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A two point connection problem for an n-th order single linear ordinary differential equation

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1. Introduction.

In this lecture, we should like to talk about a topics on a two point connection problem for an n-th order single linear ordinary differential equation with an irregular singular point of rank 2. A two point connection problem is to seek "the solutions in the large" of given ordinary differential equations. Until now, there are several investigations on this problem. G.D. Birkhoff initiated this study, and then, H.W.Knobloch, H.L.Turrittin and K.Okubo developed it to a certain extent. Some of them treated a system of ordinary differential equations with an irregular singular point of rank 1 and the other studied a single ordinary differential equation of which the coefficients of convergent solutions satisfy the two term recurrence formula.

Now, we shall explain the above last line. One method to seek"the solutions in the large" is to analyze the convergent solutions in the neighbourhood of a regular or regular singular point for the purpose of deriving the asymptotic behavior of them near an irregular singular point. Ordinarily, the convergent power series solutions can be represented by the product of a multivalued function (fractional power) and an entire function, in the case when the given ordinary differential equation has a regular singular point at origin, an irregular singular point at infinity and no other singular point elsewhere in the whole complex plane.

It is clear that the entire function will mainly contribute to the asymptotic behavior of convergent solution near infinity. Hence, we had better to investigate the asymptotic behavior of the entire function near infinity which depends on the coefficients of power series of the entire function. The coefficients of convergent power series solutions of ordinary differential equations satisfy the recurrence formula. We reduce the recurrence formula to the same difference equation by the change of integer to complex variable.

And then, if we could analyze the behavior of solutions of the difference equation, we will be able to get the behavior of the coefficients of large powers and the asymptotic behavior of the entire function, or the convergent solution near infinity.

Now, if the coefficients happen to satisfy the two term recurrence formula, the coefficients, in general, can be represented by the generalized Gamma function which have been studied in detail by E.M.Wright and others.

So, the two point connection problem for linear ordinary differential equations with an irregular singular point of rank 1 has been almost completely solved because the coefficients of convergent solutions of that equations also satisfy the two term recurrence formula.

The two point connection problem for linear ordinary differential equations with an irregular singular point of higher rank has not yet studied and it seems to be a very difficult problem, since the solutions of the reduced difference equation can not so easily analyzed as Gamma function.

Recently, K.Okubo showed results for a system of ordinary differential equations with an irregular singular point of rank 2 without complete proof in the book "Proceeding of United States-Japan Seminor on Differential and Functional Equations".

Here, we shall explain some results derived for an n-th order single linear ordinary differential equation with an irregular singular point of rank 2.

The difference between a single ordinary differential equation and a system of ordinary differential equations has not yet clarified as described in the paper "Solvable Related Equations Pertaining to Turning Point Problem" by H.L.Turrittin.

2. Properties of convergent solutions and asymptotic solutions.

An n-th order single linear differential equation with a regular singular point at origin has the following form,

$$(1) \qquad t^n \frac{d^n x}{dt^n} = \sum_{\ell=1}^{n} a_\ell(t)\, t^{n-\ell} \frac{d^{n-\ell} x}{dt^{n-\ell}}$$

where $a_\ell(t)$ $(\ell=1,2,\dots,n)$ are holomorphic at origin.

On the other hand, an n-th order single linear ordinary differential equation with an irregular singular point of rank 2 at infinity can be written down as follows,

$$(2) \qquad \frac{d^n x}{dt^n} = \sum_{\ell=1}^{n} b_\ell(t)\, t^\ell \frac{d^{n-\ell} x}{dt^{n-\ell}}$$

where $b_\ell(t)$ $(\ell=1,2,\dots,n)$ are holomorphic at infinity.

Therefor, if we consider an n-th order single linear ordinary differential equation which has a regular singular point at origin, an irregular singular point of rank 2 at infinity and no other singular point elsewhere, we have from (1) and (2),

$$(3) \qquad a_\ell(t) = t^{2\ell} b_\ell(t) \qquad (\ell=1,2,\ldots,n)$$

from which the coefficient $a_\ell(t)$ must be a polynomial of degree at most 2ℓ .

And, if we write

$$(4) \qquad a_\ell(t) = \sum_{r=0}^{2\ell} a_{\ell,r}\, t^r \qquad (\ell=1,2,\ldots,n) ,$$

we have the most general n-th order single linear ordinary differential equation with above property as follows,

$$(5) \qquad t^n \frac{d^n x}{dt^n} = \sum_{\ell=1}^{n} t^{n-\ell} \left(\sum_{r=0}^{2\ell} a_{\ell,r} t^r \right) \frac{d^{n-\ell}x}{dt^{n-\ell}}$$

Now, by the theory of ordinary differential equations, the convergent solutions in the neighbourhood of a regular singular point have the form

$$(6) \qquad x_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m)\, t^m \qquad (j=1,2,\ldots,n)$$

where ρ_j are roots of the characteristic equation

$$(7) \qquad \rho(\rho-1) \cdots (\rho-n+1) = \sum_{\ell=1}^{n} a_{\ell,0}\, \rho(\rho-1)\cdots(\rho-n+\ell+1)$$

and $\rho_j - \rho_k \neq$ integer $(j\neq k)$ are assumed because the convergent solutions involving no logarithmic polynomials are considered in this lecture.

We use the following abbreviation according to Ince's book

$$[\rho]_n = \rho(\rho-1) \cdots (\rho-n+1), \qquad [\rho]_0=1$$

and the characteristic equation (7), for example, can be written by

$$[\rho]_n = \sum_{\ell=1}^{n} a_{\ell,0} [\rho]_{n-\ell}$$

At first, we shall investigate the coefficients $G_j(m)$ of convergent solutions. Now we denote the differential operator $t\frac{d}{dt}$ by D and then we have

$$(8) \qquad t^p \frac{d^p}{dt^p} = D(D-1) \cdots (D-p+1) = [D]_p$$

If we apply the differential operator (8) to $G_j(m)t^{m+\rho_j}$, we have

$$(9) \qquad t^p \frac{d^p}{dt^p} G_j(m)t^{m+\rho_j} = G_j(m)[m+\rho_j]_p t^{m+\rho_j} .$$

Hence, if we substitute the convergent solution $x_j(t)$ into the differential equation (5), we obtain

$$(10) \qquad \sum_{m=0}^{\infty} G_j(m)[m+\rho_j]_n t^{m+\rho_j}$$

$$= \sum_{\ell=1}^{n} a_\ell(t) \sum_{m=0}^{\infty} G_j(m)[m+\rho_j]_{n-\ell} t^{m+\rho_j}$$

$$= \sum_{\ell=1}^{n} \sum_{r=0}^{2\ell} a_{\ell,r} t^r \sum_{m=r}^{\infty} G_j(m-r)[m-r+\rho_j]_{n-\ell} t^{m-r+\rho_j}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{\ell=1}^{n} \sum_{r=0}^{2\ell} a_{\ell,r} \, G_j(m-r)[m-r+\rho_j]_{n-\ell} \right) t^{m+\rho_j}$$

where we put $G_j(-r) = 0$ for $r=1,2,\ldots 2n$.

Equating the coefficients of like power of t in the both side of (10), we have the following 2n-th order recurrence formulas

$$(11) \quad \begin{cases} [\rho_j+m]_n \, G_j(m) = \displaystyle\sum_{\ell=1}^{n} \sum_{r=0}^{2\ell} a_{\ell,r}[\rho_j+m-r]_{n-\ell} \, G_j(m-r), \\ G_j(-r) \equiv 0 \qquad (r=1,2,\ldots,2n). \end{cases}$$

Specifically, if we put $m=0$, we have the characteristic equation as follows,

$$[\rho_j]_n G_j(0) = \sum_{\ell=1}^{n} \sum_{r=0}^{2\ell} a_{\ell,r} \, G_j(-r)[-r+\rho_j]_{n-\ell}$$

$$= \sum_{\ell=1}^{n} a_{\ell,0} \, [\rho_j]_{n-\ell} \, G_j(0)$$

and we can put $G_j(0)=1$ from the assumption $\rho_j-\rho_k \neq$ integer.

Next, we shall consider the formal solutions in the neighbourhood of an irregular singular point with the following form

$$(12) \qquad x^k(t) \sim \exp\left(\frac{\lambda_k}{2} t^2 + \alpha_k t\right) t^{\mu_k} \sum_{s=0}^{\infty} h^k(s) t^{-s} \quad (k=1,2,\ldots,n)$$

where the constants λ_k are roots of the characteristic equation

$$(13) \qquad \lambda^n = \sum_{\ell=1}^{n} a_{\ell,2\ell} \lambda^{n-\ell}$$

and the constants α_k, μ_k are determined later.

In order to investigate the coefficients h^k s) of formal solutions, we introduce new notation

$$(14) \qquad x_p^k(t) = t^{-p} \frac{d^p x^k}{dt^p} = \exp(\frac{\lambda_k}{2} t^2 + \alpha_k t) t^{\mu_k} \sum_{s=0}^{\infty} h_p^k(s) t^{-s}.$$

Since

$$x_p^k(t) = t^{-1} \frac{d}{dt}(t^{-p+1} \frac{d^{p-1} x^k}{dt^{p-1}}) + (p-1) t^{-2} \cdot t^{-p+1} \frac{d^{p-1} x^k}{dt^{p-1}}$$

$$= t^{-1} \frac{d}{dt} x_{p-1}^k(t) + (p-1) t^{-2} x_{p-1}^k(t) ,$$

we have

$$(15) \qquad \exp(\frac{\lambda_k}{2} t^2 + \alpha_k t) \, t^{\mu_k} \sum_{s=0}^{\infty} h_p^k(s) t^{-s}$$

$$= \exp(\frac{\lambda_k}{2} t^2 + \alpha_k t) t^{\mu_k} [\lambda_k \sum_{s=0}^{\infty} h_{p-1}^k(s) t^{-s}$$

$$+ \alpha_k \sum_{s=1}^{\infty} h_{p-1}^k(s-1) t^{-s} + \sum_{s=2}^{\infty} (\mu_k + p + 1 - s) h_{p-1}^k(s-2) t^{-s}]$$

Equating the coefficients of like power in the both side of (15), we obtain the <u>first</u> recurrence formulas

$$(16) \qquad h_p^k(s) = \lambda_k h_{p-1}^k(s) + \alpha_k h_{p-1}^k(s-1) + (\mu_k + p + 1 - s) h_{p-1}^k(s-2)$$

$$(k=1,2,\dots,n)$$

where for the moment we assume $h_p^k(s)=0$ for $s< 0$.

Now we substitute $x_p^k(t)$ into the differential equation

$$t^{-n}\frac{d^n x}{dt^n} = \sum_{\ell=1}^{n}\left(\sum_{r=0}^{2\ell} a_{\ell,r}\, t^{r-2\ell}\right) t^{-(n-\ell)}\frac{d^{n-\ell}x}{dt^{n-\ell}}$$

and we derive

$$(17)\qquad \sum_{s=0}^{\infty} h_n^k(s)t^{-s} = \sum_{\ell=1}^{n}\sum_{r=0}^{2\ell} a_{\ell,r}\sum_{s=0}^{\infty} h_{n-\ell}^k(s)t^{-s+r-2\ell}$$

$$= \sum_{\ell=1}^{n}\sum_{r=0}^{2\ell} a_{\ell,r}\sum_{s=r-2\ell}^{\infty} h_{n-\ell}^k(s+r-2\ell)t^{-s}$$

Again equating the coefficients of like power in the both side of (17), the <u>second</u> recurrence formulas are derived as follows,

$$(18)\qquad h_n^k(s) = \sum_{\ell=1}^{n}\sum_{r=0}^{2\ell} a_{\ell,r}\, h_{n-\ell}^k(s+r-2\ell) \qquad (k=1,2,\ldots,n)$$

Here, we shall try to represent $h_p^k(s)$ by $h^k(s)=h_0^k(s)$ from the first recurrence formulas (16).

After a little complicated calculations, we can represent $h_p^k(s)$ as follows,

$$(19)\qquad h_p^k(s) = \sum_{q=0}^{2p}\left\{\sum_{r=0}^{[\frac{q}{2}]}\binom{p}{q-r}\lambda_k^{p-q+r}\,\alpha_k^{q-2r}\,H(p,q,r:s)\right\} h_0^k(s-q)$$

where [ ] denotes Gauss' symbol and

$$\binom{p}{q} = \frac{p\,!}{q\,!\;(p-q)\,!} \qquad \text{for } q \leqq p$$

$$= 0 \qquad \text{for } q > p \quad .$$

With respect to the coefficient $H(p,q,r:s)$ in the brachet, it seems not to have an explicit formula in general. We can only say that $H(p,q,r:s)$ is a sum of r degree polynomial of $(\mu_k+1-s-j)$. For example,

$$H(p,q,o:s) \equiv 1$$

$$H(p,q,1:s) = c_q \sum_{j=q-1}^{p} (\mu_k+1-s+j)$$

where c_q is a constant independent of s.

But, with respect to the coefficients of the terms needed later, we can fortunately give them explicit forms and, in fact, we have

$$(20) \qquad h_p^k(s) = \lambda_k^p h_0^k(s) + p\lambda_k^{p-1} \alpha_k h_0^k(s-1)$$

$$+\left\{ \frac{p(p-1)}{2} \lambda_k^{p-2} \alpha_k^2 + \lambda_k^{p-1} \sum_{j=1}^{p} (\mu_k+1-s+j) \right\} h_0^k(s-2) +$$

$$\cdots + p\alpha_k[\mu_k-s+2p-1]_{p-1} h_0^k(s-2p+1)$$

$$+ [\mu_k-s+2p]_p h_0^k(s-2p)$$

The proof is done by induction. For $p=1$, it is evident from the first recurrence formulas

$$h_1^k(s) = \lambda_k h_0^k(s) + \alpha_k h_0^k(s-1) + (\mu_k+2-s) h_0^k(s-2) .$$

If we suppose that the formulas (20) are true for p and substitute these formulas into the first recurrence formulas for $p+1$, we have

$$h^k_{p+1}(s) = \lambda_k \left\{ \lambda^p_k \, h^k_0(s) + p\lambda^{p-1}_k \alpha_k \, h^k_0(s-1) + \left[\frac{p(p-1)}{2}\lambda^{p-2}_k \alpha^2_k + \lambda^{p-1}_k \sum_{j=1}^{p} (\mu_k+1-s+j)\right] h^k_0(s-2) + \ldots \right\}$$

$$+\alpha_k \left\{ \lambda^p_k \, h^k_0(s-1) + p\lambda^{p-1}_k \alpha_k \, h^k_0(s-2) + \ldots + [\mu_k-s+1+2p]_p h^k_0(s-2p-1) \right\}$$

$$+ (\mu_k+p+2-s) \left\{ \lambda^p_k \, h^k_0(s-2) + \ldots + p\alpha_k[\mu_k-s+2p+1]_{p-1} h^k_0(s-2p-1) + [\mu_k-s+2p+2]_p \, h^k_0(s-2p-2) \right\}$$

$$= \lambda^{p+1}_k \, h^k_0(s) + (p\lambda^p_k\alpha_k + \lambda^p_k\alpha_k) \, h^k_0(s-1)$$

$$+ \left\{ \frac{p(p-1)}{2}\lambda^{p-1}_k \alpha^2_k + p\lambda^{p-1}_k \alpha^2_k + \lambda^p_k \sum_{j=1}^{p} (\mu_k+1-s+j) + \lambda^p_k(\mu_k+p+2-s) \right\} h^k_0(s-2)$$

$$+ \ldots + \left\{ \alpha_k[\mu_k-s+1+2p]_p + p\alpha_k(\mu_k+p+2-s)[\mu_k-s+2p+1]_{p-1} \right\} h^k_0(s-2p-1)$$

$$+ (\mu_k+p+2-s)[\mu_k-s+2p+2]_p \, h^k_0(s-2p-2)$$

$$= \lambda^{p+1}_k \, h^k_0(s) + (p+1)\lambda^p_k \, \alpha_k \, h^k_0(s-1)$$

$$+ \left\{ \frac{(p+1)p}{2} \lambda^{p-1}_k \, \alpha^2_k + \lambda^p_k \sum_{j=1}^{p+1} (\mu_k+1-s+j) \right\} h^k_0(s-2)$$

$$+ \ldots + (p+1) \, \alpha_k[\mu_k-s+1+2p]_p \, h^k_0(s-2p-1)$$

$$+ [\mu_k-s+2p+2]_{p+1} \, h^k_0(s-2p-2) \, .$$

3. Invariant identity

Now we shall investigate an invariant identity of the linear ordinary differential equations with an irregular singular point of rank 2 which is described in general form in Ince's book. The invariant identity will play an important role later in the proof of independence of solutions of the recurrence formulas for $G_j(m)$.

If we assume that $h_0^k(s) = 0$ for $s < 0$, it is clear from the representation formulas (19) that for all p, $h_p^k(s)=0$ are satisfied whenever $s < 0$.

Here, we shall put $s = 0$. From the second recurrence formulas, we have under the above assumption,

$$h_n^k(0) = \sum_{\ell=1}^{n} a_{\ell,2\ell}\, h_{n-\ell}^k(0). \tag{21}$$

On the other hand, we obtain from the representation formulas,

$$h_p^k(0) = \lambda_k^p\, h_0^k(0). \tag{22}$$

Substituting (22) into (21), we have

$$\lambda_k^n\, h_0^k(0) = \left(\sum_{\ell=1}^{n} a_{\ell,2\ell} \lambda_k^{n-\ell} \right) h_0^k(0) .$$

Since λ_k is a root of the characteristic equation for an irregular singular point,

$$F(\lambda) = \lambda^n - \sum_{\ell=1}^{n} a_{\ell,2\ell}\, \lambda^{n-\ell} = 0, \tag{23}$$

the last formula is satisfied for arbitrary $h_0^k(0)$ and we can put $h_0^k(0) = 1$.

Next, we shall put $s = 1$. Again from the second recurrence formulas, we obtain

$$(24) \qquad h_n^k(1) = \sum_{\ell=1}^{n} a_{\ell,2\ell-1}\, h_{n-\ell}^k(0) + \sum_{\ell=1}^{n} a_{\ell,2\ell}\, h_{n-\ell}^k(1)$$

and from the representation formulas, we have

$$(25) \qquad h_p^k(1) = \lambda_k^p\, h_0^k(1) + p\lambda_k^{p-1}\, \alpha_k\, h_0^k(0)\ .$$

Substituting (25) and (22) into (24), we have

$$\left(\lambda_k^n - \sum_{\ell=1}^{n} a_{\ell,2\ell} \lambda_k^{n-\ell} \right) h_0^k(1)$$

$$+ \left(n\lambda_k^{n-1} - \sum_{\ell=1}^{n} a_{\ell,2\ell}\, (n-\ell)\, \lambda_k^{n-\ell-1} \right) \alpha_k\, h_0^k(0)$$

$$= \sum_{\ell=1}^{n} a_{\ell,2\ell-1} \lambda_k^{n-\ell}\, h_0^k(0).$$

Since the coefficient of $h_0^k(1)$ is the characteristic equation and $h_0^k(0)$ is not zero, the constant α_k is determined by

$$(26) \qquad \alpha_k = \sum_{\ell=1}^{n} a_{\ell,2\ell-1}\, \lambda_k^{n-\ell} \quad / \quad F'(\lambda_k)\ .$$

Here, if we integrate the function

$$\sum_{\ell=1}^{n} a_{\ell,2\ell-1}\, \lambda^{n-\ell} \quad / \quad F(\lambda)\ .$$

along the sufficiently large circle with its center at origin in the complex λ- plane,

we have the following relations from the calculation of residues,

$$(27)\qquad \sum_{k=1}^{n} \alpha_k = a_{11} \ .$$

Lastly, in order to determine the constant μ_k , we shall put $s = 2$. Again from the recurrence and representation formulas, we have

$$(28)\qquad h_n^k(2) = \sum_{\ell=1}^{n} a_{\ell,2\ell}\, h_{n-\ell}^k(2) + \sum_{\ell=1}^{n} a_{\ell,2\ell-1}\, h_{n-\ell}^k(1) + \sum_{\ell=1}^{n} a_{\ell,2\ell-2}\, h_{n-\ell}^k(0),$$

$$(29)\qquad h_p^k(2) = \lambda_k^p\, h_0^k(2) + p\lambda_k^{p-1}\, \alpha_k\, h_0^k(1) + H(p)\, h_0^k(0)$$

where

$$H(p) = \frac{p(p-1)}{2}\, \lambda_k^{p-2}\, \alpha_k^2 + \lambda_k^{p-1} \sum_{j=1}^{p} (\mu_k+1-2+j) \ .$$

Substituting (29), (25) and (22) into (28), we have

$$(30)\qquad H(n) = \sum_{\ell=1}^{n} a_{\ell,2\ell}\, H(n-\ell) + \sum_{\ell=1}^{n} a_{\ell,2\ell-1}(n-\ell)\, \lambda_k^{n-\ell}\alpha_k + \sum_{\ell=1}^{n} a_{\ell,2\ell-2}\, \lambda_k^{n-\ell}$$

because the coefficients of $h_0^k(2)$ and $h_0^k(1)$ are equal to zero from the relations determining the constants λ_k, α_k .

Now, the constant μ_k can be determined by the relation

$$(31)\quad F'(\lambda_k)\,\mu_k$$

$$= -\frac{n(n-1)}{2}(\lambda_k^{n-1}+\lambda_k^{n-2}\alpha_k^2)+\sum_{\ell=1}^{n} a_{\ell,2\ell}\frac{(n-\ell)(n-\ell-1)}{2}(\lambda_k^{n-\ell-1}+\lambda_k^{n-\ell-2}\alpha_k^2)$$

$$+\sum_{\ell=1}^{n} a_{\ell,2\ell-1}(n-\ell)\,\lambda_k^{n-\ell-1}\,\alpha_k+\sum_{\ell=1}^{n} a_{\ell,2\ell-2}\,\lambda_k^{n-\ell}.$$

By the similar calculations of residues described above, the sum of the constants μ_k will be equal to the coefficient of λ^{n-1} in the right hand side of (31).

Hence we have

$$(32)\qquad \sum_{k=1}^{n} \mu_k = -\frac{n(n-1)}{2} + a_{1,0} .$$

On the other hand, from the characteristic equation for a regular singular point

$$[\rho\ ]_n - \sum_{\ell=1}^{n} a_{\ell,0}\ [\rho\ ]_{n-\ell} = (\rho-\rho_1)(\rho-\rho_2)\ \ldots\ (\rho-\rho_n) \equiv 0 ,$$

we have one more relation as follows,

$$(33)\qquad \sum_{k=1}^{n} \rho_k = a_{1,0} + \frac{n(n-1)}{2} .$$

After all, from (32) and (33), we obtain an important "invariant identity"

$$(34)\qquad \sum_{k=1}^{n} \rho_k = \sum_{k=1}^{n} \mu_k + n(n-1) .$$