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[A] 有界領域における反射壁 Brownian motion の構成

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§1. 序

D を N 次元ユークリッド空間 R^N の有界領域とする。D の境界 ∂D の滑らかさは仮定しない。

$\alpha>0$, $x, y \in D$, $x \neq y$ に対して定義された関数 $G_\alpha(x, y)$ が次の条件を満すとき、これを D 上の <u>resolvent</u> density という。

G.1) $G_\alpha(x, y) \geq 0$, $\alpha>0$, $x, y \in D$, $x \neq y$.

G.2) $\alpha \int_D G_\alpha(x, y)\, dy \leqq 1$, $\alpha>0$, $x \in D$.

G.3) $G_\alpha(x, y) - G_\beta(x, y) + (\alpha-\beta)\int_D G_\alpha(x, z) G_\beta(z, y) = 0$, $\alpha, \beta>0$, $x, y \in D$, $x \neq y$.

G.4) $\alpha>0$ を固定すると $G_\alpha(x, y)$ は (x, y) に関し連続

G.2) に於て等号が任意の α, x について成立するとき $G_\alpha(x, y)$ を <u>conservative</u> という。

目的は

イ) D 上のある conservative resolvent density

を構成すること．

ロ）それが D を含むある compact 集合 D^* 上の拡散過程（強マルコフ過程で path が連続なもの）を決定することを示す．

ハ）ロ）の拡散過程は ∂D が充分滑らかな場合には $D \cup \partial D$ 上の所謂反射壁 Brown 運動と同等であることを示す．

これを三つの定理として述べよう．

$p(t, x, y)$，$t > 0$，$x, y \in D$，が D 上の transition density であるとはそれが次の条件を満すこと．

T.1) $p(t, x, y) \geq 0$.

T.2) $\int_D p(t, x, y)\,dy \leq 1$, $t > 0$, $x \in D$.

T.3) $\int_D p(t, x, z)\,p(s, z, y)\,dz = p(t+s, x, y)$,

$t, s > 0$, $x, y \in D$.

T.4) $p(t, x, y)$ は三変数の連続関数

T.2) に於て等号が全ての t, x について成立するとき $p(t, x, y)$ を conservative という．

さて $p^0(t, x, y)$ を D 上の吸収壁 Brown 運動の transition density とする（[8] 参照）．

1.1) $G_\alpha^0(x, y) = \int_0^{+\infty} e^{-\alpha t}\,p^0(t, x, y)\,dt$, $\alpha > 0$, $x, y \in D$，とおくと $G_\alpha^0(x, y)$ は D 上の一つの

resolvent density である。
これは次の様にもかける。

1.2) $G_\alpha^0(x, y) = \Pi_\alpha(x, y) - E_x^0(e^{-\alpha\tau}\Pi_\alpha(X_\tau, y))$,

但し

$$\Pi_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t}\frac{1}{(2\pi t)^{N/2}} e^{-\frac{|x-y|^2}{2t}}\, dt,$$

$x, y \in D$, E_x^0 は N 次元 Brown 運動の measure による平均。 τ は D からの exit time.

D 上の関数 u が <u>α-harmonic</u> とは $(\alpha - \frac{1}{2}\Delta)u(x) = 0$ が全ての $x \in D$ について成立することとする。但し Δ は Laplace 微分作用素。 D 上の関数 u, v に対し

1.3) $(u, v) = \int_D u(x)v(x)\,dx$,

$\mathbb{D}(u, v) = \int_D (\operatorname{grad} u, \operatorname{grad} v)(x)\,dx$

とおく。

1.4) $\mathbb{H}_\alpha = \{u : u$ は D 上で α-harmonic 且つ $(u, u) < +\infty$, $\mathbb{D}(u, u) < +\infty\}$ とおく

$\boxed{\text{定理 1}}$

(1) 各 $\alpha > 0$, $x \in D$ について次の性質を満す $\mathbb{H}_\alpha$ の元 $R_\alpha^x(y) = R_\alpha(x, y)$ が一意的に存在する。

1.5) $\mathbb{D}(R_\alpha^x, v) + 2\alpha(R_\alpha^x, v) = 2v(x)$, $\forall v \in \mathbb{H}_\alpha$.

(2) $G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$ は D 上の conservative resolvent density である

$G_\alpha(x,y)$ は x, y に関し対称である.

(3) $B(D)$ ($\mathbb{C}(D)$) を D 上の有界(有界連続)関数の全体とする.

1.6) $G_\alpha f(x) = \int_D G_\alpha(x,y) f(y) dy$ で定義される. G_α は $\mathbb{B}(D)$ を $\mathbb{C}(D)$ に移す. 又,

$\lim_{\alpha \to +\infty} \alpha G_\alpha f(x) = f(x)$, $\forall f \in \mathbb{C}(D)$, $\forall x \in D$.

(4) $K_1 \subset D_1 \subset K_2 \subset D$ とする. 但し, K_1, K_2 は compact, D_1 は open. このとき,

$$\sup_{x \in K_1,\, y \in D - K_2} G_\alpha(x, y) < +\infty.$$

(5) 次の条件 1.7) を満す D 上の transition density $p(t, x, y)$ が一意的に存在する.

1.7) $G_\alpha(x,y) = \int_0^{+\infty} e^{-\alpha t} p(t, x, y) dt$,

$\alpha > 0$, $x, y \in D$, $x \neq y$.

この $p(t, x, y)$ は conservative, x, y に関し対称, $\int_D p(t, x, y) f(y) dy$ は (t, x) について連続 (但し $f \in \mathbb{B}(D)$).

$\boxed{\text{定義}}$ compact set D^* が D の compact 化

$\Longleftrightarrow$ (i) D は D^* に含まれ, そこで open, dense.

(ii) D^* での D の topology は Euclidean topology と同等.

$\boxed{定理 2}$

(1) $\exists D^*$: D の compact 化.

$p(t,x,y)$ (in 定理 1) は $x \in D^*$ まで拡張され 拡張された関数を改めて $p(t,x,y)$ と書くと T.1), T.2), T.3) が $x \in D^*$, $y \in D$ について成立する。 又 $\int_D p(t,x,y) f(y) dy$ は $f \in \mathbb{B}(D)$, $x \in D^*$ のとき t について連続

(2) $\exists \mathbb{X} = \{X_t, P_x, x \in D^*\}$ Markov process on D^*. $\mathbb{X}$ は次の条件を満す.

(a) $\forall A$: Borel set of D_1^*,

$$P_x(X_t \in A) = \int_{D \cap A} p(t,x,y)\, dy, \quad t>0,\ x \in D^*.$$

(b) $\mathbb{X}$ は連続 : $P_x(X_t$ is continuous in $\forall t \geq 0) = 1$, $x \in D^*$.

(c) $\mathbb{X}$ は強マルコフ過程.

(d) $\exists D_1^*$ Borel set in D^*. $D \subset D_1^*$.

$$P_x(X_0 = x) = 1, \quad x \in D_1^*.$$

$$P_x(X_t \in D_1^*,\ \forall t \geq 0) = 1, \quad x \in D_1^*.$$

(e) $P_x(X_t \in A,\ t < \tau) = \int_A p^0(t,x,y)\, dy$,

$t>0$, $x \in D$, A は D の Borel set,

$\tau = \inf\{t \,;\, X_t \in D^* - D\}$.

$\boxed{\text{定理 3}}$

∂D が C^3-級と仮定する。 このとき、

(1) $\exists\, \Phi$: homeomorphism from $D \cup \partial D$ to D_1^*.

(2) $\dot{X}_t = \Phi^{-1}(X_t)$, $t \geq 0$.

$\dot{P}_x = P_{\Phi(x)}$, $x \in D \cup \partial D$ とおくと

$\dot{X} = \{\dot{X}_t, \dot{P}_x, x \in D \cup \partial D\}$ は $D \cup \partial D$ 上の conservative 拡散過程であり次の条件を満す.

$$\dot{P}_x(\dot{X}_t \in A) = \int_{A \cap D} \dot{p}(t, x, y)\,dy, \quad t > 0,\ x \in D \cup \partial D,$$

A は $D \cup \partial D$ の Borel set.

但し $\dot{p}(t, x, y)$, $t > 0$, $x \in D^*$, $y \in D$ は

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u(t, x) = 0, \quad \frac{\partial}{\partial n_x} u(t, x) = 0,\ x \in \partial D,$$

の基本解である. n_x は inner normal.

$\boxed{\text{定義}}$

定理 2, (2) の $X = \{X_t, P_x, x \in D^*\}$ を D^* 上の反射壁 Brown 運動という.

以上は

M. Fukushima ; A construction of reflecting barrier Brownian motions for bounded domains, to appear. の内容の紹介である.

数理研シンポジュームでは 定理1の証明を中心に報告した。 それは上の論文の§2の部分であり ここではそれを英文のままのせる。
定理2,3の証明は概略を報告した。

定理2について

$G_\alpha(x,y)$ を定理1のそれとする。 定理2,(1) の D^* として D の $G_1(x,y)$ による Martin-倉持型の completion をとる。 この D^* は次の性質によって特徴づけられるものである。

(D*.1) D^* は D の compact化

(D*.2) $\{G_1 f,\ f \in C_0(D)\}$ の元は D^* に連続に拡張され、拡張されたこの関数の族は D^* の2点を分離する。

但し $C_0(D)$ は台が compact な D 上の連続関数の全体を表わす。

勿論 $G_1(x,y)$ は各 $y \in D$ について x の関数として D^* 上に連続に拡張される。

$\forall \alpha > 0$, $\forall x \in D^* - D$,

1.8) $G_\alpha(x,y) = G_1(x,y) - (\alpha-1)\int_D G_\alpha(x,z)G_\alpha(z,y)\,dz$

とおくと、 $G_\alpha(x,y)$, $x \in D^*$, $y \in D$ は G.1), G.2), G.3) を満すことがわかる。

特に $G_\alpha(x,y)$, $\alpha>0$, $x, y\in D$ の conservativity により

(1.9) $\quad \alpha\int_D G_\alpha(x,y)dy = \int_D G_1(x,y)dy \leq 1$, $\forall\alpha>0$, $\forall x\in D^*$, が従がう。 次に定理2, (1) を満す $p(t,x,y)$, $t>0$, $x\in D^*$, $y\in D$ は

(1.10) $\quad G_\alpha(x,y) = \int_0^{+\infty} e^{-\alpha t}p(t,x,y)dt$,

$\alpha>0$, $x\in D^*$; $y\in D$,

なる関係を満すものとして見つけることが出来る。

さて、定理2, (2), (a) の条件を満す Markov process $X=(X_t, P_x, x\in D^*)$ を考えるとその適当な version が強マルコフ過程となることがわかる。 この証明は先に記した G_1 の性質 ($G_1 f$, $f\in \mathbb{C}_0(D)$ の連続性) に注意して行なわれる。 X_t としては右連続 ($t\geq 0$), 且つ左極限を有する ($t>0$) ものがとれることがわかる。

Ray [20], 国田・渡辺(毅) [10] [11] で扱かわれている場合と同じく、D^* は branching point を含むかもしれない。 $P_x(X_0=x)<1$ なる x を branching point と呼びその全体を Δ_0 とする。 $\Delta_0\subset D^*-D$ である。 $D^*-\Delta_0 = D_1^*$ とおく。 次のことが示される。

$P_x(X_t\notin\Delta_0, X_{t-}\notin\Delta_0, \forall t\geq 0)=1$, $\quad \forall x\in D^*$.

更に X について定理2, (2), (e) の性質が証明され　その結果

$P_x(X_t$ is continuous at such t that X_t or $X_{t-} \in D) = 1$, $\quad x \in D^*$, がわかる。

定理2の証明で鍵となっている本質的なことは path が境界で jump を持たないことの証明である。

これは次の様な定性的な方法で示される。

$\boxed{\text{定義}}$ $u \geq 0$ on D^* が 1-excessive とは

$e^{-t}\int_D p(t,x,y)u(y)dy \uparrow u(x)$, $(t \downarrow 0)$, $x \in D^*$

が成立すること。

$\boxed{\text{定理 4}}$

u が 1-excessive 且つ $\int_D u(x)dx < +\infty$ ならば　$\exists 1\, \nu$: D^*_1 上の測度

$$u(x) = \int_{D^*_1} G_1(x,y)\nu(dy), \quad \forall x \in D^*.$$

但し $G_1(x,y)$, $x \in D^*$, は y について D から D^* に 1-excessive に拡張されたもの。

この ν を u に対応する canonical measure という

例　$\mathcal{R} = G_{1/2}(\mathcal{B}(D^*))$ とおく。

$u \in \mathcal{R}$, $\quad u = G_{1/2}f$ に対し

$A^u_t = e^{-t/2}u(X_t) - u(X_0) + \int_0^t e^{-s/2}f(X_s)ds$, $t>0$,

$\nu_u(x) = E_x((A^u_{+\infty})^2)$, $x \in D^*$ とおくと
ν_u は 1-excessive 且つ $\int_D \nu_u(x) < +\infty$.

$\boxed{定義}$ $u \in \mathcal{R}$ に対し ν_u に対応する canonical measure を μ_u とする。
$|||u|||_X = \sqrt{\mu_u(D_1^*)}$ を u の X に関する Dirichlet norm という。

$\boxed{定理 5}$ $u \in \mathcal{R}$ のとき
$|||u|||_X = \int_D (\mathrm{grad}\, u, \mathrm{grad}\, u)(x)\, dx$ が成立つ。
且つ $\mu_u(D_1^* - D) = 0$ である。

定理5は反射壁Brown運動の特性である。本質、渡辺(信)[18]の結果による。
定理5は次のことを意味していることがわかる。
$E_x(A_t) = 0$, $E_x(A_t^2) < +\infty$ なる X の任意の additive functional A に対し確率積分 $\int \chi_{D_1^*-D} \cdot dA$ は恒等的に 0 に等しい。
ここで $\chi_{D_1^*-D}$ は D_1^* の indicator function.
これは path が D^*-D で jump する可能性を否定する。

§2. Construction of a resolvent density (Proof of Theorem 1).

From now on, we fix an arbitrary bounded domain D of R^N. The following criterion for a function on D to be α-harmonic is easily verified and will be frequently used in this paper.

Lemma 2.1. Let α be positive number. A function u on D is α-harmonic, if and only if, for each ball B with closure contained in D, it holds that

$$u(x) = \int_{\partial B} h_\alpha^B(x, y)\, u(y)\, \sigma(dy), \qquad x \in B,$$

where $\sigma(dy)$ is the surface Lebesque measure of ∂B and

$$h_\alpha^B(x, y) = \frac{1}{2} \frac{\partial}{\partial n_y} G_\alpha^B(x, y), \quad x \in B,\ y \in \partial B,$$

$G_\alpha^B(x, y)$ being the resolvent density defined by 1.1) for the Ball B.

For functions u and v on D, let $\mathbb{D}(u, v)$ and (u, v) be quantities defined by 1.3) if they have the meaning anyway. For each $\alpha > 0$, consider the function space $\mathbb{H}_\alpha$ defined by 1.4). $\mathbb{H}_\alpha$ contains the function which is identically zero on D. Later, we shall show that it contains also non-trivial elements. Let us put for $u, v \in \mathbb{H}_\alpha$

2.1) $\quad \mathbb{D}_\alpha(u, v) = \mathbb{D}(u, v) + 2\alpha(u, v).$

<u>Lemma 2.2</u>. For each $\alpha > 0$, $\mathbb{H}_\alpha$ forms a real Hilbert space with the inner product $\mathbb{D}_\alpha(u, v)$. Moreover, any Cauchy sequence of functions in $\mathbb{H}_\alpha$ with respect to the norm $\sqrt{\mathbb{D}_\alpha(u, u)}$ converges on D uniformly on any compact subset of D.

<u>Proof.</u> Suppose that $u_n \in \mathbb{H}_\alpha$, $n = 1, 2, \ldots,$ and

$$\mathbb{D}_\alpha(u_n - u_m, u_n - u_m) \xrightarrow[n,m \to +\infty]{} 0.$$

Let K be any compact subset of D. We can choose $\varepsilon > 0$ such that the closed ball $K_\varepsilon(x)$ with radius ε centered at any point $x \in K$ is contained in D. Applying Lemma 2.1 to the α-harmonic function $u_n - u_m$, we have

2.2) $\quad u_n(x) - u_m(x)$

$$= \frac{1}{V_\varepsilon}\int_{K_\varepsilon(x)} \eta_\alpha(|y-x|)(u_n(y) - u_m(y))dy, \quad x \in K,$$

where V_ε is the volume of $K_\varepsilon(x)$, $|y - x|$ is the distance between x and y, and $\eta_\alpha(r)$ is a function of real $r > 0$ which depends only on $\alpha > 0$ and satisfies $0 < \eta_\alpha(r) < 1$. The Schwarz inequality applied to 2.2) leads to

$$(u_n(x) - u_m(x))^2 \leq \frac{1}{V_\varepsilon}(u_n - u_m, u_n - u_m)$$

$$\frac{1}{2\alpha V_\varepsilon}\mathbb{D}_\alpha(u_n - u_m, u_n - u_m), \quad x \in K.$$

Thus, u_n converges to a function u on D uniformly on any compact subset of D. By virtue of Lemma 2.1, u is also α-harmonic on D and the derivatives of u_n converge to those of u uniformly on any compact subset of D. On the other hand, since u_n, $n = 1, 2, \ldots$, form a Cauchy sequence with respect to $\mathbb{D}_\alpha(\cdot, \cdot)$, one can find, for any $\varepsilon > 0$, a compact subset $K \subset D$ such that

$$\int_{D-K} |\text{grad } u_n|^2(x)dx + 2\int_{D-K} u_n(x)^2 dx < \varepsilon$$

uniformly in n. Hence, $u \in \mathbb{H}_\alpha$ and $\mathbb{D}_\alpha(u_n - u, u_n - u) \xrightarrow[n \to +\infty]{} 0$.

Lemma 2.3. Let $\alpha > 0$ be fixed.

1) For each $x \in D$, there exists a function $u^{(x)} \in \mathbb{H}_\alpha$ uniquely such that

2.3) $\mathbb{D}_\alpha(u^{(x)}, v) = 2v(x)$, for any $v \in \mathbb{H}_\alpha$.

2) $u^{(x)}$ in 1) is a unique element of $\mathbb{H}_\alpha$ minimizing the value of the functional $\Psi(u) = \mathbb{D}_\alpha(u, u) - 4u(x)$ on $\mathbb{H}_\alpha$.

Proof. 1). For a fixed $x \in D$, define the linear mapping Φ from $\mathbb{H}_\alpha$ to R^1 by $\Phi(v) = 2v(x)$, $v \in \mathbb{H}_\alpha$. Φ is bounded, because, if $v_n \xrightarrow[n \to +\infty]{} 0$ in $\mathbb{H}_\alpha$, then, by Lemma 2.2, $2v_n(x) \xrightarrow[n \to +\infty]{} 0$.

The Riesz representation theorem implies 1).

2). We have only to notice the equality $\Psi(u) = \Psi(u^{(x)}) + \mathbb{D}_\alpha(u - u^{(x)}, u - u^{(x)})$, $u \in \mathbb{H}_\alpha$.

Definition 1. For $\alpha > 0$ and $x \in D$, denote by $R^x_\alpha(y) = R_\alpha(x, y)$, $y \in D$, the function $u^{(x)}(y)$, $y \in D$, determined by Lemma 3.

Definition 2. Let $G^0_\alpha(x, y)$ be the resolvent density defined by 1.1). Define the function $G_\alpha(x, y)$, $\alpha > 0$, $x, y \in D$, by $G_\alpha(x, y) = G^0_\alpha(x, y) + R_\alpha(x, y)$.

Before examening prperties of $G_\alpha(x, y)$ stated in Theorem 1, we prepare three lemmas.

An exhaustion of D is a sequence of domains D_n, $n = 1, 2, \ldots,$ such that the closure of D_n is contained in D_{n+1} and D_n converges monotonically to D. An exhaustion $\{D_n\}$ of D is called regular if ∂D_n are of class C^3.

Lemma 2.4. Let $\alpha > 0$ be fixed.

1) Any non-negative α-harmonic function on D is either identically zero on D or strictly positive on D.

2) The function $w = 1 - \alpha G^0_\alpha 1$ is strictly positive on D. Moreover w is the unique element in $\mathbb{H}_\alpha$ satisfying

2.4) $\mathbb{D}_\alpha(w, v) = 2\alpha(1, v)$ for all $v \in \mathbb{H}_\alpha$.

Proof.

1) Since Lemma 2.1 implies that the value of an α-harmonic function at any point of D is a weighted volume mean on the ball centered at the point, 1) is verified in the same manner as in the case of harmonic functions.

2) It is evident, by the expression 1.2) of G_α^0, that w is α-harmonic and strictly positive on D. In order to show 2.4), consider a regular exhaustion $\{D_n\}$ of D.

Put $w_n = \chi_{D_n} - \alpha\, {}^nG_\alpha^0 \chi_{D_n}$, where χ_{D_n} is the indicator function of D_n, ${}^nG_\alpha^0 \chi_{D_n}(x) = \int_{D_n} {}^nG_\alpha^0(x, y)dy$ and ${}^nG_\alpha^0(x, y)$ is the resolvent density defined by 1.1) for D_n. w_n is α-harmonic in D_n, converges to w monotonically and (consequently) uniformly on any compact subset of D. On account of Lemma 2.1, the derivatives of w_n converge to those of w on D. Denote by $\mathbb{D}_\alpha^n(\ \ ,\ \ )$ the integral 2.1) on D_n. Since w_n belongs to $\mathbb{C}^1(D_n \cup \partial D_n)$, we can apply Green's formula to w_n and $v \in \mathbb{H}_\alpha$, obtaining $\mathbb{D}_\alpha^n(w_n, v) = 2\alpha(\chi_{D_n}, v)$. This equality implies the inequality

$$\mathbb{D}_\alpha^n(w_n, w_n) - 4\alpha(\chi_{D_n}, w_n) \leqq \mathbb{D}_\alpha^n(v, v) - 4\alpha(\chi_{D_n}, v)$$

for all $v \in \mathbb{H}_\alpha$. Letting n tend to infinity and using Fatou's Lemma, we obtain

$$\mathbb{D}_\alpha(w, w) - 4\alpha(1, w) \leqq \mathbb{D}_\alpha(v, v) - 4\alpha(1, v).$$

Thus, $w \in \mathbb{H}_\alpha$, and if we put, instead of v, $w + \varepsilon v$ in the inequality above, we arrive at 2.4). Proof of the uniqueness is straightforward.

<u>Lemma 2.5.</u> Take an exhaustion $\{D_n\}$ of D arbitrarily. Let ${}^nR_\alpha^x(y)$ and ${}^nG_\alpha(x, y)$, $\alpha > 0$, $x, y \in D_n$ be the functions defined by Definition 1 and Definition 2 for the

domain D_n. Then, $\lim_{n \to +\infty} {}^nG_\alpha(x, y) = G_\alpha(x, y)$, $\alpha > 0$, $x, y \in D$, $x \neq y$. Moreover, for each $x \in D$, the equality

2.5) $$\lim_{n \to +\infty} {}^nR^x_\alpha(y) = R^x_\alpha(y), \quad y \in D,$$

holds and the convergence is uniform in y on any compact subset of D.

<u>Proof.</u> Let ${}^nG^0_\alpha(x, y)$ be the resolvent density defined by 1.1) for the domain D_n. Since ${}^nG^0_\alpha(x, y)$ increases to $G^0_\alpha(x, y)$, we have only to show 2.5) together with the uniformity of the convergence.

Let us fix $x \in D$. We can assume that x is in D_1. Let us denote by superscript n that we are concerned with D_n instead of D; for instance, $\mathbb{H}^n_\alpha$ and $\mathbb{D}^n_\alpha$. It is clear that, if $m < n$, the restriction of the function of $\mathbb{H}^n_\alpha$ to D_m is an element of $\mathbb{H}^m_\alpha$.

If $m < n$, we have

$$\mathbb{D}^m_\alpha({}^nR^x_\alpha - {}^mR^x_\alpha, {}^nR^x_\alpha - {}^mR^x_\alpha)$$

$$= \mathbb{D}^m_\alpha({}^nR^x_\alpha, {}^nR^x_\alpha) - 2\,\mathbb{D}^m_\alpha({}^mR^x_\alpha, {}^nR^x_\alpha) + \mathbb{D}^m_\alpha({}^mR^x_\alpha, {}^mR^x_\alpha).$$

We can apply Lemma 2.3 to each term of the last expression. The first term is not greater than $\mathbb{D}^n_\alpha({}^nR^x_\alpha, {}^nR^x_\alpha) = 2\,{}^nR^x_\alpha(x)$. The second and the third terms are equal to $-4\,{}^nR^x_\alpha(x)$ and $2\,{}^mR^x_\alpha(x)$, respectively. Therefore, for each N, it holds that

2.6) $$0 \leqq \mathbb{D}^N_\alpha({}^nR^x_\alpha - {}^mR^x_\alpha, {}^nR^x_\alpha - {}^mR^x_\alpha) \leqq 2({}^mR^x_\alpha(x) - {}^nR^x_\alpha(x))$$

for any m and n such that $N \leqq m < n$. 2.6) implies that

${}^nR^x_\alpha(x)$ is non-increasing in n and since ${}^nR^x_\alpha(x) = \frac{1}{2}\mathbb{D}^n_\alpha({}^nR^x_\alpha, {}^nR^x_\alpha)$ is non-negative, ${}^nR^x_\alpha(x)$ converges. Thus, 2.6) and Lemma 2.1 show that ${}^nR^x_\alpha(y)$ converges to an α-harmonic function $\widetilde{R}^x_\alpha(y)$ on D uniformly on any compact subset of D, and for each N, the restriction of ${}^nR^x_\alpha$ to D_N converges to that of $\widetilde{R}^x_\alpha$ in the norm $\mathbb{D}^N_\alpha$.

Let us prove that $\widetilde{R}^x_\alpha(y) = R^x_\alpha(y)$, $y \in D$. Since R^x_α belongs to $\mathbb{H}_\alpha{}^n$, Lemma 2.3, 2) implies

$$\mathbb{D}^n_\alpha({}^nR^x_\alpha, {}^nR^x_\alpha) - 4\,{}^nR^x_\alpha(x) \leqq \mathbb{D}^n_\alpha(R^x, R^x) - 4R^x_\alpha(x).$$

Letting n tend to infinity, we have, for each N,

$$\mathbb{D}^N_\alpha(\widetilde{R}^x_\alpha, \widetilde{R}^x_\alpha) - 4\widetilde{R}^x_\alpha(x) \leqq \mathbb{D}_\alpha(R^x_\alpha, R^x_\alpha) - 4R^x_\alpha(x).$$

Let N tend to infinity, then

$$\mathbb{D}_\alpha(\widetilde{R}^x_\alpha, \widetilde{R}^x_\alpha) - 4\widetilde{R}^x_\alpha(x) \leqq \mathbb{D}_\alpha(R^x_\alpha, R^x_\alpha) - 4R^x_\alpha(x).$$

Thus, we see that $\widetilde{R}^x_\alpha \in \mathbb{H}_\alpha$ and that, by Lemma 2.3, 2), the inequality above is just the equality and $\widetilde{R}^x_\alpha(x) = R^x_\alpha(y)$, $y \in D$. The proof of Lemma 2.5 is complete.

We have seen in §1 (in the paragraph below the description of Theorem 1) that, if ∂D_n is of class $\mathbb{C}^3$, ${}^nG_\alpha(x, y)$ is nothing but the Laplace transform of the fundamental solution of the heat equation with the reflecting barrier boundary condition for the domain D_n and the latter is a transition density on D_n. Hence, we have

<u>Lemma 2.6.</u> Let $\{D_n\}$, $\{{}^nR_\alpha(x, y)\}$ and $\{{}^nG_\alpha(x, y)\}$ be those in Lemma 5. If $\{D_n\}$ is regular, then for each n, we have,

2.7) $\quad {}^nG_\alpha(x, y) \geqq 0, \quad \alpha > 0,\ x,y \in D_n,\ x \neq y.$

2.8) $\quad {}^nR_\alpha(x, y) \geqq 0, \quad \alpha > 0,\ x,y \in D_n.$

2.9) $\quad \alpha \int_{D_n} {}^nG_\alpha(x, y)dy \leqq 1, \quad \alpha > 0, \quad x \in D_n.$

2.10) $\quad {}^nG_\alpha(x, y) - {}^nG_\beta(x -y) + (\alpha - \beta) \int_{D_n} {}^nG_\alpha(x, z)\,{}^nG_\beta(z, y)dz = 0, \quad \alpha, \beta > 0,\ x,y \in D_n, \quad x \neq y.$

We note that 2.8) follows from 2.7).

Now, let us complete the proof of Theorem 1 by the following series of lemmas.

<u>Lemma 2.7.</u> $R_\alpha(x, y)$ is non-negative for $\alpha > 0$, $x, y \in D$ and $\alpha \int_D G_\alpha(x, y)dy \leqq 1$, for $\alpha > 0$, $x \in D$. $G_\alpha(x, y)$ is symmetric in $x, y \in D$ and continuous in (x, y) on $D \times D$ off the diagonal.

<u>Proof.</u> The first part of Lemma 2.7 is an immediate consequence of Lemma 2.5 and Lemma 2.6. It is well known that $G^0_\alpha(x, y)$ is symmetric in $x, y \in D$ and continuous in $(x, y) \in D \times D$ off the diagonal set. $R_\alpha(x, y)$ is symmetric because $\mathbb{D}_\alpha(R^x_\alpha, R^y_\alpha) = 2R^x_\alpha(y) = 2R^y_\alpha(x)$, $x, y \in D$.
We shall show that $R_\alpha(x, y)$ is continuous in $(x, y) \in D \times D$. Since $R_\alpha(x, y)$ is α-harmonic in x and in y, applying Lemma 2.1 for any $x, y \in D$ and for sufficiently small balls B_1 and B_2 containing x and y, respectively, we have
$R_\alpha(x, y) = \int_{\partial B_1} \int_{\partial B_2} h^{B_1}_\alpha(x, z)\, R_\alpha(z, z')\, h^{B_2}_\alpha(z, z')\sigma_1(dz)\sigma_2(dz')$, where $\sigma_1(dz)$ and $\sigma_2(dz')$ are the surface Lebesgue

measures of ∂B_1 and ∂B_2, respectively. While, $R_\alpha(z, z')$ being continuous in z' for each z, $\int_{\partial B_2} R_\alpha(z, z')\, \sigma_2(dz')$ is finite and α-harmonic in z. Thus,

$$\int_{\partial B_1} \int_{\partial B_2} R_\alpha(z, z')\, \sigma_1(dz)\, \sigma_2(dz') < +\infty .$$

Since R_α is non-negative, Lebesgue's convergence theorem implies continuity of $R_\alpha(x,y)$. The proof of the latter half of Lemma 2.7 is complete.

We will show (4) of Theorem 1.

Lemma 2.8. Let K_1 and K_2 be compact subsets of D such that K_1 and the closure of $D-K_2$ are disjoiht. Then, $\sup_{x \in K_1,\, y \in D-K_2} G_\alpha(x, y)$ is finite.

Proof. Without loss of generality, we can assume that $S = \partial(D - K_2) \cap D$ is sufficiently regular. Consider an regular exhaustion $\{D_n\}$ of D such that $D_1 \supset K_2$. Let x be fixed in K_1. For a fixed n, set $D' = D_n - K_2$ and $u(y) = {}^nG_\alpha(x, y)$, $y \in D' \cup \partial D'$. Since $\frac{\partial}{\partial n_y} u(y) = 0$, $y \in \partial D_n$, we see by Green's formula that $\mathbb{D}'_\alpha(u, v-u) = 0$ holds if $v \in \mathbb{C}^1(D' \cup \partial D')$ and $v = u$ on S.[1] Hence, the equality

$$2.11) \qquad \mathbb{D}'_\alpha(u, u) = \mathbb{D}'_\alpha(v, v) - \mathbb{D}'_\alpha(u - v, u - v)$$

is valid for each v belonging to $\tilde{\mathfrak{D}}_u = \{v;\ v$ is square summable on D', v has square summable weak-derivatives on D', $v \in \mathbb{C}(D' \cup S)$ and $v = u$ on $S\}$.[2] Set $\delta = \sup_{y \in S} u(y)$ and $u_1(y) =$

min $(u(y), \delta)$, $y \in D' \cup S$. Obviously, $\mathbb{D}'_\alpha(u, u) \geqq \mathbb{D}'_\alpha(u_1, u_1)$. But, since $u_1 \in \tilde{\mathcal{D}}_u$, 2.11) holds for $v = u_1$ and consequently $\tilde{u}_1(y) = u(y)$ on D'.

We have proved that, if $x \in K_1$ and $y \in D_n - K_2$, then ${}^nG_\alpha(x, y) \leqq \sup_{y \in S} {}^nG_\alpha(x, y)$. Letting n tend to infinity, we see by virtue of Lemma 2.5, $G_\alpha(x, y) \leqq \sup_{y \in S} G_\alpha(x, y)$, $x \in K_1$, $y \in D - K_2$. Thus,

$$\sup_{x \in K_1,\ y \in D - K_2} G_\alpha(x, y) \leqq \sup_{x \in K_1,\ y \in S} G_\alpha(x, y) .$$

The right hand side above is finite by Lemma 2.7.

Let us show (3) of Theorem 2.

Lemma 2.9. The operator G_α defined by 1.6) maps $\mathbb{B}(D)$ into $\mathbb{C}(D)$. Moreover, if $f \in \mathbb{C}(D)$, $\lim_{\alpha \to +\infty} \alpha G_\alpha f(x) = f(x)$, $x \in D$.

Proof. We note that G^0_α has the properties of Lemma 2.9.[3)] For $f \in \mathbb{B}(D)$, $R_\alpha f(x) = \int_D R_\alpha(x, y) f(y) dy$ is α-harmonic and bounded on account of Lemma 2.1 and Lemma 2.7. Moreover, we see by Lemma 2.1 that for any $x \in D$ and sufficiently small ball B containing x,

$$|\alpha R_\alpha f(x)| \leqq \int_{\partial B} h^B_\alpha(x, y)\, |\alpha R_\alpha f(y)|\, \sigma(dy)$$

$$\leqq \sup_{x \in D} |f(x)| \int_{\partial B} h^B_\alpha(x, y)\, \sigma(dy) \xrightarrow[\alpha \to +\infty]{} 0 .$$

The proof of Lemma 2.9 is complete.

The following lemmas are (2) and (5) of Theorem 1.

<u>Lemma 2.10.</u> $G_\alpha(x, y)$ is a conservative resolvent density on D. $R_\alpha(x, y)$ is strictly positive.

Proof. We must prove that $G_\alpha(x, y)$ satisfies the conditions G.1) $\sim$ G.4) stated in the beginning of §1 and the conservativity condition. G.1), G.2) and G.3) are already proved in Lemma 2.7.

<u>Proof of resolvent equation G.4).</u> Take a regular exhaustion $\{D_n\}$ of D. Let f and g be non-negative continuous functions on D with compact supports. Owing to 2.10) of Lemma 2.6, we have for sufficiently large n,

2.12) $$(f, {}^nG_\alpha g)_n - (f, {}^nG_\beta g)_n + (\alpha - \beta)({}^nG_\alpha f, {}^nG_\beta g)_n = 0,$$

where $(u, v)_n$ denotes the integral of $u \cdot v$ on D_n.

Note that $0 \leq {}^nG_\alpha f(x)\,{}^nG_\beta g(x) \leq \dfrac{1}{\alpha\beta} \sup\limits_{x\in D} f(x) \cdot \sup\limits_{x\in D} g(x)$

and that ${}^nG_\alpha g$ converges to $G_\alpha g$ on D (since, ${}^nG^0_\alpha g$ increases to $G^0_\alpha g$ and ${}^nR^x_\alpha(y)$ converges uniformly on any compact subset). Hence, we can delete both superscript and subscript n in 2.12). Owing to Lemma 2.8 and Lemma 2.9, the left hand side of G.4) is, for each $x \in D$, continuous in $y \in D - \{x\}$, and we can see that G.4) is valid.

<u>Proof of conservativity.</u> If we show that $R_\alpha 1 \in \mathbb{H}_\alpha$ and that

2.13) $$\mathbb{D}_\alpha(\alpha R_\alpha 1, v) = 2\alpha(1, v),$$

holds for all $v \in \mathbb{H}_\alpha$, then, we have, by (2) of Lemma 2.4, $1 - \alpha G^0_\alpha 1 = \alpha R_\alpha 1$ and $\alpha G_\alpha 1 = 1$.

Let $\{D_n\}$ be an exhaustion of D. Integrating $\mathbb{D}_\alpha(R^x_\alpha, R^y_\alpha) = 2R_\alpha(x, y)$ by $dxdy$ on $D_m \times D_n$, we obtain

$$2.14) \qquad \mathbb{D}_\alpha(R_\alpha \chi_{D_m}, R_\alpha \chi_{D_n}) = 2 \int_{D_m}\int_{D_n} R_\alpha(x, y)dy.$$

Here, we have used the Fubini theorem, which is possible, because, if $m \leqq n$,

$$\int_{D_m}\int_{D_n} dxdy \int_D \left|(\mathrm{grad}_z\, R^x_\alpha(z),\ \mathrm{grad}_z\, R^y_\alpha(z))\right| dz$$

$$\leqq \int_{D_n}\int_{D_n} \sqrt{\mathbb{D}_\alpha(R^x_\alpha, R^x_\alpha)}\ \sqrt{\mathbb{D}_\alpha(R^y_\alpha, R^y_\alpha)}\,dxdy$$

$$= \left(\int_{D_n} \sqrt{2R_\alpha(x, x)}\ dx\right)^2 \leqq 2\int_{D_n} R_\alpha(x, x)dx \times \text{Lebesgue}$$

measure of D_n, the integral in the last expression being finite by Lemma 2.7. In view of Lemma 2.7, $R_\alpha(x, y) \geqq 0$ and $\int_D\int_D R_\alpha(x, y)dxdy \leqq \frac{1}{\alpha} \times$ Lebesgue measure of D. Therefore, $R_\alpha \chi_{D_n}$ forms a Cauchy sequence in $\mathbb{H}_\alpha$ and, by Lemma 2.2, converges to $R_\alpha 1$ in $\mathbb{H}_\alpha$. We have $\mathbb{D}_\alpha(R_\alpha 1, R_\alpha 1) = 2(1, R_\alpha 1)$. In the same way, 2.13) is obtained. Strict positivity of $R_\alpha(x,y)$ follows from Lemma 2.4.

<u>Lemma 2.11.</u> There is a transition density $P(t, x, y)$ on D uniquely which satisfies the following conditions.

$$(1) \qquad G_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t}P(t,x,y)dt, \qquad \alpha > 0.$$

(2) For each $t > 0$, $f \in \mathbb{B}(D)$,

$\int_D P(t,x,y)f(y)dy$ is continuous in $(t,x) \in (0, +\infty) \times D$.

(3) $P(t,x,y)$ is symmetric in $x, y \in D$ and it is conservative.

(4) Set $\gamma(t,x,y) = P(t,x,y) - P^0(t,x,y)$, then

$$\frac{1}{t}\int_D \gamma(t,x,y)dy \xrightarrow[t \to 0]{} 0 \quad \text{uniformly in } x \text{ on}$$

any compact subset of D.

<u>Proof.</u> First of all, we will show the existence of a non-negative function $\gamma(t,x,y)$ continuous in $t > 0$, satisfying

$$2.15) \qquad R_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t}\gamma(t,x,y)dt , \qquad \alpha > 0,\ x,y \in D.$$

If $x \neq y$, $R_\alpha(x, y)$ is completely monotonic in $\alpha \in (0, +\infty)$. In fact, by the resolvent equations G. 4) for G_α and G_α^0, we have, if $x \neq y$,

$$2.16) \qquad (-1)^n \frac{d^n}{d\alpha^n} R_\alpha(x, y) = n!\,[G_\alpha^{[n+1]}(x,y) - (G_\alpha^0)^{[n+1]}(x, y)],$$

$n = 0, 1, 2, \ldots\ldots$ Here $G_\alpha^{[1]}(x, y) = G_\alpha(x, y)$ and $G_\alpha^{[n+1]}(x, y) = \int_D G_\alpha^{[n+1]}(x, z)G_\alpha(z, y)dz$, $n = 1,2,\cdots$. $(G_\alpha^0)^{[n]}$ is similarily defined. Evidently, the right hand side of 2. 16) is non-negative and, by Lemma 2.8, finite. Hence, $R_\alpha(x, y)$ is expressed by a measure on $[0, +\infty)$ as

$$2.17) \qquad R_\alpha(x, y) = \int_0^{+} e^{-\alpha s}\gamma(ds, x, y),\ x \neq y,\ \alpha > 0.$$

Take a ball B with closure contained in D. Since $R_\alpha(x, y)$ is α-harmonic in x, we see, by Lemma 2.1, for any $x \in B$ and any $y \in D$,

2. 18) $R_\alpha(x, y) = \int_{\partial B} h_\alpha^B(x, z) R_\alpha(z, y)\, \sigma(dz).$

Note that $h_\alpha^B(x, z)$ is written in the form

2. 19) $h_\alpha^B(x, z) = \int_0^{+\infty} e^{-\alpha t}\, h^B(t,x,z)\,dt, \quad x \in B, \ z \in \partial B,$

where $h^B(t, x, z) = \frac{1}{2} \frac{\partial}{\partial n_z} P_B^0(t,x,z)$, P_B^0 being the transition density P^0 for B. Let us put, for $t > 0$, $x \in B'$ and $y \in D$,

2. 20) $\gamma(t,x,y) = \int_{\partial B} \int_0^t h^B(t-s, x, z)\gamma(ds, z, y)\,dt\, \sigma(dz).$

Owing to 2. 17), 2.18) and 2. 19), $\gamma(t,x,y)$ of 2. 20) satisfies the desired equation 2. 15). On the other hand, for any ball B' such as $B' \cup \partial B' \subset B$, the obvious idenity

$$h^B(t,x,z) = \int_{\partial B'} \int_0^t h^{B'}(t-s, x, z')h^B(s, z', z)\,ds\ \ \sigma'(dz'),$$

$x \in B'$, $z \in \partial B$, leads us to

2. 21) $\gamma(t,x,y) = \int_{\partial B'} \int_0^t h^{B'}(t-s, x, z')\,\gamma(s, z', y)\,ds\,\sigma'(dz'),$

$t > 0$, $x \in B'$, $y \in D$, which implies the continuity of $\gamma(t, x, y)$ in $(t,x) \in (0, +\infty) \times B'$.

Here, we have used the following estimate which is a comsequence of 2. 17), 2. 20) and Lemma 2, 8.

2. 22) $\sup_{0 < t \leqq T,\ x \in B',\ y \in D} \gamma(t,x,y) \leqq C \cdot e^{T} \cdot \sup_{z \in \partial B,\ y \in D} R_1(z, y)$

$< +\infty$, where T is an arbitrary positive number and C is a constant determined by T, B and B'. Hence, we see that, for any x and y in D, $\gamma(t,x,y)$ defined by 2. 20) is independent of the ball B such that $x \in B$ and $B \cup \partial B \subset D$, because it satisfies 2. 15) and it is continuous in t. It is symmetric in x, y because of the symmetricity of $R_\alpha(x, y)$ (Lemma 2. 7). Henceforce, it is continuous in y and 2. 21) and 2. 22) imply its continuity in $(t,x,y) \in (0, +\infty) \times D \times D$. In view of 2. 22), we see that $\int_D \gamma(t,x,y)f(y)dy$ is continuous in $(t,x) \in (0, +\infty) \times D$ for each $f \in \mathbb{B}(D)$.

Now put, for $t > 0$, $x, y \in D$,

2. 23) $$P(t,x,y) = P^0(t,x,y) + \gamma(t,x,y).$$

Then, $P(t,x,y)$ is continuous in $(t,x,y) \in (0, +\infty) \times D \times D$ and satisfies (1), (2) and the first half of (3) of Lemma 2. 11. Particularily, $\int_D P(t,x,y)dy$ is continuous in t, and so, the conservativity of $P(t,x,y)$ follows from that of $G_\alpha(x, y)$. For each $x, y \in D$, $P(t+s, x, y)$ and $\int_D P(t, x, z)P(s, z, y)dz$ are continuous in $(t, s) \in (0, +\infty) \times (0, +\infty)$, and so, they are identical by virtue of G. 4) for $G_\alpha(x, y)$. Thus, $P(t, x, y)$ is a transition density. (4) of Lemma 11 follows from 2. 21) and the inequality $\int_D \gamma(t, x, y)dy \leqq 1$, $t > 0$, $x \in D$.

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