

Characteristic classes for spherical fiber spaces.

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1. Statement of results. Let $SF = SG = \lim_n SG(n)$, $SG(n) = \{ f : S^n \rightarrow S^n, \text{degree } 1 \}$, B_{SF} be the classifying space of SF . Our purpose is to determine $H_*(B_{SF}, Z_p)$ as a Hopf-algebra over Z_p , where p is an odd prime number. Coefficient is always Z_p , and we omit it later. Let $Q_0(S^0) = \varinjlim_n \Omega_0^n S^n$. Then $Q_0(S^0)$ has the same homotopy type of SF . Let $i : Q_0(S^0) \rightarrow SF$ be the homotopy equivalence. Dyer-Lashof determined $H_*(Q_0(S^0))$ as a algebra over Z_p . $H_*(Q_0(S^0))$ is a free commutative algebra generated by x_J , $J \in H$, where $H = \{ J = (\varepsilon_1, j_1, \dots, \varepsilon_r, j_r) \}$ satisfies the following properties: 1) $r \geq 1$ 2) $j_i \equiv 0, (p-1)$ 3) $j_r \equiv 0, (2(p-1))$ 4) $(p-1) \leq j_1 \leq j_2 \leq \dots \leq j_r$ 5) $\varepsilon_i = 0$ or 1 6) if $\varepsilon_{i+1} = 0$ then $j_i/p-1$ and $j_{i+1}/p-1$ are even parity, if $\varepsilon_{i+1} = 1$ then $j_i/p-1$ and $j_{i+1}/p-1$ are odd parity. There is a continuous map $h_0 : L_p \rightarrow Q_0(S^0)$, and $x_j \equiv h_{0*}(e_{2j(p-1)})$, where $e_i \in H_i(L_p)$ is a generator, and $x_I \equiv x^{(\varepsilon_1, j_1, \dots, \varepsilon_r, j_r)} \equiv \beta_p^{\varepsilon_1} Q_{j_1} \dots \beta_p^{\varepsilon_{r-1}} Q_{j_{r-1}} \beta_p^{\varepsilon_r} x_{j_r}$, where Q_j is the extended power operation defined by Dyer-Lashof. We identify $H_*(Q_0(S^0))$ and $H_*(SF)$ by i_* as a Z_p -module and we denote $\tilde{x} = i_*(x)$, if $x \in H_*(Q_0(S^0))$.

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Theorem I. $H_*(SF)$ is a free commutative algebra generated by \tilde{x}_j ; $J \in H$. Even though i_* is not a ring homomorphism.

Let H_1 be the subset of H consisting of $J = (\varepsilon_1, j_1, \dots, \varepsilon_r, j_r)$, such that $j_1 \neq p-1$, and $r \geq 2$. Let $H_2 = \{ (\varepsilon, p-1, 1, j) \} \subseteq H$. And let $H_1^+ = \{ J \in H_1, \deg(x_j) = \text{even} \}$, $H_1^- = \{ J \in H_1, \deg(x_j) = \text{odd} \}$ $i = 1, 2, \dots$. Let $j : B_{SO} \rightarrow B_{SF}$ be the inclusion map. Then by Peterson-Toda, $H_*(B_{SO})/\ker j^* \cong Z_p [z_1, z_2, \dots]$, where $\deg(z_j) = 2j(p-1)$, and $\Delta(z_j) = \sum_{j_1+j_2=j} z_{j_1} \otimes z_{j_2}$, $z_0 = 1$. Let $\tilde{z}_j = j_*(z_j) \in H(B_{SF})$.

Theorem II. $H_*(B_{SF}) = Z_p [\tilde{z}_1, \tilde{z}_2, \dots] \otimes \wedge (\sigma \tilde{x}_1, \sigma \tilde{x}_2, \dots) \otimes C_*$. C_* is a free commutative algebra generated by \tilde{x}_j , $J \in H_1 \cup H_2$. $\sigma : H_*(SF) \rightarrow H_*(B_{SF})$ is suspension. $\sigma \tilde{x}_j, \sigma \tilde{x}_j$ are primitive elements, and $\Delta(\tilde{z}_j) = \sum_{j_1+j_2=j} \tilde{z}_{j_1} \otimes \tilde{z}_{j_2}$.

$$H^*(B_{SF}) = Z_p [q_1, q_2, \dots] \otimes \wedge (\Delta q_1, \Delta q_2, \dots) \otimes C. \quad C = \bigotimes_{J \in H_1^- \cup H_2^+}$$

$$\wedge ((\sigma \tilde{x}_1)^*) \otimes \bigotimes_{J \in H_1^- \cup H_2^-} \Gamma_p [(\sigma \tilde{x}_J)^*], \text{ where } ()^* \text{ denote dual elements.}$$

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2. H-structures on $Q_0(S^0)$. Let $SF(n) = \{ f : (S^n, *) \rightarrow (S^n, *) \}$, degree 1. Then, $SG(n)$, and $SF(n)$ become H-spaces by composition of maps.

Let $SF(n) \times SF(n) \xrightarrow{\wedge} SF(2n)$, $SG(n) \times SG(n) \xrightarrow{*} SG(2n)$ be the map defined by reduced join and join respectively, then these three maps \wedge , $*$, are homotopic in the stable range. Let $i_n : \Omega_0^n S^n \rightarrow \Omega_{-1}^n S^n = SF(n)$ be the map defined by $i_n(\mathcal{L}) = (i_n \vee \mathcal{L})$, and $i : Q_0 S^0 \rightarrow SF$ be the limit of i_n .

Proposition 2-1. The following deagram is homotopy commutative.

$$\begin{array}{ccccc}
 Q_0 S^0 \times Q_0 S^0 & \xrightarrow{i \times i} & SF \times SF & \xrightarrow{\wedge} & SF & \xleftarrow{i} & Q_0 S^0 \\
 \downarrow \Delta \times \Delta & & & & & & \uparrow V \\
 (Q_0 S^0 \times Q_0 S^0) \times (Q_0 S^0 \times Q_0 S^0) & \xrightarrow{id \times T \times id} & (Q_0 S^0 \times Q_0 S^0) \times (Q_0 S^0 \times Q_0 S^0) & \xrightarrow{V \times \wedge} & Q_0 S^0 \times Q_0 S^0 & &
 \end{array}$$

where $V : Q_0 S^0 \times Q_0 S^0 \rightarrow Q_0 S^0$ be loop multiplication, and $\wedge : Q_0 S^0 \times Q_0 S^0 \rightarrow Q_0 S^0$ be the map defined by reduced join.

If K is a CW-complex, we put $Q(K) = \varinjlim_n \Omega^n S^n K$. $\tilde{e} : W \times_{\pi_p} Q(K)^p \rightarrow Q(K)$ be the map defined by Dyer-Lashof. Let $Q(K) \times Q(L) \rightarrow Q(K \wedge L)$ be the map defined by reduced join

Proposition 2-2. The following deagram is homotopy commutative.

$$\begin{array}{ccccc}
 Q(K) \times (W \times_{\pi_p} Q(L)^p) & \xrightarrow{id \times \tilde{e}} & Q(K) \times Q(L) & \xrightarrow{\wedge} & Q(K \wedge L) \\
 \downarrow & & & & \uparrow \tilde{e} \\
 W \times_{\pi_p} (Q(K) \times Q(L)^p) & \xrightarrow{id \times (\Delta \times id)} & W \times_{\pi_p} (Q(K) \times Q(L))^p & \xrightarrow{id \times (\wedge)^p} & W \times_{\pi_p} Q(K \wedge L)^p
 \end{array}$$

Let $h : L_p = W/\pi_p \rightarrow Q(S^0) = \varinjlim_n \Omega^n S^n$ be the map defined by

$$\begin{aligned}
 h : L_p = W/\pi_p &\longrightarrow W \xrightarrow{\pi_p} W \xrightarrow{\pi_p} Q(S^0) \xrightarrow{\theta} Q(S^0), \quad w \in Q_1(S^0), \\
 h_0 : L_p &\xrightarrow{h} Q_p(S^0) \xrightarrow{(-p, id)} Q_0(S^0).
 \end{aligned}$$

(P) 小文字

Proposition 2-3. The following diagram is homotopy commutative.

$$\begin{array}{ccc}
 Q(K) \times L_p & \xrightarrow{id \times \Delta} & Q(K) \times Q(S^0) \xrightarrow{\lambda} Q(K \wedge S^0) = Q(K) \\
 \downarrow \tau & & \uparrow \cong \\
 L_p \times Q(K) & \xrightarrow{id \times \Delta} & W \times Q(K) \xrightarrow{\pi_p} Q(K)
 \end{array}$$

(P) 小文字

3. Proof of Theorem I. We introduce a filtration into $H_*(Q_0(S^0))$.
 $H_*(Q_0(S^0)) = G_0 \supseteq G_1 \supseteq G_2 \dots$ satisfy following properties. 1) $G_1 = \ker \varepsilon$,
 $\varepsilon : H_*(Q_0(S^0)) \longrightarrow Z_p$ is the augmentation. 2) $G_i \otimes G_j \longrightarrow G_{i+j}$
 3) $x_j \in G_{p^r-1}$ where $J = (\varepsilon_1, j_1, \dots, \varepsilon_r, j_r) \in H$, and $x_j \notin G_{p^r-1+1}$

Proposition 3-1. There exists unique filtration in $H_*(Q_0(S^0))$ satisfying the properties 1), 2), 3), and for $x \in H_*(Q_0(S^0))$, if $x \in G_j$ and $\Delta x = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$, then x', x'' belong G_j .

Proposition 3-2. Let $E_0 H_*(Q_0(S^0))$ be the algebra associated to the above filtration. Then $H_*(Q_0(S^0))$ and $E_0 H_*(Q_0(S^0))$ are isomorphic as algebras.

Proposition 3-3. $\Lambda_*(x \otimes y) \in G_{pij}$, if $x \in G_i$ and $y \in G_j$.

Then Theorem I follows from Prop.2-1, Prop3-1, Prop3-2, and Prop3-3.

4. H_p - structure on B_{SF} . Let $f_n \longrightarrow B_{SG(n)}$ be the universal oriented spherical fiber space with fiber S^{n-1} . \sum_p denotes the

permutation group of p -element. $J^m_{\Sigma_p} = \Sigma_p \times \dots \times \Sigma_p$ denote m -th join of Σ_p .

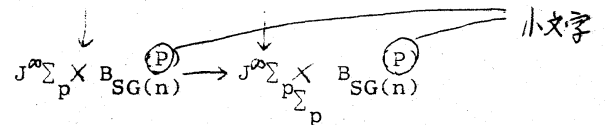
Let $f_n^{(p)} \rightarrow B_{SG(n)}^{(p)}$ be exterior p -th Whitney join of f_n . Let

$\pi_2: f_n \rightarrow J^m_{\Sigma_p} \times B_{SG(n)}^{(p)}$ denote the induced fibering of $f_n^{(p)}$ by

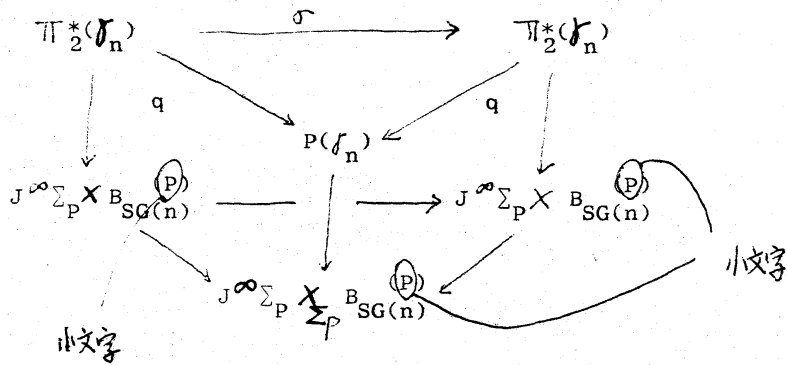
$$\pi_2: J^m_{\Sigma_p} \times B_{SG(n)}^{(p)} \rightarrow B_{SG(n)}^{(p)}.$$

Proposition 4-1. There exists a spherical fibring $P(f_n) \rightarrow J^{\infty}_{\Sigma_p} \times_{\Sigma_p} B_{SG(n)}^{(p)}$

with fiber S^{p-1} , and bundle map $q: \pi_2^*(f_n) \rightarrow P(f_n)$



They satisfy following commutative diagram, $\forall \sigma \in \Sigma_p$



Let $E_{SG(n)} \rightarrow B_{SG(n)}$ be the principal fibering associated with f_n ,

$$\text{i.e. } E_{SG(n)} = \left\{ f: S^{n-1} \rightarrow f_n \text{ ; oriented fiber map } \right\}.$$

$$\begin{array}{ccc} S^{n-1} & \rightarrow & f_n \\ \downarrow & & \downarrow \\ * & \rightarrow & B_{SG(n)} \end{array}$$

$P_0(f_n) \rightarrow J^m_{\Pi_p} \times_{\Pi_p} B_{SG(n)}^{(p)}$ denote restricted fibering of $P(f_n)$, where Π_p

denote cyclic group of order p .

$\bar{j} : J^m_p \times_{\pi_p} B_{SG(n)}^{(p)} \rightarrow B_{SG(pn)}$ be the classifying map of $\mathcal{P}_n^{(p)}$ (小文字)

As the map $\bar{\theta} : J^m_p / \pi_p \rightarrow J^m_p \times_{\pi_p} (e_0)^p \rightarrow J^m_p \times_{\pi_p} B_{SG(n)}^{(p)} \xrightarrow{\bar{j}} B_{SG(pn)}$ (小文字)

$e_0 \in B_{SG(n)}$, is induced by the n-times of the regular representation :

$$\pi_p \rightarrow SO(pn) \rightarrow SG(pn),$$

by the result of Kambe, we may suppose the above map is homotopic to constant map for suitable m , and n . And we

may take m , sufficiently large for a suitably sufficient large n . So

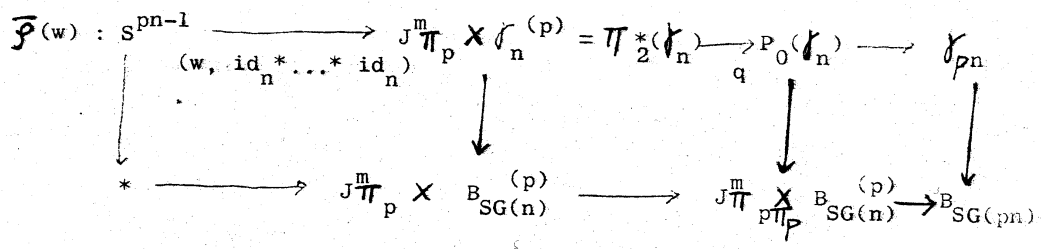
we may assume $\bar{\theta} (J^m_p / \pi_p) = e_0 \in B_{SG(pn)}$.

We define a map $\bar{f} : J^m_p \rightarrow SG(pn)$ in the following way. We

identify $SG(n) = (E_{SG(n)})_{e_0} = \pi^{-1}(e_0)$, and $SG(pn) = (E_{SG(pn)})_{e_0} = \pi^{-1}(e_0)$,

respectively. We fix $i_n \in (E_{SG(n)})_{e_0}$ and for $w \in J^m_p$, $\bar{f}(w)$

represents a following map.



We define $\bar{\theta}' : J^m_p \times E_{SG(n)}^{(p)} \rightarrow E_{SG(pn)}$ be the following commutative

diagram, for $(w, f_1, \dots, f_p) \in J^m_p \times E_{SG(n)}^{(p)}$, $\bar{\theta}'(w, f_1, \dots, f_p) :$

$$\begin{array}{ccccccc}
 S^{pn-1} & \xrightarrow{\bar{p}^{-1}(w)} & S^{pn-1} & \xrightarrow{(w, f_1 * \dots * f_p)} & J_{\pi_p}^m \times \delta_n^{(p)} & \xrightarrow{q} & P_0(\delta_n) \longrightarrow \delta_{pn} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & * & \longrightarrow & J_{\pi_p}^m \times B_{SG(n)}^{(p)} & \longrightarrow & J_{\pi_p}^m \times B_{\pi_p}^{(p)} \xrightarrow{B_{SG(n)}} B_{SG(pn)}
 \end{array}$$

Proposition 4-2. $\bar{\theta}'$ is π_p -equivariant, we obtain following commutative diagram.

$$\begin{array}{ccc}
 \bar{\theta} : J_{\pi_p}^m \times_{\pi_p} E_{SG(n)}^{(p)} & \longrightarrow & E_{SG(pn)} \\
 \downarrow & & \downarrow \\
 J_{\pi_p}^m \times_{\pi_p} B_{SG(n)}^{(p)} & \longrightarrow & B_{SG(pn)}
 \end{array}$$

And $\bar{\theta}(J_{\pi_p}^m \times_{\pi_p} SG(n)^{(p)}) \subseteq SG(pn) \subseteq E_{SG(pn)}$, and $\bar{\theta}(w, f_1, \dots, f_p) = \bar{p}(w)(f_1 * \dots * f_p) \bar{p}(w)^{-1}$, for any $(w, f_1, \dots, f_p) \in J_{\pi_p}^m \times_{\pi_p} SG(n)^{(p)}$.

5. Decomposition of $\bar{\theta}$. Let $\Lambda = \{ J = (\varepsilon_1, \dots, \varepsilon_p), \varepsilon_i = 0 \text{ or } 1 \}$,

$|J|$ = number of $\{ \varepsilon_i = 1, J = (\varepsilon_1, \dots, \varepsilon_p) \}$. π_p operates on Λ by permutation. We introduce in Λ an total ordering by the lexicographic order, for example, $(0, 1, \dots) \preceq (1, \dots)$. Let $\bar{\Lambda} = \Lambda / \pi_p$. We define the map $\bar{\Lambda} \xrightarrow{\pi} \Lambda$, by $\pi(\{J\}) =$ the first element in $\{J\}$.

Λ_0 denote the image of π . For each element $J_0 \in \Lambda_0$, we define $\mathcal{N}_{J_0} :$

$$(\Omega_0^{n-1} \times S^{n-1})^{(p)} \xrightarrow{\text{小文字}} G(pn) \text{ as follows, where } G(pn) = \{ f : S^{pn-1} \rightarrow S^{pn-1} \},$$

$$\varphi_2 : S^{n-1} \rightarrow S_0^{n-1} \vee S_0^{n-1}$$

For $(l_1, \dots, l_p) \in (\Omega_0^{n-1} S^{n-1})^p$, $\eta_{J_0}(l_1, \dots, l_p)$ represents following map.

$$\eta_{J_0}(l_1, \dots, l_p) : S^{n-1} * \dots * S^{n-1} \xrightarrow{\varphi_2 * \dots * \varphi_2} (S_0^{n-1} \vee S_1^{n-1}) * \dots * (S_0^{n-1} \vee S_1^{n-1})$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$S^{pn-1} \xleftarrow{\textcircled{*}} \bigvee_{J \in \Lambda_0} S_J^{pn-1}$$

$\textcircled{*}$ is the map as follows, $\textcircled{*}|_{S_J} : S_J \rightarrow S$ represents, a) $0 * 0$, if $J \neq \sigma J_0$ for any $\sigma \in \Pi_p$, $0 : S^{n-1} * \dots * S^{n-1} \rightarrow S^{pn-1}$ b) $l_1^{\varepsilon_1} * \dots * l_p^{\varepsilon_p}$, if $J = \sigma J_0 = (\varepsilon_1, \dots, \varepsilon_p)$ for some $\sigma \in \Pi_p$, where $l_i^0 = \text{id}$, $l_i^1 = l_i$.

And $S_J^{pn-1} = S_{\varepsilon_1}^{n-1} * \dots * S_{\varepsilon_p}^{n-1}$.

We define $\bar{\theta}'_{J_0} : J\pi_p^m \times (\Omega_0^{n-1} S^{n-1})^p \rightarrow G(pn)$, for each $J_0 \in \Lambda_0$,

as $\bar{\theta}'_{J_0}(w, l_1, \dots, l_p) = \bar{P}(w) \eta_{J_0}(l_1, \dots, l_p) \bar{P}(w)^{-1}$.

Proposition 5-1. $\bar{\theta}'_{J_0} : J\pi_p^m \times (\Omega_0^{n-1} S^{n-1})^p \rightarrow G(pn)$, is

Π_p -equivariant, therefore it define a following map $\bar{\theta}_{J_0} : J\pi_p^m \times_{\Pi_p} (\Omega_0^{n-1} S^{n-1})^p \rightarrow G(pn)$.

Let $i : G(pn) \rightarrow \Omega_0^{pn+1} S^{pn+1}$ be the inclusion

Proposition 5-2. $i\bar{\theta}$ and $\bigvee_{J_0 \in \Lambda_0} i\bar{\theta}_{J_0}$ are homotopic on

$(pn-5)$ -skelton as a map $J\pi_p^m \times_{\Pi_p} (\Omega_0^{n-1} S^{n-1})^p \rightarrow \Omega_0^{pn+1} S^{pn+1}$, where \vee denote loop multiplication on $\Omega_0^{pn+1} S^{pn+1}$.

For $J_0 \in \Lambda_0$, $|J_0| \neq 0, p$, we define $h_{J_0} : J\pi_p^m \rightarrow G(pn)$ as follows, for $w \in J\pi_p^m$,

$$\begin{array}{ccc}
 h_{J_0}(w) : S^{pn-1} & \xrightarrow{\bar{f}(w)^{-1}} & S^{pn-1} \times_{\pi_p} \dots \times_{\pi_p} \mathbb{R}^p \\
 \downarrow & & \downarrow \otimes \\
 S^{pn-1} & \xleftarrow{\bar{f}(w)} & S^{pn-1}
 \end{array}$$

where $\otimes|_{S_J} : S_J^{pn-1} \rightarrow S^{pn-1}$ represents a) $0^* \times 0$, if $J \neq J_0$, for any $\sigma \in \pi_p$. b) id_{pn-1} , if $J = \sigma J_0$, for some $\sigma \in \pi_p$. h_{J_0} is well defined.

Proposition 5-3. For $J_0 \in J_0 = (\varepsilon_1, \dots, \varepsilon_p)$, $0 \neq |J| \neq p$, the following diagram is homotopy comutative.

$$\begin{array}{ccc}
 J^m \pi_p \times_{\pi_p} \Omega_0^{n-1} S^{n-1} & \xrightarrow{\text{id} \times \Delta_p} & J^m \pi_p \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^{(p)} \longrightarrow G_0(pn) \\
 h_{J_0} \times \downarrow & & \downarrow * \text{id}_{pn-1} \\
 G(pn) \times G_0(pn) & \xrightarrow{\quad \quad \quad} & G_0(2pn)
 \end{array}$$

where $(\)^{\varepsilon_1 * \dots * \varepsilon_p} : \Omega_0^{n-1} S^{n-1} \rightarrow G(pn)$ is the map defined by $l \mapsto (l)^{\varepsilon_1 * \dots * \varepsilon_p}$.

We define $\bar{\theta}_p : J^m \pi_p \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p \rightarrow G(pn)$ by $\bar{\theta}_p(w, l_1, \dots, l_p) = \bar{f}(w) (l_1^*, \dots, l_p^*) \bar{f}(w)^{-1}$.

Proposition 5-4. $\bar{\theta}_p \cong \bar{\theta}(0, \dots, 0)$; homotopic.

6. Proof of Theorem II. $\bar{\theta} : J^{\infty} \pi_p \times_{\pi_p} SF^p \rightarrow SF$, $\bar{\theta} : J^{\infty} \pi_p \times_{\pi_p} B_{SF}^p \rightarrow B_{SF}$ are the maps corresponding to $\bar{\theta} : J^m \pi_p \times_{\pi_p} SG(n)^p \rightarrow SG(pn)$,

$\bar{\theta} : J_{\pi_p}^m \times_{\pi_p} B_{SG(n)}^p \longrightarrow B_{SG(pn)}$ for large m and n . We define

$$\bar{Q}_j : H_*(SF) \longrightarrow H_*(SF), \quad \bar{Q}_j : H_*(B_{SF}) \longrightarrow H_*(B_{SF}), \quad j = 1, 2, \dots,$$

by the $\bar{Q}_j(x) = \bar{\theta}_*(e_j \otimes x^p)$, for $x \in H_*(SF)$, or $\in H_*(B_{SF})$.

Proposition 6-1. In the homology spectral sequence associated with following fibering $SF \longrightarrow E_{SF} \longrightarrow B_{SF}$. $E_{**}^2 = H_*(B_{SF}) \otimes H_*(SF)$.

If $x \in E_{2n,0}^2$ is transgressive. $y \in E_{0,2n-1}^2$, $\tau(x) = \{y\}$, then we obtain

$$\text{the following relations. } \{\tau \bar{Q}_0(x)\} = \{\tau(x^p)\} = \{\bar{Q}_{p-1}(y)\}$$

$$\text{in } E_{0,2np-1}^{2np}, \text{ and } \{\tau(x^{p-1} \otimes y)\} = \{\bar{Q}_{p-2}(y)\} \text{ in } E_{0,2n(p-1)}^{2n(p-1)}.$$

Proposition 6-2. Let $\bar{h}_1 = h_{(1,0,\dots,0)} : J_{\pi_p}^m / \pi_p \longrightarrow G(pn)$,

and $\bar{H}_1 : J_{\pi_p}^m \times S^{pn-1} \longrightarrow S^{pn-1}$ be the representative of \bar{h}_1 . Then

in $H_*(C_{\bar{H}_1})$, $P^j(s)$ and $\Delta P^j(s)$ are non zero, where $s \in H^{pn-1}(C_{\bar{H}_1}) \cong Z_p$

is a generator, and $1 \leq j \leq m-1 / 2(p-1)$.

Proposition 6-3. If $\bar{x}_I \in H_*(SF)$ belongs to G_{p^j} , $j \geq 1$, where $I \in H$,

then $\bar{Q}_{p-2}(x_I)$, $\bar{Q}_{p-1}(x_I)$ belong to $G_{p^{j+1}}$, and as elements of $G_{p^{j+1}}/G_{p^{j+1}+1}^{\text{decomp}}$.

they coincide with $\beta_{p^{p-1}} \bar{Q}_{p-1}(x_I)$, $\bar{Q}_p(x_I)$ respectively.

This proposition is proved by using the following lemmas.

Let $\bar{h}_1 : J_{\pi_p}^\infty \longrightarrow G(\infty) = \{f : S^\infty \longrightarrow S^\infty\}$, represent

$$\bar{h}_1 : J_{\pi_p}^m \longrightarrow G(pn) \text{ for large } m, n. \text{ And } \bar{h}_{1,0} : J_{\pi_p}^\infty \xrightarrow{\bar{h}_1} G_p(\infty) \xrightarrow{((-p \cdot \text{id}))} Q_0 S^0.$$

Lemma 6-1. $\bar{h}_{1,0_*}(e_{2i(p-1)}) = cx_i - x'$, $c \neq 0$, $x' \in G_2$,

$\bar{h}_{1,0_*}(e_{2i(p-1)-1}) = c\beta_p x_i + x''$, $c' \neq 0$, $x'' \in G_2$. This proposition is proved by using prop. 6-2.

Lemma 6-2. In $H_*(SF)$, we obtain the following relations.

$$\langle \tilde{x}_j, \sigma(\Delta q_j) \rangle \neq 0, \langle \beta_p x_j, \sigma(q_j) \rangle \neq 0, \text{ for } x \in G_2, \langle x, \sigma(\Delta q_j) \rangle = 0,$$

$$\langle x, \sigma(q_j) \rangle = 0.$$

> Lemma 6-3 \otimes 下行.

Lemma 6-4. For any $x_i \in G_{pj}$, $I \in H$, $\bar{\theta}_1 * (e_i \otimes x_i^p) \in G_{pj+1}$. And as an element of $G_{pj+1}/G_{pj+1+1} + \text{decop.}$, it coincide with $Q_i(x_i)$.

Lemma 6-5. For any $x_i \in G_{pj}$, $I \in H$, $j \geq 1$. $\bar{\theta}(1, \dots, 1) * (e_i \otimes x_i^p)$

belongs G_{pj+1+1} , if $i \leq p-1$.

We consider $j_* : \underbrace{H_*(SO)}_{\wedge H_*(SF)} \rightarrow H_*(SO)/\ker j_* \cong \mathbb{A}(y_1, y_2, \dots)$

$\deg(y_i) = 2i(p-1)-1$. Let $\tilde{y}_i \in H_*(SF)$, be $j_*(y_i)$.

Proposition 6-4. $H_*(SF)$ is a free commutative algebra generated by $\tilde{x}_j, \tilde{y}_j, j = 1, 2, \dots, \tilde{x}_I, I \in H_1^+ \cup H_2^+, \bar{Q}_{p-1} \dots \bar{Q}_{p-1}(\tilde{x}_I), I \in H_1^- \cup H_2^-$,

\bar{Q}_{p-1} operate on x_I k-times, $k \geq 0$. $\bar{Q}_{p-2} \bar{Q}_{p-1} \dots \bar{Q}_{p-1}(\tilde{x}_I), I \in H_1^- \cup H_2^-$,

\bar{Q}_{p-2} operates on $\bar{Q}_{p-1} \dots \bar{Q}_{p-1}(\tilde{x}_I)$ exactly one times, and \bar{Q}_{p-1} operates

on \tilde{x}_I , k-times, $k \geq 0$.

This proposition is proved by using prop, 6-3, and structure of $H_*(SF)$ as an algebra. Then Theorem II follows from prop. 6-1, prop. 6-4 and the comparison theorem for spectral sequence.

Lemma 6-3, If $J_0 \in \mathbb{A}_0$, and $|J_0| \neq 0, 1, p$, then for any $x_i \in H_*(\mathbb{Q}_0(S^p)), x_i \in G_{pj}, I \in H, \bar{\theta}_{J_0} * (e_i \otimes x_i^p) \in G_{pj+|J_0|}$.

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