\alpha_n) \cap V$ is a neighborhood of x. Let x' be an arbitrary point of $U(\alpha_n) \cap V$. Since $\{ \mathfrak{U}_i \}$ is saturated, there exists an element $\beta = (\beta_i) \in (\alpha|n) \cap M$ such that $x' \in \cap U(\beta_i)$. Hence $(\beta,x') \in W \cap Y$ and $f(\beta,x') = x'$. Thus $f(W \cap Y) \supset U(\alpha_n) \cap V$, which proves that f is open.

Let $\alpha = (\alpha_i)$ be an arbitrary element of M. Since $\{ U(\alpha_i) \}$ is a k-sequence, we can pick a point x from $\cap U(\alpha_i)$. Then $(\alpha,x) \in Y$ and $g(\alpha,x) = \alpha$. Thus g is onto. Since $g^{-1}(\alpha) = \{ \alpha \} \times (\cap U(\alpha_i))$ and $\cap U(\alpha_i)$ is compact, each point-inverse under g is compact.

To prove g is closed let F be a closed set of Y. Pick a point $\alpha = (\alpha_i) \in M - g(F)$. Since $g^{-1}(\alpha) = \{ \alpha \} \times (\cap U(\alpha_i))$, $g^{-1}(\alpha) \cap F = \emptyset$, $\{ U(\alpha_i) \}$ is a k-sequence, and $g^{-1}(\alpha)$ is compact, then there exists an n with $((\alpha|n) \times U(\alpha_n)) \cap F = \emptyset$.

Pick an arbitrary element $\beta = (\beta_i)$ from $(\alpha|n) \cap M$. Let (β,x) be an arbitrary point of $g^{-1}(\beta)$. Since $\beta_i = \alpha_i$ for $i=1,\cdots,n$, then $x \in \cap U(\beta_i) \subset U(\alpha_n)$. Thus $(\beta,x) \in (\alpha|n) \times U(\alpha_n)$ and hence $g^{-1}(\beta) \cap F = \emptyset$. This equality implies that $(\alpha|n) \cap g(F) = \emptyset$ and that g is closed.

Since g is now a perfect mapping of Y onto M and M is a complete metric space, Y is a paracompact Čech complete space by Frolík[9]. That completes the proof.

By Propositions 2.1, 2.3, 2.6 and Theorem 3.3 we get the following.

3.4. THEOREM. (mcc) = OPP^{-1}(mcc) = O(Čech complete) = O(paracompact Čech complete).

When an mcc-structure in the proof of Theorem 3.3 is replaced with an mcm-structure, each point-inverse under g is a singleton. Hence in this case the perfect mapping g of Y to M has to be a homeomorphism and Y is a complete metric space. Thus we get the following.

3.5. THEOREM. <u>A space is an</u> mcm-<u>space if and only if it is the image of a complete metric space under an open mapping</u>.

By Propositions 2.1, 2.6 and Theorem 3.5 we get the following, since each complete metric space is evidently an mcm-space.

3.6. THEOREM. (mcm) = OP(mcm) = O(complete metric).

3.7. THEOREM. <u>Each inductively open mapping</u> f <u>of an</u> mcc-<u>space</u> X <u>is compact-covering</u>.

Proof. Let $f:X \to Y$ be such a mapping and K a compact set of Y. By Theorem 3.3 there exist a Čech complete space Z and an open mapping $g:Z \to X$. Then $fg:Z \to Y$ is an inductively open mapping. Since by Nagami[17, Theorem 1] each inductively open mapping of a Čech complete space is compact-covering, there exists a compact set L of Z with $fg(L) = K$. Since $g(L)$ is a compact set of X, f is compact-covering. That completes the proof.

3.8. THEOREM. Let $f:X \to Y$ be an inductively open mapping of an mcc-space X. Then Y is an mcc-space.

Proof. Let X' be a subset of X such that $f(X') = Y$ and $f|X'$ is open. Let $\{ \mathcal{U}_i = \{ U(\alpha_i): \alpha_i \in A_i \} ; \varphi^i_j \}$ be a closure-refining mcc-structure of X. Set

$$V(\alpha_i) = f(U(\alpha_i) \cap X'),\ \alpha_i \in A_i,$$
$$B_i = \{ \alpha_i \in A_i : V(\alpha_i) \neq \emptyset \},$$
$$\mathcal{V}_i = \{ V(\beta_i): \beta_i \in B_i \},$$
$$\psi^i_j = \varphi^i_j | B_i.$$

Since $\varphi^{i+1}_i(B_{i+1}) \subset B_i$, then $\{ B_i ; \psi^i_j \}$ forms an inverse subsystem of $\{ A_i \}$ and $\{ \mathcal{V}_i ; \psi^i_j \}$ is a structure of Y.

To prove $\{ \mathcal{V}_i \}$ is a weak mcc-structure, let (β_i) be an arbitrary element of inv lim B_i. For each i let G_i be a non-empty set of Y such that $V(\beta_i) \supset G_i \supset \overline{G_{i+1}}$. Set $H_i = f^{-1}(\overline{G_i}) \cap \mathrm{Cl}\, U(\beta_i)$. Since $(\beta_i) \in$ inv lim A_i, $\{ \mathrm{Cl}\, U(\alpha_i) \}$ is a k-sequence. Since $H_i \neq \emptyset$, $H_i \supset H_{i+1}$ and H_i is closed, $\{ H_i \}$ is a k-sequence by Lemma 1.3. Since $f(H_i) = \overline{G_i}$, $\{ \overline{G_i} \}$

is a k-sequence by Lemma 1.4 and hence $\{G_i\}$ is also a k-sequence. Thus $\{\mathcal{V}_i\}$ is a weak mcc-structure. Since Y has now an mcc-structure by Lemma 1.7. That completes the proof.

By Nagami[17, Theorem 2] Baire's category theorem is true for each open image of a Čech complete space. Hence by Theorem 3.3 we get at once the following.

3.9. THEOREM. *Any mcc-space cannot be the countable sum of nowhere dense subsets.*

Recall that a sequence $\{\mathcal{V}_i\}$ of open coverings of a space X is said to be a *development* if for each point $x \in X$, $\{\mathcal{V}_i(x)\}$ forms a neighborhood base of x, where $\mathcal{V}_i(x)$ is the star of x with respect to $\mathcal{V}_i$. A space(i.e. a regular space in this paper) with a development is said to be a *Moore space*. A development $\{\mathcal{V}_i\}$ is said to be *complete*, if $V_i \in \mathcal{V}_i$ and $\{V_i\}$ has the finite intersection property then $\cap \overline{V}_i = \emptyset$. A space with a complete development is said to be a *complete Moore space*.

3.10. PROPOSITION. *A complete Moore space X is an mcm-space.*

Proof. Let $\{\mathcal{V}_i\}$ be a complete development of X. Let $\{\mathcal{U}_i\}$ be a closure-refining structure of X such that $\mathcal{U}_i < \mathcal{V}_i$ for each i. Then $\{\mathcal{U}_i\}$ is an mcm-structure. That completes the proof.

By this argument it is also true that each Moore space is an mm-space. The author had not known whether each Moore space is completely regular(cf. [16, p.262]). Professors Hodel and Younglove kindly informed that there exists a Moore space which is not completely regular by Jones[10]. Armentrout [3] and Younglove[30] also give such a counter-example with some additional conditions desired.

3.11. THEOREM. For an mcc-space X the following two conditions are equivalent.

(1) X is an mcm-space.

(2) X is the image of a metric space (Y,d) under an open mapping f such that $(f^{-1}(x),d)$ is complete for each $x \in X$.

Proof. Theorem 3.5 implies the implication (1) $\rightarrow$ (2).

(2) $\rightarrow$ (1): Let $\{ \mathcal{U}_i = \{ U(\alpha_i): \alpha_i \in A_i \} ; \varphi^i_j \}$ be a closure-refining mcc-structure of X. Let $\{ \mathcal{V}_i = \{ V(\beta_i): \beta_i \in B_i \} ; \psi^i_j \}$ be a closure-refining structure of Y with mesh $\mathcal{V}_i < 1/i$. Set

$$W(\alpha_i,\beta_i)=U(\alpha_i) \cap f(V(\beta_i)),$$
$$C_i = \{ (\alpha_i,\beta_i) \in A_i \times B_i : W(\alpha_i,\beta_i) \neq \emptyset \},$$
$$\sigma^i_j(\alpha_i,\beta_i) = (\varphi^i_j(\alpha_i), \psi^i_j(\beta_i)),$$
$$\mathcal{W}_i = \{ W(\gamma_i): \gamma_i \in C_i \}.$$

Then $\{ \mathcal{W}_i ; \sigma^i_j \}$ is a weak mcc-structure of X. To see it is an mcm-structure pick an element $(\gamma_i) \in \text{inv lim} \{ C_i ; \sigma^i_j \}$, where $\gamma_i = (\delta_i, \varepsilon_i)$. Pick two points x and x' from $\cap W(\gamma_i)$. Since $\cap V(\varepsilon_i)$ is not empty and hence is a singleton $\{ y \}$. Since $f^{-1}(x) \cap V(\varepsilon_i) \neq \emptyset$ and $f^{-1}(x') \cap V(\varepsilon_i) \neq \emptyset$ for any i, then $y \in f^{-1}(x)$ and $y \in f^{-1}(x')$ at the same time, which implies $x = x'$. Thus $\{\mathcal{W}_i\}$ is a weak mcm-structure and X is an mcm-space by Lemma 1.7. That completes the proof.

The following is a theorem due to Pasynkov[18]. We present a proof using our notions for the reader's convenience.

3.12. THEOREM. A paracompact mcc-space X is Čech complete.

Proof. Let $\{\mathcal{U}_i = \{U(\alpha_i): \alpha_i \in A_i\}\,;\, \varphi^i_j\}$ be a saturated closure-refining mcc-structure of X. By the paracompactness of X there exists an inverse subsystem $\{B_i\}$ of $\{A_i\}$ satisfying the following three conditions.

(1) $U(\beta_i) \neq \emptyset$, $\beta_i \in B_i$.

(2) $\mathcal{V}_i = \{U(\beta_i): \beta_i \in B_i\}$ is a locally finite covering of X.

(3) For each i and each $\beta_{i+1} \in B_{i+1}$, $U(\beta_{i+1})$ intersects at most a finite number of elements of $\mathcal{V}_i$.

Let $\mathcal{F}$ be a non-empty collection of closed sets of X with the finite intersection property such that for each i there exists an element F of $\mathcal{F}$ refining $\mathcal{V}_i$. Set

$$C_i = \{\beta_i \in B_i: U(\beta_i) \text{ contains an element of } \mathcal{F}\}.$$

Then $C_i \neq \emptyset$ and $\{C_i\}$ forms an inverse subsystem of $\{B_i\}$.

To see C_i is finite let β_{i+1} be an arbitrary element of C_{i+1}. Then $U(\beta_{i+1}) \cap U(\beta_i) \neq \emptyset$ for any $\beta_i \in C_i$, which implies that C_i is finite by (3). Pick an element (γ_i) from inv lim C_i. Set $K = \bigcap U(\gamma_i)$. Then K is compact and non-empty. Since $\{U(\gamma_i)\}$ is a k-sequence, $K \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. Thus $\bigcap\{F: F \in \mathcal{F}\} \neq \emptyset$ and X is Čech complete by Lemma 3.1. That completes the proof.

4. Monotonic non-complete spaces.

By Propositions 2.2, 2.3 and 2.6 we get at once the following.

4.1. THEOREM. (mp) = $OCPP^{-1}$(mp). (mm) = OCP(mm).

4.2. THEOREM. *An open compact mapping* $f:X \to Y$ *of an mp-space* X *is compact-covering.*

Proof. Let $\{ \mathfrak{U}_i = \{ U(\alpha_i): \alpha_i \in A_i \} ; \varphi^i_j \}$ be a saturated closure-refining mp-structure of X. Then $\{ f(\mathfrak{U}_i); \varphi^i_j \}$ is a saturated structure of Y. Let K be a non-empty compact set of Y and B_i a finite subset of A_i satisfying the following four conditions.

(1) $\mathfrak{V}_i = \{ f(U(\alpha_i)): \alpha_i \in B_i \}$ covers K.

(2) $U(\alpha_i) \cap K \neq \emptyset$, $\alpha_i \in B_i$.

(3) $\{ B_i \}$ forms an inverse subsystem of $\{ A_i \}$.

(4) $\varphi^{i+1}_i | B_{i+1}$ is a closure-refining transformation from $\mathfrak{V}_{i+1}$ to $\mathfrak{V}_i$.

Let (β_i) be a generic element of inv lim B_i. By (2) and (4), $\cap f(U(\beta_i)) \neq \emptyset$. Since f is compact and $\{ \mathfrak{U}_i \}$ is closure-refining, then $\cap U(\beta_i) \neq \emptyset$, which implies that $\{ U(\beta_i) \}$ is a k-sequence. Set

$$L = \bigcap_{i=1}^{\infty} (\cup \{ U(\alpha_i): \alpha_i \in B_i \}).$$

Then L is compact by Lemma 2.5.

Let y be an arbitrary point of K. Let (γ_i) be an element of inv lim B_i such that $y \in \cap f(U(\gamma_i))$. Then $(\cap U(\gamma_i)) \cap f^{-1}(y) \neq \emptyset$, which implies that $f(L) \supset K$. That completes the proof.

4.3. THEOREM. *An mp-space is of countable type.*

Proof. Assume that f in the above theorem is the identity mapping of X to X. Since L has the countable character by Lemma 2.5, X is of countable type. That completes.

A weak p-space defined in [16, Definition 8.1] is an mp-space as can easily be verified. Thus OCP(weak p-spaces) is a subfamily of $OCPP^{-1}$(mp). Hence Theorem 4.2 gives a simple proof of [16, Theorem 4.1] asserting that an OCP-mapping of a (weak) p-space is compact-covering.

The following is proved to be true by a quite similar argument to Arhangel'skiĭ[31, Theorem 4.2].

4.4. THEOREM. *Let X be a completely regular mp-space and BX a* (T_2) *compactification of X. Then the outer weight of X in BX is equal to the network weight of X.*

4.5. LEMMA(Burke-Stoltenberg[4, Theorem 2.2]). *A completely regular space X is a strict p-space if and only if X has a sequence* $\{\mathcal{V}_i\}$ *of open coverings of X such that for each* $x \in X$, $\{\bigcap_{i=1}^{n} \mathcal{V}_i(x): n=1,2,\cdots\}$ *is a k-sequence.*

4.6. THEOREM. *A completely regular,* θ*-refinable, mp-space X is a strict p-space.*

Proof. Let $\{\mathcal{U}_i = \{U(\alpha_i): \alpha_i \in A_i\}; \varphi^i_j\}$ be a closure-refining mp-structure of X. By an easy application of induction we get a saturated closure-refining mp-structure

$$\{\mathcal{V}_i = \{V(\beta_i): \beta_i \in B_i\}; \psi^i_j\}$$

of X, a sequence

$$\mathfrak{V}_{ij} = \{V_j(\beta_i) : \beta_i \in B_i\},\ i=1,2,\cdots,\ j=1,2,\cdots,$$

of open coverings of X, a sequence

$$\rho_i : B_i \rightarrow A_i,\ i=1,2,\cdots,$$

of transformations, and a sequence

$$F_{ij},\ i=1,2,\cdots,\ j=1,2,\cdots,$$

of closed sets of X satisfying the following six conditions for each i.

(1) $\rho_i \psi^{i+1}_i = \varphi^{i+1}_i \rho_{i+1}$.

(2) ρ_i is a closure-refining transformation from $\mathfrak{V}_i$ to $\mathfrak{U}_i$.

(3) $V_j(\beta_i) \subset V(\beta_i)$, $\beta_i \in B_i$.

(4) $\bigcup_{j=1}^{\infty} F_{ij} = X$.

(5) $\mathfrak{V}_{ij} | F_{ij}$ is point-finite for each j.

(6) $V(\beta_i) \cap F_{jk} \neq \emptyset$, $j < i$, $k < i$, implies $\mathrm{Cl}\, V(\beta_i) \subset V_k(\psi^i_j(\beta_i))$.

To show that $\{\mathfrak{V}_i\}$ satisfies the condition in Lemma 4.5 let x be an arbitrary point of X. Define numbers n(1), n(2), $\cdots$ and m(1), m(2),$\cdots$ by the following two conditions.

(7) $n(1) = 1$, $m(i) = \sum_{j=1}^{i} n(j)$, $i=1,2,\cdots$.

(8) $x \in F_{m(i)n(i+1)}$ and $x \notin F_{m(i)n}$ for $n < n(i+1)$.

Since $m(i) < m(i+1)$ and $n(i+1) < m(i+1)$ by (7), and $x \in F_{m(i)n(i+1)}$ by (8), then by (6)

(9) $\mathfrak{V}_{m(i+1)}(x) \subset \mathfrak{V}_{m(i)n(i+1)}(x)$.

Moreover by (5) and (8)

(10) $\mathfrak{V}_{m(i)n(i+1)}(x)$ is the sum of elements of a finite subcollection of $\mathfrak{V}_{m(i)n(i+1)}$.

Since $\{ \mathfrak{V}_i \}$ is closure-refining, by (3), (9) and (10) we get

(11) $\mathrm{Cl}\ \mathfrak{V}_{m(i+1)}(x) \subset \mathrm{Cl}\ \mathfrak{V}_{m(i)n(i+1)}(x) \subset \mathfrak{V}_{m(i)-1}(x)$.

Set

(12) $D_{m(i)} = \{ \beta \in B_{m(i)} : x \in V_{n(i+1)}(\beta) \}$.

Then by (10)

(13) $D_{m(i)}$ is finite and non-empty.

By (3), (6) and (8), $\psi^{m(i+1)}{}_{m(i)}(D_{m(i+1)}) \subset D_{m(i)}$. Hence

(14) $\{ D_{m(i)};\ \psi^{m(i+1)}{}_{m(i)} | D_{m(i+1)} \}$ forms an inverse system.

Pick an arbitrary element $(\gamma_{m(i)})$ from inv lim $D_{m(i)}$. Then by (3) and (6),

$$x \in \mathrm{Cl}\ V(\gamma_{m(i+1)}) \subset V_{n(i+1)}(\gamma_{m(i)}) \subset V(\gamma_{m(i)}).$$

Hence by (1) and (2)

(15) $\{ V(\gamma_{m(i)}) \}$ is a k-sequence.

In consideration of (11), (13), (14) and (15), we conclude by Lemma 2.5 that

(16) $\{ \mathfrak{V}_i(x) \}$ is a k-sequence.

Since the condition in Lemma 4.5 is now satisfied by (11) and (16), X is a strict p-space. That completes the proof.

4.7. LEMMA(Miščenko[15]). <u>Let</u> X <u>be a set and</u> $\mathfrak{B}$ <u>a collection of subsets of</u> X <u>with</u> ord $\mathfrak{B} \leq \aleph_\lambda$. <u>Then the power of the family of all finite minimal coverings of</u> X <u>by elements of</u> $\mathfrak{B}$ <u>does not exceed</u> $\aleph_\lambda$.

4.8. THEOREM. Let X be a completely regular mp-space with $\aleph_\lambda$ -Lindelöf property and with a point $\aleph_\lambda$ base $\circledB$. Then $w(X) \leqq \aleph_\lambda$.

Proof. Since $\aleph_\lambda$ -Lindelöf property is inherited to cozero sets, applying an analogous argument in the proof of Lemma 1.5, X admits an mp-structure $\{\circledU_i = \{U(\alpha_i): \alpha_i \in A_i\}; \varphi^i_j\}$ such that each $U(\alpha_i)$ is a cozero set of X and such that

(1) $|A_i| \leqq \aleph_\lambda$.

Pick an $\alpha_i \in A_i$. Let $\circledV(\alpha_i)$ be the family of all finite minimal coverings of $U(\alpha_i)$ by elements of $\circledB|U(\alpha_i)$. Then by Lemma 4.7

(2) $|\circledV(\alpha_i)| \leqq \aleph_\lambda$.

Set

$$\circledV = \cup\{\circledV(\alpha_i): \alpha_i \in A_i, i=1,2,\cdots\}.$$

Then by (1) and (2) we get

(3) $|\circledV| \leqq \aleph_\lambda$.

To show that $\circledV$ is a base of X, let x be an arbitrary point of X and U an arbitrary open neighborhood of x. Let (α_i) be an element of inv lim A_i with $x \in \cap U(\alpha_i)$. Set $K = \cap U(\alpha_i)$. Since K is compact, there exists a finite subcollection $\circledB' = \{B_1, \cdots, B_n\}$ of $\circledB$ such that $x \in B_1 \subset U$ and such that

(4) $\circledB'|K$ is minimal.

Set $V = \bigcup_{i=1}^{n} B_i$. Since $\{U(\alpha_i)\}$ is converging to K, there exists an m with $U(\alpha_m) \subset V$. Since $\circledB'|U(\alpha_m) \subset \circledV(\alpha_m)$ by (2), then $B_1 \cap U(\alpha_m) \in \textcircled{V}(\alpha_m)$. Since $x \in B_1 \cap U(\alpha_m) \subset U$, $\textcircled{V}$ is a base of X. Thus by (3), $w(X) \leqq \aleph_\lambda$. That completes the proof.

4.9. COROLLARY.

If X is an mm-space with $\aleph_\lambda$-Lindelöf property, $w(X) \leqq \aleph_\lambda$.

Proof. For this case $\{\textcircled{U}_i\}$ in the preceding proof can be assumed to be an mm-structure. Since $\cup\, \textcircled{U}_i$ is a base of X whose power is at most $\aleph_\lambda$, we get the desired inequality. That completes the proof.

Recall that a base $\textcircled{B}$ of a space X is of *countable order* if $\{U_i\} \subset \textcircled{B}$, $x \in \cap U_i$ and $U_1 \supsetneqq U_2 \supsetneqq \cdots$, then $\{U_i\}$ is a local base at x.

4.10. THEOREM. *A space X has a base of countable order if and only if X is an mm-space.*

Proof. Since it is evident that the condition is necessary, let us prove the condition is sufficient. Let $\{\textcircled{U}_i = \{U(\alpha_i): \alpha_i \in A_i\}\,;\, \varphi^i_j\}$ be an mm-structure of X. Assume that each A_i is well ordered in such a way that, for each i, if $\varphi^{i+1}_i(\alpha_{i+1}) < \varphi^{i+1}_i(\alpha'_{i+1})$ in A_i, then $\alpha_{i+1} < \alpha'_{i+1}$ in A_{i+1}. By an easy application of induction, we get, for each i, a subset B_i of A_i satisfying the following four conditions.

(1) $\{U(\alpha_i): \alpha_i \in B_i\}$ covers X.

(2) $U(\alpha_i) - \cup\{U(\alpha'_i): \alpha'_i < \alpha_i\} \neq \emptyset$, $\alpha_i \in B_i$.

(3) $\varphi^{i+1}_i(B_{i+1}) \subset B_i$.

(4) $\cup \{ U(\alpha'_i) : \alpha'_i < \alpha_i \} = \cup \{ U(\alpha'_{i+1}) : \alpha'_{i+1} < \min(\varphi^{i+1}_i)^{-1}(\alpha_i) \}$, $\alpha_i \in B_i$.

Set

$$\mathfrak{U} = \{ U(\alpha_i) : \alpha_i \in B_i,\ i=1,2,\cdots \}.$$

To show that $\mathfrak{U}$ is a base of countable order let $\{U_i\}$ be a sequence of elements of $\mathfrak{U}$ such that $\cap U_i$ contains a point x and such that $U_i \supsetneqq U_{i+1}$ for each i. Let β_i be the minimal element of B_i such that $U(\beta_i)$ contains some member of $\{U_1, U_2, \cdots\}$. Then by (2), (3) and (4), (β_i) is an element of inv lim $\{ B_i ; \varphi^i_j | B_i \}$. Since $\cap U(\beta_i)$ contains x, $\{U(\beta_i)\}$ is a local base at x. Hence $\{U_i\}$ is a local base at x. That completes the proof.

5. <u>Complete mappings</u>.

5.1. DEFINITION. Consider the following condition for a mapping $f : X \to Y$ defined on an mp-space X.

There exists an mp-structure $\{\mathfrak{U}_i\}$ of X such that, for each $y \in Y$, the restriction of $\{\mathfrak{U}_i\}$ to $f^{-1}(y)$ is an mcc-structure of $f^{-1}(y)$.

If it is the case, then f is said to be <u>monotonically complete</u> or, simply, <u>complete</u> (with respect to $\{\mathfrak{U}_i\}$).

By virtue of Lemma 1.5 and the remark following 2.4, if f is complete with respect to $\{U_i\}$, then f is complete with respect to a saturated closure-refining mp-structure $\{\mathfrak{V}_i\}$ of X.

Here is another note which can easily be seen: When X is especially metrizable, f is complete if and only if there exists a metric d of X agreeing the topology of X such that $(f^{-1}(y),d)$ is complete is the usual sense for each $y \in Y$.

It can also easily be seen that when X is an mm-space, f is complete if and only if there exists an mm-structure of X with respect to which f is complete.

5.2. THEOREM. Let $f:X \longrightarrow Y$ be an open complete mapping of an mp-/mm-space X. Then f is compact-covering and Y is an mp-/mm-space.

Proof. The first half of the theorem is proved quite analogously to Theorem 4.2. Since a closure-refining mp-/mm-structure of X with respect to which f is complete is mapped by f to an mp-/mm-structure of Y, Y has to be an mp-/mm-space. That completes the proof.

5.3. THEOREM. A space X is an mp-/mm-space if and only if X is the image of a paracompact p-/metric space Y under an open complete mapping f.

Proof. The sufficiency was proved in the above theorem.

Necessity. Let $\{\mathfrak{U}_i = \{U(\alpha_i): \alpha_i \in A_i\}; \varphi^i_j\}$ be a saturated closure-refining mp-/mm-structure of X. As was done in Theorem 3.3 set

$$M = \text{inv lim } A_i,$$

$$Y = \{((\alpha_i),x) \in M \times X: x \in \cap U(\alpha_i)\},$$

and let f and g be the restrictions of the corresponding projections from Y to X and M. Then f is open and $f(Y) = X$. Set $g(Y) = T$. g is perfect and T is dense in M. Let d be the

Baire's metric on T. Then for each $x \in X$,

(1) $(gf^{-1}(x), d)$ is complete.

Let $\pi_i : M \to A_i$ be the projection. Set

(2) $W(\alpha_i) = \pi_i^{-1}(\alpha_i) \times U(\alpha_i)$, $\alpha_i \in A_i$.

Then it is easy to see that $W(\alpha_i) \cap Y \neq \emptyset$ for any $\alpha_i \in A_i$. Set

$$\mathcal{V}_i = \{ V(\alpha_i) = W(\alpha_i) \cap Y : \alpha_i \in A_i \}.$$

To see that $\{\mathcal{V}_i; \varphi^i_j\}$ is an mp-structure of Y let (α_i) be an element of inv lim A_i such that $\cap V(\alpha_i) \neq \emptyset$. Then $\cap W(\alpha_i) \neq \emptyset$ and hence by (2), $\cap U(\alpha_i) \neq \emptyset$. Set $K = \cap U(\alpha_i)$. Let x be an arbitrary point of K. Since $((\alpha_i), x) \in Y$, $\{(\alpha_i)\} \times K \subset Y$. Since $\{(\alpha_i)\} \times K$ is compact and $\cap V(\alpha_i) = \{(\alpha_i)\} \times K$, then $\{V(\alpha_i)\}$ is a k-sequence. By (1) and (2), we know that f is complete with respect to $\{\mathcal{V}_i\}$.

When $\{\mathcal{U}_i\}$ is especially an mm-structure, g is a homeomorphism and the completeness of f with respect to $\{\mathcal{V}_i\}$ implies the completeness of f with respect to the natural metric induced by d. That completes the proof.

5.4. THEOREM. Let be given a sequence $\{X_i\}$ of mp-spaces and a sequence $\{f_i : X_i \to Y_i\}$ of complete mappings. Then $\prod f_i : \prod X_i \to \prod Y_i$ is complete.

Proof. Let f_i be complete with respect to a closure-refining mp-structure $\{\mathcal{U}_{ij} : j = 1, 2, \cdots\}$ of X_i. Set

$$\mathcal{U}_i = \prod_{j=1}^{i} \mathcal{U}_{ji} \times \prod_{j=i+1}^{\infty} X_j.$$

Then $\prod f_i$ is complete with respect to $\{\mathcal{U}_i\}$ accompanied by the naturally defined refine-transformations. That completes the proof.

5.5. LEMMA. *Let* $f:X \to Y$ *be a closed, complete mapping with respect to a (saturated) closure-refining* mp-*structure* $\{\mathcal{U}_i = \{U(\alpha_i): \alpha_i \in A_i\};\ \varphi^i_j\}$ *of* X. *Assume that* Y *admits a closure-refining* mcc-*structure* $\{\mathcal{V}_i = \{V(\beta_i): \beta_i \in B_i\};\ \psi^i_j\}$. *Set*

$$W(\alpha_i, \beta_i) = U(\alpha_i) \cap f^{-1}(V(\beta_i)),$$

$$C_i = \{(\alpha_i, \beta_i) \in A_i \times B_i : W(\alpha_i, \beta_i) \neq \emptyset\},$$

$$\mathcal{W}_i = \{W(\gamma_i): \gamma_i \in C_i\},$$

$$\sigma^i_j(\alpha_i, \beta_i) = (\varphi^i_j(\alpha_i), \psi^i_j(\beta_i)),\ (\alpha_i, \beta_i) \in C_i.$$

Then $\{\mathcal{W}_i;\ \sigma^i_j\}$ *is a (saturated) closure-refining* mcc-*structure of* X.

Proof. $\{\mathcal{W}_i\}$ is obviously a (saturated) closure-refining structure of X. To prove that it is an mcc-structure, let (γ_i) be an arbitrary element of inv lim $\{C_i;\ \sigma^i_j\}$, where $\gamma_i = (\alpha_i, \beta_i)$. Set $K = \bigcap V(\beta_i)$.

Assume that for each point $y \in K$, there exists an $i(y)$ with $y \notin f(U(\alpha_{i(y)}))$. Since f is closed and $\{\mathcal{U}_i\}$ is closure-refining, $\{K - f(\mathrm{Cl}\ U(\alpha_i)): i=1,2,\cdots\}$ is an open covering of K. Since K is compact, $K - f(\mathrm{Cl}\ U(\alpha_m)) = K$ for some m. Since $\{V(\beta_i)\}$ is converging to K, there exists an n with $f(\mathrm{Cl}\ U(\alpha_m)) \cap V(\beta_n) = \emptyset$. Then $U(\alpha_m) \cap f^{-1}(V(\beta_n)) = \emptyset$, which implies that $W(\gamma_k) = U(\alpha_k) \cap f^{-1}(V(\beta_k)) = \emptyset$ for $k \geqq \max\{m, n\}$. This contradiction implies that there exists a point $z \in K$ such that $z \in f(U(\alpha_i))$ for each i. Since

$f^{-1}(z) \cap U(\alpha_i) \neq \emptyset$ for any i and f is complete, $(\cap U(\alpha_i)) \cap f^{-1}(z) \neq \emptyset$ and hence $\cap W(\gamma_i) \neq \emptyset$. Thus $\{W(\gamma_i)\}$ is a k-sequence. That completes the proof.

5.6. THEOREM. <u>The composition of closed, complete mappings is still closed and complete.</u>

Proof. It suffices to prove the theorem for the composition of two such mappings $f:X \to Y$ and $g:Y \to Z$. Let $\{\mathfrak{U}_i\} / \{\mathfrak{V}_i\}$ be a closure-refining mp-structure of X/Y with respect to which f/g is respectively complete. Pick an arbitrary point $z \in Z$. Then $\{\mathfrak{V}_i\} | g^{-1}(z)$ is an mcc-structure. Since $f|f^{-1}g^{-1}(z)$ is complete with respect to $\{\mathfrak{U}_i\} | f^{-1}g^{-1}(z)$, we can apply Lemma 5.5 and the composition gf is complete. The closedness of gf is obvious. That completes the proof.

6. <u>Examples</u>.

6.1. EXAMPLE. <u>A first-countable, Čech complete space which is not an</u> mcm-<u>space</u>.

Set $X = I \times D$, where I is the unit closed interval and D is the set of two points $\{0,1\}$. Introduce the interval topology with respect to the lexicographic order of $I \times D$ into X. As is well known X is a first-countable, compact space which is not metrizable. Thus X is clearly Čech complete.

Assume that X is an mcm-space. Then by Theorem 3.5 there exists a complete metric space Y and an open mapping $f:Y \to X$. Since f is compact-covering by Theorem 3.7, there exists a compact set L of Y with $f(L) = X$. Then X is metrizable as the perfect image of a metric space. This contradiction shows that X is not an mcm-space.

6.2. EXAMPLE. An mcc-space which is neither an mcm-space nor a Čech complete space.

Let ω or ω_1 be respectively the first infinite or the first uncountable ordinal. Let $[0,\omega]$ or $[0,\omega_1]$ be respectively the space of all ordinals $\leqq \omega$ or $\leqq \omega_1$, with the interval topology. Consider the Tychonoff plank:

$$X = [0,\omega] \times [0,\omega_1] - (\omega , \omega_1).$$

The following construction is well known. Set

$$A = \{\omega\} \times [0,\omega_1] - (\omega , \omega_1),$$

$$B = [0,\omega] \times \{\omega_1\} - (\omega , \omega_1).$$

For each positive integer i consider a copy X_i of X with edges A_i and B_i corresponding respectively to A and B.

Let $f_i : X \rightarrow X_i$ be the identity mapping. Consider the topological sum $\cup X_i$ of X_i's. In $\cup X_i$, for each n, identify A_{2n-1} with A_{2n}, and B_{2n} with B_{2n+1}. By this identification $\cup X_i$ is deformed to Y. Let $f: \cup X_i — Y$ be the natural projection. By this f, Y enjoys the quotient topology. Set $g_i = ff_i$. Let Z be the disjoint sum of Y and a singleton p. Set

$$W(n,p) = Z - f(\bigcup_{i \leqq n} X_i).$$

The topology of Z is defined in such a way that each open set of Y is open in Z and $\{W(n,p): n=1,2,\cdots\}$ is an open neighborhood base of p. Then Z is a space(i.e. a regular space) which is not completely regular(cf. e.g. Willard [27, 18G, p.134]). Hence Z is not Čech complete. Since Z is not first-countable, Z is not an mcm-space.

Let us prove that Z is an mcc-space. Set

$$U(a) = [0,\omega] \times [0,\alpha],\ a = (\omega,\alpha) \in A,$$

$$V(b) = [0,n] \times [0,\omega_1],\ b = (n,\omega_1) \in B.$$

Set further

$$W(a,2i-1) = g_{2i-1}(U(a)) \cup g_{2i}(U(a)),\ a \in A,\ i=1,2,\cdots,$$

$$W(b,1) = g_1(V(b)),\ b \in B,$$

$$W(b,2i) = g_{2i}(V(b)) \cup g_{2i+1}(V(b)),\ b \in B,\ i=1,2,\cdots.$$

Then for each n we have an open covering

$$\mathfrak{U}_n = \{W(n,p)\} \cup \{W(a,2i-1): a \in A,\ i=1,2,\cdots\}$$
$$\cup \{W(b,1): b \in B\} \cup \{W(b,2i): b \in B,\ i=1,2,\cdots\}$$

of Z. Set

$$C_n=\{(n,p)\} \cup \{(a,2i-1),(b,1),(b,2i): a\in A, b\in B, i=1,2,\cdots\}.$$

Then we can write

$$\mathfrak{U}_n = \{W(\gamma): \gamma \in C_n\}.$$

Set

$$D_{n+1} = \{\gamma \in C_{n+1}: W(\gamma) \subset W(n,p)\}.$$

Define $\varphi^{n+1}{}_n : C_{n+1} \to C_n$ by:

$$\varphi^{n+1}{}_n|(C_{n+1} - D_{n+1}) = \text{identity},$$

$$\varphi^{n+1}{}_n(\gamma) = (n,p),\ \gamma \in D_{n+1}.$$

Then it can easily be seen that $\{\mathfrak{U}_n; \varphi^{n+1}{}_n\}$ forms an mcc-structure of Z.

6.3. EXAMPLE. <u>An</u> mcm-<u>space which is not complete metric</u>. See Wicke-Worrell[24, p.260].

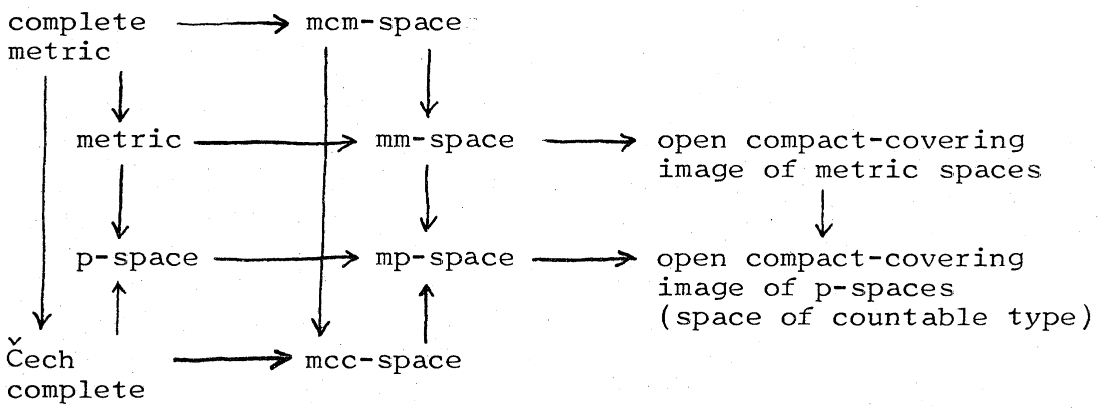
6.4. EXAMPLE. <u>An</u> mp-<u>space which is neither a</u> p-<u>space nor an</u> mcc-<u>space</u>.

Consider the space X in Example 6.1. Let $f: X \to I$ be the projection and J a subset of I which is not G_δ. Set $Y = f^{-1}(J)$. Then Y is a paracompact p-space. Let Z be the space given in Example 6.2. The product $T = Y \times Z$ is the desired space.

6.5. EXAMPLE. A paracompact space of countable type which is not an mp-space.

Michael line[12] is the desired. It is to be noted that a space is of countable type if and only if it is the image of a p-space under an open compact-covering mapping (cf. Čoban[7]).

The following is a diagram of implications among the classes of monotonic spaces. The converse of each implication is not true by the above examples and by well known ones.



7. Acknowledgement.

The four kinds of monotonic spaces and complete mappings defined in this paper, which are intrinsic modifications of those in [5], which are intrinsic modifications of those in [5], have the preceding equivalents by Wicke and Worrell as indicated in the following table.

mcc-space·············	λ_b-space[21], condition $\mathcal{K}$[25]
mp-space··············	β_b-space[21]
mcm-space·············	monotonically complete space with a base of countable order[19]
mm-space··············	space with a base of countable order[23], space with a λ-base[24]
complete mapping······	uniformly λ-complete mapping[21], uniformly monotonically complete mapping[24]

Some theorems in this paper were already proved essentially or proved partly by Wicke and Worrell as exhibited in the following.

Theorem 2.7················[23, Theorem 1]

Theorems 2.8, 2.9 and 2.10········[26]

Theorem 3.4················[25, Theorem 4]

Theorem 3.6······[19, Theorem 4] and [29, Theorems 4, 5]

Proposition 3.10··········[24, Remark 7, p.256]

Theorem 4.1················[21, Theorem 4.4]

Theorem 4.6················[22, Theorem 7.4]

Theorem 4.10··············[19, Theorem 5]

Theorem 5.3 ··············· [21, Theorem 4.6]

Theorem 5.6 ··············· [24, Theorem 7]

Hitherto we have assumed that all spaces are regular. This assumption played a role to avoid needless complexity. It is easy to verify that the regularity can be weakened to T_1-axiom in the theorems for mcm-spaces and mm-spaces. Finally it should be noted that the proof of Theorem 3.3 is based on the beautiful idea of Wicke[20].

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8. Appendix.

Wicke[21] considered the conditions λ_c and β_c. For the reader's convenience let us define their monotonic equivalents. A mapping is said to be quasi-perfect if it is closed and every point-inverse is countably compact. A space is said to be an M-space(by K. Morita, Products of normal spaces with metric spaces, Math. Annalen 154(1964)365-382, Theorem 6.1) if it is the preimage of a metric space under a quasi-perfect mapping. A space is said to be a complete M-space(by Wicke [21, Definition 3.2]) if it is the preimage of a complete metric space under a quasi-perfect mapping. A decreasing sequence $\{U_i\}$ of subsets of a space X is said to be a q-sequence(by Michael[13, Definition 1.2]) if $\cap U_i$ is a non-empty countably compact set and $\cap U_i \subset U$ with U open implies $U_m \subset U$ for some m. $\{U_i\}$ is said to be a weak q-sequence if the inequalities $U_i \supset F_i \neq \emptyset$ and $F_i \supset \overline{F}_{i+1}$ (i=1,2,$\cdots$) imply that $\{F_i\}$ is a q-sequence.

8.1. DEFINITION. Consider the following two conditions for a structure $\{\mathfrak{U}_i = \{U(\alpha_i): \alpha_i \in A_i\}, \varphi^i_j\}$ of a space X, where (α_i) is a generic element of inv lim A_i.

(1) $\{U(\alpha_i)\}$ is a q-sequence.

(2) $\{U(\alpha_i)\}$ is a q-sequence or $\cap U(\alpha_i) = \emptyset$.

A structure satisfying (1)/(2) is said respectively an mcM-/mM-structure. If a 'q-sequence' in this definition is replaced with a 'weak q-sequence', we get a concept of a weak mcM-/weak mM-structure. A space having an mcM-/mM-structure is said respectively to be a monotonically complete M-/ monotonic M-space(abbreviated by an mcM-/mM-space).

By this definition each θ-refinable mcM-/mM-space is an mcc-/mp-space.

Complete mappings defined in Definition 5.1 should be weakened as was defined by Wicke[21, Definition 3.1] as follows.

8.2. DEFINITION. A mapping $f:X \to Y$ of a space X with a structure $\{\mathcal{U}_i\}$ is said to be complete if for each point $y \in Y$, $\{\mathcal{U}_i | f^{-1}(y)\}$ is a weak mcM-structure of $f^{-1}(y)$.

8.3. THEOREM(Wicke[21, Theorem 4.2 and 4.3]). A space is an mcM-space if and only if it is the open image of a complete M-space.

8.4. THEOREM(Wicke[21, Theorem 4.7]). The perfect image of an mcM-/mM-space is respectively an mcM-/mM-space.

8.5. THEOREM(Wicke[21, Theorems 4.2 and 4.4]). A space is an mM-space if and only if it is the image of an M-space under an open complete mapping. The image of an mM-space under an open complete mapping is still an mM-space.

8.6. THEOREM. A closed set of an mcM-/mM-space is an mcM-/mM-space. A G_δ set of an mcM-/mM-space is an mcM-/mM-space. A locally mcM-/mM-space is an mcM-/mM-space.

8.7. THEOREM. A space with a weak mcM-/weak mM-structure is an mcM-/mM-space.

8.8. THEOREM. An mcM-space which is an mp-space at the same time is an mcc-space.

These theorems are easily verified by analogous arguments used in the proof of 1.7, 2.6, 2.7, 2.9, 2.10, 5.3, or are almost evident.

Just after completing this manuscript the author noticed that the theorems for monotonic complete spaces can be generalized from a more general stand point of view. Such a gneral theory will be given elsewhere.