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43巻 1968年 1-12

INVARIANT IMBEDDING AND MULTIPLE SCATTERING PROCESSES

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I. Introduction : In problems of solving classical equations of mathematical physics two types of difficulties are inseparably associated, i.e., difficulties of analytical character and of computational nature, because the classical methods reduce problems to the solution of systems of linear equations. If we use the invariant imbedding in a systematic fashion, we shall try to reduce problems to the iteration of non-linear transformation. It will permit us to avoid such untractable matters. Such problems will be encountered in the fields of radiative transfer, neutron diffusion, rarefied gas dynamics , random walk and wave propagation. Particularly, this approach is powerful to treat with the interaction of photons and gas particles in stochastic media (e.g., shocks, turbulence , convection, and others), allowing for the magnetic field.

Then, the characteristic of the invariance principles consists in the transformation of the two-points boundary value problems to the initial value problems. In the field of radiative transfer Chandrasekhar has developed elegantly the theory of invariance principles due to originally Ambarzumian. Whereas the angular distribution of emergent radiation is evaluated with the aid of this technique, however, the determination of the internal radiation also is reduced to the above problem by the use of the initial value method(cf. Reference ,Books (4)).

A summary of some recent developments of this approach is presented by Bailey and Wing (cf.J.Math.Analy. and Appl.,) . The generalized Riccati transformation provides another manipulative derivation of the result (cf.Rybicki and Usher,Ap.J.146 ,871(1966)).

2. The rod model

2.1 The stationary multiple scattering processes

2.1.1 The transport equation : Consider an inhomogeneous rod extending from z=a to z=b. The interval is thought of as being rod material capable of transporting particles, whereas these particles can move only to the right or to the left (See Fig.1).

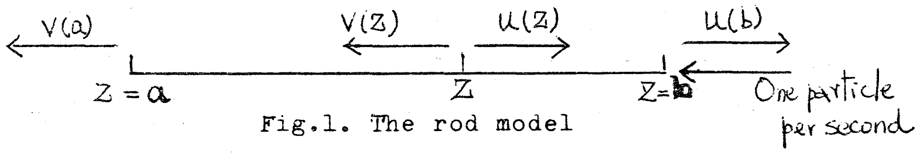

Fig.1. The rod model

Let

u(z)= expected number density of particles at z and moving to the right,

v(z)=expected number density of particles at z and moving to the left.

The transport equation appropriate to this case is written in the form

(1) $du/dz = \alpha(z)u(z) + \beta(z)v(z), \quad dv/dz = \beta(z)u(z) + \alpha(z)v(z),$

where

(2) $\alpha(z)= \sigma(z)(p(z) - 1), \quad \beta(z)= \sigma(z)q(z) \quad (0 \leqslant z \leqslant x).$

In the above $\sigma(z)$ is the cross-section, p(z) represents an expected total of particles moving in the direction of the original particle at z in each collision, and q(z) arises going in the direction opposed.

Eq.(1) should be solved subject to the boundary conditions

(3) $u(a)=0, \qquad v(b)=1.$

2.1.2 Invariant imbedding equations : Let

R(b,a)=expected number of particles emerging to the right each second at b due to a flux of one particle per second injected at z=b,

T(b,a)= expected number of particles emerging to the left each second at z=a due to a flux of one particle per second injected at z=b.

We shall call R and T functions the reflection and transmission functions, respectively. Furthermore, R(a,b) and T(a,b) functions represent the above global quantities when one particle is incident on z=a, because of the polarity of an inhomogeneous properties of the rod.

It is evident that

(4) $R(b,a)=u(b), \qquad T(b,a)=v(a).$

Add an infinitesimal length Δ to the rod at z=b. As the incident flux passes through the interval $(b,b+\Delta)$, some of the particles cause scattering and others pass through unaffected to become incident upon (a,b). When a scattering occurs in Δ, a scattered particle emerges at $b+\Delta$, whereas the other becomes a part of the incident flux at b. Some of particles reflected from (a,b) may cause scattering while passing through $(b,b+\Delta)$. The products of this scattering yield a contribution to the reflected flux at $b+\Delta$ and furnish another source of particles incident upon (a,b). By taking Δ to be an infinitesimal, all other events have a probability of occurence of order Δ^2 or higher, apart from those taken account of above.

Adding up the various effects and their associated probabilities, we get the equation

(5) $R(b+\Delta,a)= \beta(b)\Delta + \beta(b)R(b,a)\Delta R(b,a) + 2\alpha(b)R(b,a)\Delta + R(b,a)+\mathcal{O}(\Delta).$

If we let $\Delta\to 0$, we derive a Riccati type of first-order differential equation. This type of quadratically non-linear equation is characteristic of the equation given by invariant imbedding technique. It is provided by

(6) $dR(b,a)/db = \beta(b) + 2\alpha(b)R(b,a) + \beta(b)R(b,a)R(b,a),$

together with the boundary condition $R(b,b)=0$.

Eq.(6) gives directly the value of the reflected intensity from the right end due to a unit input without the necessity of finding the internal flux of the rod.

Similarly, we find the functional equations for R- and T-functions as below:

(7) $dT(b,a)/db = \alpha(b)T(b,a) + \beta(b)T(b,a)R(b,a)$,

(8) $-dR(b,a)/da = \beta(a)T(b,a)T(a,b)$,

(9) $-dT(b,a)/da = \alpha(a)T(b,a) + \beta(a)T(b,a)R(a,b)$,

together with the boundary condition $T(b,b)=1$.

It is of interest to mention that eqs.(6)-(9) consist of half of all differential equations for R- and T-functions.

2.2 Time-dependent multiple scattering processes

2.2.1 The transfer equation

Consider a one-dimensional homogeneous medium of optical thickness $z=x$, illuminated by radiation of time-dependent specific intensity $w(t)$ incident on the right-hand boundary $z=x$.(See Fig.2). Scattering of light in either direction is assumed equally probable.

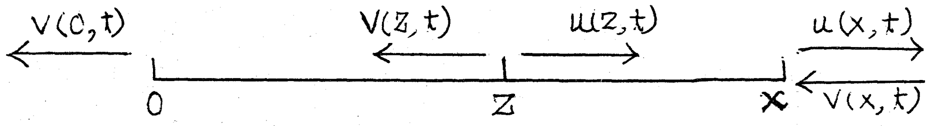

Fig.2. A time-dependent transport process

Let $u(z,t)$ and $v(z,t)$ denote respectively the specific intensities of radiation at level z at time t, directed towards the boundaries $z=x$ and $z=0$.

The equation of transfer appropriate to the case of relaxation is expressed in the form

(10) $\partial u/\partial z + t_2\, \partial u/\partial t = -u + B(z,t)$,

(11) $-\partial v/\partial z + t_2\, \partial v/\partial t = -v + B(z,t)$,

where t_2 is the mean free time, and the source function B is given by

(12) $B(z,t) = \frac{a}{2}\int_{-\infty}^{t}\left\{u(z,t') + v(z,t')\right\}\exp\left\{-(t-t')/t_1\right\} dt'/t_1$.

In eq.(12) a is the albedo for single scattering,i.e. the probability of photon survival, and t_1 is the duration of temporal capture, which corresponds to the mean molecular interaction time in kinetic theory of dilute gases.

Eqs.(10) and (11) should be solved subject to the boundary and initial conditions

(13) $u(0,t)=0, \quad v(x,t)=w(t),$

(14) $u(z,0)=0, \quad v(z,0)=0 \quad (0 \leq z < x).$

The quantities $u(x,t)$ and $v(0,t)$ are called respectively the reflected and transmitted intensities, and $v(x,t)$ is the intensity of radiation incident on the boundary $z=x$ at time t.

2.2.2 <u>The reflected intensity</u> : Let $R(x,t)$ denote the reflection function. Define

(15) $$u(x,t) = \int_{-\infty}^{t} R(x,t-t')\hat{v}(x,t')dt',$$

where $\hat{v}(x,t)$ is given by eq.(13).

We shall seek an integral equation for the reflection function $R(x,t)$ making use of the invariant imbedding technique in perturbation scheme.

Imbedding the rod of optical thickness x in position and time, we get

(16) $$u(x+\Delta,t+t_2\Delta)= u(x,t) +\Delta\left\{ -u(x,t)+B(x,t)\right\} + \mathcal{O}(\Delta),$$

where $\mathcal{O}(\Delta)$ is of the order of magnitude of the infinitesimal Δ^2.

From eq.(11) we obtain

(17) $$\hat{v}(x,t)=v(x+\Delta,t-t_2\Delta)-v(x,t)+\Delta +B(x,t)\Delta + \mathcal{O}(\Delta).$$

By initial condition (14), eq.(17) becomes

(18) $$\hat{v}(x,t)=w(t-t_2\Delta)-w(t)\Delta +\frac{a}{2}\Delta\int_{-\infty}^{t}u(x,t')e^{-(t-t')/t_1}dt'/t_1 + \frac{a}{2}\Delta\int_{-\infty}^{t}w(t')e^{-(t-t')/t_1}dt'/t_1 + \mathcal{O}(\Delta).$$

On making use of eqs.(11) and (18), we obtain

(19) $$u(x,t)=\int_{-\infty}^{t}R(x,t-t')w(t'-t_2\Delta)dt' - \Delta\int_{-\infty}^{t}R(x,t-t')w(t')dt' + \frac{a}{2}\Delta\int_{-\infty}^{t}R(x,t-t')dt'\int_{-\infty}^{t'}u(x,t'')e^{-(t'-t'')/t_1}dt''/t_1 + \frac{a}{2}\Delta\int_{-\infty}^{t}R(x,t-t')dt'\int_{-\infty}^{t'}w(t'')e^{-(t'-t'')/t_1}dt''/t_1 + \mathcal{O}(\Delta).$$

On the other hand, we have

(20) $$u(x+\Delta,t+t_2\Delta)= \int_{-\infty}^{t+t_2\Delta} R(x+\Delta,t+t_2\Delta-t')w(t')dt'.$$

With the aid of eqs.(19) and (20), eq.(16) becomes

$$(21)\quad \int_{-\infty}^{t+t_2\Delta} R(x+\Delta, t+t_2\Delta - t')w(t')dt' = \int_{-\infty}^{t} R(x,t-t')w(t'-t_2\Delta)dt' - 2\Delta\int_{-\infty}^{t} R(x,t-t')w(t')dt' + \frac{a}{2}\Delta\int_{-\infty}^{t} R(x,t-t')dt'\int_{-\infty}^{t'} u(x,t'')e^{-(t'-t'')/t_1} \cdot dt''/t_1 + \frac{a}{2}\Delta\int_{-\infty}^{t} R(x,t-t')dt'\int_{-\infty}^{t'} w(t'')e^{-(t'-t'')/t_1}dt''/t_1 + + \frac{a}{2}\Delta\int_{-\infty}^{t} u(x,t')e^{-(t-t')/t_1}dt'/t_1 + \frac{a}{2}\Delta\int_{-\infty}^{t} w(t')e^{-(t-t')/t_1}dt'/t_1 + O(\Delta).$$

First, we shall consider the case of Dirac delta time-dependent function, $w(t)=f\,\delta(t)$, where F is a constant and δ is the Dirac delta function. The substitution of w(t) into eq.(21) provides (after letting $\Delta\to 0$)

$$(22)\quad \partial R(x,t)/\partial x + 2t_2\partial R/\partial t + 2R = a\Big\{ e^{-t/t_1}/2t_1 + \int_{-\infty}^{t} R(x,t-t') \cdot e^{-t'/t_1}dt'/t_1 + \frac{1}{2}\int_{-\infty}^{t} dt'\int_{-\infty}^{t'} R(x,t-t')R(x,t'')e^{-(t'-t'')/t_1}dt''/t_1\Big\}.$$

The condition imposed on R are

$$(23)\quad R(x,0)=0 \text{ for } 0\geqslant t;\quad R(0,t)=0 \text{ for } t\geqslant 0.$$

Eq.(21) is the requisite integral equation governing the reflection function.

Furthermore, we consider a fluorescence problem for which the diffusely reflected light decreases for a long time after the sudden switching-off of the external radiation field incident on the boundary x, assuming no emitting source within the medium.

Writing

$$(24)\quad v(x,t) = FH^*(t),$$

where

$$(25)\quad H^*(t) = \begin{cases} 0 & t>0 \\ 1 & t<0, \end{cases}$$

We find the requisite intensity u(x,t), reflected by the end z=x at time t, is given by

$$(26)\quad u(x,t) = F\int_{-\infty}^{t} R(x,t-t')H^*(t')dt' = F\int_{t}^{\infty} R(x,y)dy,$$

where R-function is given by eq.(22).

2.2.3 The transmitted intensity

Let T(x,t) denote the transmission function. Then, we have

$$(27)\quad v(0,t) = \int_{-\infty}^{t^*} T(x,t-t')v(x,t')dt',$$

where $t^* = t-xt_2$.

We inquire into an integral equation for $T(x,t)$. In a manner similar to that used in a previous section, we have

(28) $v(0,t+t_2\Delta)=v(0,t) + \mathcal{O}(\Delta)$.

From eq.(27) we obtain

(29) $v(0,t+t_2\Delta)=\int_{-\infty}^{t^*} T(x+\Delta,t+t_2\Delta-t')w(t'-t_2\Delta)dt'$.

On the other hand, using eq.(18), we see that the transmitted intensity $v(0,t)$ is provided by

(30) $$v(0,t)=\int_{-\infty}^{t^*}T(x,t-t')\hat{v}(x,t')dt'=\int_{-\infty}^{t^*}T(x,t-t')w(t'-t_2\Delta)dt' - \Delta\int_{-\infty}^{t^*}T(x,t-t')w(t')dt' + \frac{a}{2}\Delta\int_{-\infty}^{t^*}T(x,t-t')dt'\int_{-\infty}^{t'}u(x,t'')e^{-(t'-t'')/t_1}dt''/t_1 + \frac{a}{2}\Delta\int_{-\infty}^{t^*}T(x,t-t')dt'\int_{-\infty}^{t'}w(t'')e^{-(t'-t'')/t_1}dt''/t_1 + \mathcal{O}(\Delta).$$

The, recalling eqs.(14),(29), and (30), we find that eq.(28) becomes

(31) $$\int_{-\infty}^{t^*}T(x+\Delta,t+t_2\Delta-t')w(t'-t_2\Delta)dt' = \int_{-\infty}^{t^*}T(x,t-t')w(t'-t_2\Delta)dt' - \Delta\int_{-\infty}^{t^*}T(x,t-t')w(t')dt' + \frac{a}{2}\Delta\int_{-\infty}^{t^*}T(x,t-t')dt'\int_{-\infty}^{t'}u(x,t'')e^{-(t'-t'')/t_1}dt''/t_1 + \frac{a}{2}\Delta\int_{-\infty}^{t^*}T(x,t-t')dt'\int_{-\infty}^{t'}w(t'')e^{-(t'-t'')/t_1}dt''/t_1 + \mathcal{O}(\Delta).$$

Inserting $w(t)=F\delta(t)$ into eq.(31) and letting $\Delta\to 0$, we have

(32) $$\partial T(x,t)/\partial x + t_2\,\partial T/\partial t + T = \frac{a}{2}\Big\{\int_{-\infty}^{t^*}T(x,t-t')e^{-t'/t_1}dt'/t_1 + \int_{-\infty}^{t^*}dt'\int_{-\infty}^{t'}T(x,t-t')R(x,t'')e^{-(t'-t'')/t_1}dt''/t_1\Big\},$$

along with the boundary and initial conditions

(33) $T(x,t)=0$ $(x>0,\ 0\geq t$ or $t<xt_2)$, $T(0,t)=F\delta(t)$ $(t\geq 0)$.

Eq.(32) is the requisite integral equation for the T-function in the case of Dirac delta-function time-dependence.

Furthermore, consider the same quenching fluorescence problem as that treated in the preceding section.

Under the incident intensity $v(x,t)$, given by eq.(13), the required intensity transmitted from the boundary $z=0$ at time t is provided by

(34) $v(0,t)=F\int_{-\infty}^{t^*}T(x,t-t')H^*(t')dt'=F\int_{t}^{\infty}T(x,y)dy$,

where $T(x,y)$ satisfies eq.(32).

3. The slab model

3.1 Stationary multiple scattering processes

3.1.1 The equation of transfer

Let a parallel beam of radiation of net flux πF per unit area normal to the direction of the propagation be incident on the upper surface z=x of the atmosphere at polar angle $\cos^{-1}\mu$ with the inwards normal and an azimuth φ $(0<\mu\leqslant 1, 0\leqslant\varphi\leqslant 2\pi)$. Consider an inhomogeneous plane-parallel atmosphere of finite optical thickness x with anisotropic scattering and an internal source distribution B, whose bottom surface reflects radiation isotropically. The optical altitude is measured from the bottom.

Let the intensity of radiation at altitude z directed towards the top surface z=x be denoted by $I_+(z,\Omega,x)$, where Ω stands for $(0<\mu\leqslant 1, 0\leqslant\varphi\leqslant 2\pi)$, and let the intensity of radiation directed towards the bottom be denoted by $I_-(z,\Omega,x)$. The direction of the beam is specified by its direction $\cos^{-1}\mu$ with respect to the outward normal to the atmosphere at z=x. The albedo for single scattering λ depends upon z, where $0\leqslant z\leqslant x$.

We shall determine the angular distribution of diffusely reflected light emerging from the top of the atmosphere, i.e. the solution of the Chandrasekhar planetary problem with thermal emission.

The equation of transfer is written in the form

$$\mu dI(z,\Omega,x)/dz + I = \lambda(z)\frac{1}{4\pi}\int \gamma(z,\Omega,\Omega')I(z,\Omega',x)d\Omega' + B(z,\Omega) + \lambda(z)Fe^{-(x-z)/\mu_0}\gamma(z,\Omega,\Omega_0)/4, \tag{35}$$

where the phase function $\gamma(z,\Omega,\Omega_0)$ is normalized to 4π on the unit sphere, and $B(z,\Omega)$ represents the internal source.

Eq.(35) should be solved subject to the boundary conditions

$$I_-(x,\Omega,x)=0, \tag{36}$$

$$I_+(0,\Omega,x)=\frac{A}{\pi}\int_0^1\int_0^{2\pi} I_-(0,\Omega',x)\mu' d\mu' d\varphi' + FA\mu_0 e^{-x/\mu_0}, \tag{37}$$

where A is a constant.

3.1.2 The reflected intensity

Let the principle of invariant imbedding be

(38) $I_+(z,\Omega,x) = \hat{I}_+(z,\Omega,z) + \int_- S(z,\Omega,\Omega')I_-(z,\Omega',x)d\Omega'/4\pi\mu$,

where S represents the scattering function and the subscript on the integral indicates the integration over negative values of μ only. In eq.(38) $\hat{I}_+(z,\Omega,z)$ represents the intensity of radiation emitted at level z if there is no layer from x to z , and $I_-(z,\Omega,x)$ represents the intensity of radiation diffusely reflected at level z directed towards the bottom. The downward directed radiation consists of the diffuse radiation field and the radiation field due to the emitting source $\mathcal{B}$.

In the limit z=x we have

(39) $I_+(x,\Omega,x)=\hat{I}_+(x,\Omega,x)=I_+^e(x,\Omega,x) + FS(x;\Omega,\Omega_o)/4\mu$.

On differentiating eq.(38), with respect to z, passing to the limit z=x, and making use of the above relation, we get

(40) $$dI_+(x,\Omega,x)/dx + I_+(x,\Omega,x)/\mu = \frac{1}{\mu}\Big\{ B(x,\Omega) + \lambda(x)F\Upsilon(x,\Omega,\Omega_o)/4 + \lambda(x)\int_+ \Upsilon(x,\Omega,\Omega')I_+(x,\Omega',x)d\Omega'/4\pi \Big\} + (4\pi\mu)^{-1}\int S(x;\Omega,\Omega')\Big\{ B(x,\Omega') + \lambda(x)\int_+ \Upsilon(x,\Omega',\Omega'')I_+(x,\Omega'',x)d\Omega''/4\pi + \frac{\lambda(x)}{4}F\Upsilon(x,\Omega',\Omega_o)\Big\} d\mu' d\varphi'/\mu' - FS(x;\Omega,\Omega_o)/4\mu\mu_o .$$

It is the requisite invariant imbedding equation, whose boundary condition is

(41) . $I_+(0,\Omega,0)=FA\mu_o \quad (0<\mu_o\leq 1)$.

The scattering function S is governed by

(42) $$\partial S(x;\Omega,\Omega)/\partial x + (1/\mu + 1/\mu_o)S = \lambda(x)\Big\{ \Upsilon(x;\Omega,\Omega_o) + \int_+ \Upsilon(x,\Omega,\Omega')S(x;\Omega',\Omega_o)d\Omega'/4\pi\mu' + \int_- S(x;\Omega,\Omega'')\Upsilon(x,\Omega'',\Omega_o)d\Omega''/4\pi\mu'' + (16\pi^2)^{-1}\int_-\int_+ S(x;\Omega,\Omega')\Upsilon(x,\Omega',\Omega'')S(x;\Omega'',\Omega_o)d\Omega' d\Omega''/\mu'\mu'' \Big\}.$$

In the absence of an internal source, putting

(43) $I_+(x,\Omega,x)=FS(x;\Omega,\Omega_o)/4\mu$,

from eq.(40) we get eq.(42). The boundary condition is given by

(44) $S(0;\Omega,\Omega_c)= 4A\mu\mu_o$.

3.2 <u>Time-dependent multiple scattering processes</u>

3.2.1 The equation of transfer and invariant imbedding equation

Consider the time-dependent diffuse reflection of parallel rays by a homogeneous, non-emitting and isotropically scattering slab. We shall determine the time history of radiation which is diffusely reflected from a slab as a result of a constant incident flux starting at time zero(see Fig . 3).

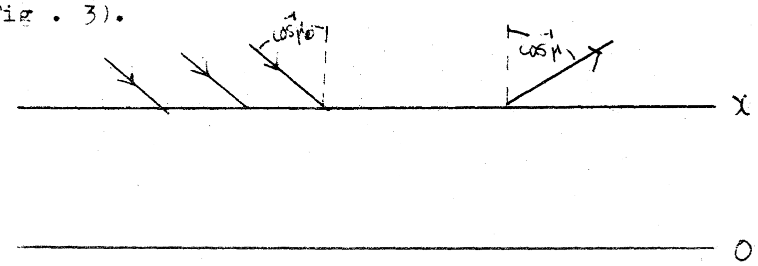

Fig.3 Incident and reflected beam for a slab of thickness x

The equation of transfer is

(45) $$\mu \partial I(z,\mu,t)/\partial z + (1/c)\partial I/\partial t + \sigma I = (\lambda\sigma/2)\int_{-1}^{+1} I(z,\mu',t')d\mu',$$

where c is the velocity of light, σ is the volume attenuation coefficient, and λ is the albedo for single scattering, together with the boundary and initial conditions

(46) $$I(0,+\mu,t)=0, \quad \mu>0; \quad I(x,-\mu,t)=\pi H(t)\,\delta(\mu-\mu_0), \quad \mu>0.$$

In a manner similar to the stationary case, we find that an invariant imbedding equation for $\bar{S}$-function is given by

(47) $$\partial\bar{S}(x,t;\mu,\mu_0)/\partial x + (1/\mu+1/\mu_0)(\partial/\partial t+1)\bar{S} = \lambda\Big\{H(t)/4\mu + (1/2)\int_0^1 \bar{S}(x,t;\mu,\mu')d\mu'/\mu' + (1/2\mu)\int_0^1 \bar{S}(x,t;\mu',\mu_0)d\mu' + \int_0^t dt' \int_0^1 \bar{S}(x,t';\mu',\mu_0)d\mu' \int_0^1 \partial\bar{S}(x,t-t';\mu,\mu'')/\partial t\, d\mu''/\mu''\Big\},$$

together with the initial condition

(48) $$\bar{S}(x,0;\mu,\mu_0)=0.$$

In eqs.(46) and (47) H is the Heaviside unit step function

(49) $$H(t) = 0, \quad t<0, \qquad = 1, \quad t>0.$$

In the above S is the required solution of the non-linear integro-differential equation of convolution type (see figures which show the general way in which the reflected intensities build up to their limiting values).

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