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An analytic study of a pseudo-complex-structure.

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0. Introduction

In this paper we give an analytic study of a pseudo-complex-structure, which is roughly speaking, the abstract substitute of an imbedding of a real manifold into a complex manifold. Until now the main tool of this analysis was the theory of partial differential equations developed by Kohn [4] and Hörmander [2]. But the results obtained so far seem to be applied only to the 2-regular case. (See Definition 2 for the concept of the regularity.)

On the other hand, the geometric study of pseudo-complex-structures was already made by N. Tanaka [8], [9], [10]. And this formulation, when combined with a recent work of Hörmander [3], yields a powerfull theorem (Theorem 1.5) which is a partial generalization of Kohn-Hörmander theory to the general μ-regular case.

This theorem is of particular importance in the study of a complex manifold. In fact it follows from this the finite-dimensionality of the space of gloval sections of every analytic vector bundle over a complex manifold having some nice real submanifold, which is in general neither compact, nor pseudo-concave.

I am greatly indebted to Professor N. Tanaka not only for the formulation of basic geometric materials, but also for a number of stimulating conversations out of which especially Theorems 1 - 2 grew. Therefore this paper might be regarded as a collaboration with him. I thank also Professor S. Matsuura for his serious interest and encouragement.

1. The concept of a pseudo-complex-structure and many other concepts related to it were already introduced in [9] [10]. But we reproduce these here. We assume the differentiability

of class C^∞. Let E be a (real or complex) vector bundle over a manifold M. We shall denote by $\underline{E}$ the sheaf of germs of local C^∞ sections of E. For $p \in M$, E_p denotes the fiber of E over p. Similarly $\mathscr{S}_p$ is a stalk of $\mathscr{S}$ over p if $\mathscr{S}$ is a sheaf over M. $\Gamma_\Omega(\mathscr{S})$ is the set of sections over Ω and $\Gamma(\mathscr{S})$ denotes the set of global sections. For simplicity we use $\Gamma_\Omega(E)$, $\Gamma(E)$ instead of $\Gamma_\Omega(\underline{E})$, $\Gamma(\underline{E})$ respectively.

Let M be a real submanifold of a complex manifold $\tilde{M}$ and let $T^C(M)$, $T^C(\tilde{M})$ denote the complexified tangent bundles of M, $\tilde{M}$ respectively. As usual, $T_p^C(M)$ is regarded as a subspace of $T_p^C(\tilde{M})$ for any p of M. Now we set

$$(1.1) \qquad S_p = T_p^C(M) \cap T_p^{(1,0)}(\tilde{M}) \qquad p \in M$$

where $T_p^{(1,0)}(\tilde{M})$ is the holomorphic part of $T_p^C(\tilde{M})$. Then there uniquely exists a (complex) vector bundle S over M whose fiber over p is just S_p provided that

$$(1.2) \qquad \dim S_q = \dim S_{q'}, \qquad q, q' \in M .$$

The pair (M,S) has the following properties:

(i) S is a subbundle of $T^C(M)$.

(ii) $S_p \cap \overline{S}_p = (0)$

(iii) S is completely integrable, i.e. $[\underline{S}, \underline{S}] \subset \underline{S}$.

Thus a real submanifold M of a complex manifold $\tilde{M}$ gives rise to a pair (M,S) with the properties (i), (ii), (iii) if (1.2) is satisfied.

Conversely, if M is an arbitrary real analytic manifold and if moreover S is a real analytic subbundle of $T^C(M)$ with properties (ii), (iii), then there exists a complex manifold $\tilde{M}$ of which M is a real submanifold and for which (1.1) holds.

Definition 1. A pair (M,S) *with the properties (i), (ii), (iii) is called a pseudo-complex-structure over* M .

In general the concept of a pseudo-complex-structure will provably be much wider than that of a real submanifold of a complex manifold satisfying (1.2). Especially a pseudo-complex-structure (M,S) constructed as above from an imbedding of M into a complex manifold will be called the pseudo-complex-structure *induced* by the imbedding under consideration.

Now let (M,S) be a pseudo-complex-structure and put

$$D_p = \{\mathrm{Re}\, x \,;\, x \in S_p\}$$

where $\mathrm{Re}\, x$ denotes the real part of x . We call the Pfaffian system D whose fiver over p is D_p *the first Pfaffian system* of (M,S) .

Definition 2. A pseudo-complex-structure (M,S) *is called*

μ-regular if there exists a series of Pfaffian systems $D = D^1 \subsetneqq D^2 \subsetneqq \cdots \subsetneqq D^\mu = T(M)$ *such that* $\underline{D}^i = [\underline{D}^{i-1}, \underline{D}] + \underline{D}^{i-1}$ $(2 \leq i \leq \mu)$.

The above D^i is uniquely determined for a given μ-regular pseudo-complex-structure (M,S) . So we call D^i *the i-th Pfaffian system* of (M,S) . By succesive use of the Jacobi identity

$$[\underline{D}^i, \underline{D}^j] \subseteq \underline{D}^{i+j} ,$$

where $D^i = T(M)$ for $i > \mu$. Thus the usual bracket operation for vector fields gives rise to a Lie algebra structure of $\Gamma(\underline{\;\;})$ ($\mathfrak{m} = \sum_{i=1}^{\mu} \underline{\mathfrak{g}}^i$ $\mathfrak{g}^i = D^i/D^{i-1}$) and the bracket operation of this shall be denoted by $[\ ,\ ]^*$. But we have then

$$[fX, gY]^* = fg[X, Y]^* \qquad X, Y \in \Gamma(\underline{\mathfrak{m}})$$

where f,g are C^∞ functions. Therefore $[\ ,\ ]^*$ induces also a Lie algebra structure of $\mathfrak{m}_p = \sum_{i=1}^{\mu} \mathfrak{g}_p^i$ for any point p of M , whose bracket operation we shall denote by $[\ ,\ ]_p^*$. This Lie algebra $\mathfrak{m}_p$ has the properties:

(i) $\mathfrak{m}_p$ is finite-diminsional.

(ii) $[\mathfrak{g}_p^i, \mathfrak{g}_p^j]_p^* \subseteq \mathfrak{g}_p^{i+j}$ $\qquad (\mathfrak{g}_p^i = 0$ for $i > \mu$)

(iii) $\mathfrak{m}_p$ is generated by $\mathfrak{g}_p^1$.

(iv) $[\mathrm{Re}\, ix, \mathrm{Re}\, iy]^*_p = [\mathrm{Re}\, x, \mathrm{Re}\, y]^*_p \qquad i=\sqrt{-1} \quad x,y \in S_p$.

We call this niopotent Lie algebra $\mathfrak{m}_p$ *the Levi-Tanaka form* of (M,S) at p (which will be short refered to L-T form at p). Put

$$L_p(x,y) = [\mathrm{Re}\, x, \mathrm{Re}\, iy]^*_p \qquad x,y \in S_p \ .$$

Then L_p is a $\mathscr{G}^2_p$-valued symmetric bilinear form by (iv).

Definition 3. Let (M,S) *be a regular pseudo-complex-structure and let* $\mathfrak{m}_p = \sum_{i=1} \mathscr{G}^i_p$, L_p *be as above. We say that* (M,S) *is totally indefinite at* p *if, for any non-zero linear form* α *of* $\mathscr{G}^2_p$ *, the (real-valured) symmetric bilinear form*

$$\langle \alpha, L_p(x,y) \rangle$$

is indefinite (not semi-definite). When (M,S) *is totally indefinite at every point of* M *, we simply call* (M,S) *totally indefinite.*

Total indefiniteness was suggested by the condition of sub-ellipticity in Hörmander [1] .

In order to state our main theorem we need still the concept of an analytic vector bundle over a pseudo-complex-structure. First of all we define the analyticity of a function.

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Definition 4. Let (M,S) *be a pseudo-complex-structure and let* f *be a* C^∞ *function on an open set* Ω *of* M *. We say that* f *is* (M,S)*-analytic in* Ω *if* $Xf = 0$ *for any* $X \in \Gamma_\Omega(\overline{S})$.

The (M,S)-analyticity is a local property, that is, f is (M,S)-analytic in $\Omega = \bigcup_\lambda \Omega_\lambda$ if and only if f is (M,S)-analytic in each Ω_λ .

Definition 5. Let (M,S) *is a pseudo-complex-structure and let* E *be a vector bundle over* M *and* $\mathcal{S}$ *be a subsheaf of* $\underline{E}$.

The pair $(E,\mathcal{S})$ *is called an analytic vector bundle over* (M,S) *if it satisfies the following conditions:*

(i) , if the values in E_p *of sections* $s_1,\cdots,s_m$ *of* $\mathcal{S}$ *over an open set* Ω *are linearly independent at every* p *of* Ω*, then* $\sum_{j=1}^{m} f_j s_j$ *is a section of* $\mathcal{S}$ *if and only if* $f_1,\cdots,f_m$ *are all* (M,S)*-analytic in* Ω.

(ii), for any $p \in M$ *, there exist a series of local sections* $s_1,\cdots,s_e$ *(*e = fiber dim. of E*) of* $\mathcal{S}$ *whose values in* E_p *are linearly independent.*

Remark 1. Let (M,S) be the induced pseudo-complex-structure by an imbedding of M into a complex manifold $\tilde{M}$ and let $\tilde{E}$ be an analytic vector bundle over $\tilde{M}$ and $\tilde{\mathcal{S}}$ denote the sheaf of germs of local analytic sections of $\tilde{E}$. Denote by $\mathcal{S}$ the smallest subsheaf of $\underline{E}$ $(E = \tilde{E}|_M)$ containing $\tilde{\mathcal{S}}|_M$ with property (i) of definition 5.

Then $(E,\mathscr{S})$ is an analytic vector bundle over M and we call $(E,\mathscr{S})$ *the restriction of* $\tilde{E}$.

Our main theorem is now stated as follows:

Theorem 1. Let (M,S) *be a totally indefinite regular pseudo-complex-structure over a compact manifold* M *and let* $(E,\mathscr{S})$ *be an analytic vector bundle over* (M,S) . *Then* $\Gamma(\mathscr{S})$ *is a finite-dimensional vector space (over* C *)* .

Now we proceed to formulate another important theorem. Let (M,S) be a pseudo-complex-structure. Then $\underline{\overline{S}}_p$ is a Lie subalgebra of $\underline{T^C(M)}_p$ since S is completely integrable. Then there exists a unique Lie algebra sheaf $\mathscr{A}'$ whose stalk $\mathscr{A}'_p$ over p is the normalizer of $\underline{\overline{S}}_p$ in $\underline{T^C(M)}_p$. The $\underline{\overline{S}}$ is a subsheaf of $\mathscr{A}'$ such that $\underline{\overline{S}}_p$ is an ideal of $\mathscr{A}'_p$. So $\mathscr{A}=\mathscr{A}'/\underline{\overline{S}}$ is again a Lie algebra sheaf of (M,S). We call this sheaf $\mathscr{A}$ *the tangential sheaf of* (M,S) . Our second theorem is the following.

Theorem 2. Let (M,S) *be a totally indefinite regular pseudo-complex-structure over a compact manifold* M *and let* $\mathscr{A}$ *denote its tangential sheaf. Then* $\Gamma(\mathscr{A})$ *is finite-dimensional.*

Remark 2. $\mathscr{A}$ is , of course, a subsheaf of $\underline{T^C(M)/S}$. If (M,S) is induced by an imbedding of M into a complex manifold $\tilde{M}$, $E = T^C(M)/\overline{S}$ can be naturally identified with the restriction to M of the (real) tangent bundle $T(\tilde{M})$ of $\tilde{M}$ and moreover $(E, \mathcal{A})$ is the restriction of $T(\tilde{M})$ in the sense of Remark 1. Therefore, in this case, Theorem 2 is a consequence of Theorem 1.

The key of the proofs of Theorems 1-2 is a consequence of Hörmander [3] stated as below.

Lemma 1. Let (M,S) *be a* μ*-regualr pseudo-complex-structure on* M *and suppose that* $X_p^1, \cdots, X_p^m$ $(X^i \in \Gamma(S))$ *span* S_p *for every* p *of* M. *Then, for any compact set* K *of* M *and for any* $0 < \varepsilon < \frac{1}{\mu}$, *there exists a positive constant such that*

$$\|u\|^2_{(\varepsilon)} \leq C\Big(\sum_{j=1}^{m} (\|X^j u\|^2_{(0)} + \|\overline{X}^j u\|^2_{(0)}) + \|u\|^2_{(0)}\Big) \qquad u \in C_0^\infty(K)$$

But what we really need is the following refinement of Lemma 1 for a totally indefinite regular pseudo-complex-structure.

Lemma 2. In addition to the hypothesis of lemma 1, assume that (M,S) *is totally indefinite. Then, for any compact subset* K *of* M *and for any* $0 < \varepsilon < \frac{1}{\mu}$, *there exists a positive constant* C *such that*

$$\|u\|^2_{(\varepsilon)} \leq C\Big(\sum_{j=1}^{m} \|\overline{X}^j u\|^2_{(0)} + \|u\|_{(0)}\Big)^2 \qquad u \in C_0^\infty(K)$$

This lemma follows from Lemma 1 by generalizing a technique of Kohn [4]. Now, in view of Lemma 2, the proof of Theorem 1 is almost evident. The proof of Theorem 2 still needs a minor differential geometric trick. We give these proofs in paragraph 3 .

2. Now we are in a position to apply Theorems 1-2 to the study of the automorphism group of a pseudo-complex-structure and to the study of an analytic vector bundle over a complex manifold.

Definition 6. Let (M,S) , (M',S') *be two pseudo-complex-structures* A *diffeomorphism* f *of* M *onto* M' *is called an isomorphism of* (M,S) *onto* (M',S') *if* $(df)_p$ *maps* S_p *isomorphically onto* $S'_{f(p)}$ *for any* $p \in M$. *An isomorphism of* (M,S) *onto itself is called an automorphism of* (M,S) .

As an application of Theorem 2 we have

Theorem 3. Let (M,S) *be a totally indefinite regular pseudo-complex-structure over a compact manifold* M. *Then the automorphism group of* (M,S) *is a Lie transformation group over* M *with respect to some natural topology.*

Proof. Let $\mathcal{H}$ denote the infinitesimal automorphism group of (M,S) (i.e. the Lie algebra of generators of 1-parameter subroups of the automorphism group of (M,S)). Let $\mathcal{A}$ be the tangential sheaf of (M,S) and ρ denote the natural projection of $\Gamma(T^C(M))$ to $\Gamma(T^C(M)/\overline{S})$. Then

$$\rho(\mathscr{U}) = \Gamma(\mathscr{A}) \cap \rho(\Gamma(T(M))$$

But ρ is one-to-one on $\Gamma(T(M))$ since $S_p \cap \bar{S}_p = (0)$. Thus $\mathscr{U}$ is finite-dimensional. Now a theorem of Palais [6] implies the conclusion of Theorem 3 .

Remark 3. This theorem was proved by Naruki [4] when (M,S) is 2-regular. It was shown by N. Tanaka that the finite-dimensionality of the automorphism group of a pseudo-complex-structure follows (without compactness assumption) under the assumptions of strong-regularity and non-degeneracy of (M,S). For all of these, we refer to [9] .

Now let (M,S) be the pseudo-complex-structure induced by an imbedding of M into a complex manifold $\tilde{M}$ and let $(E,\mathscr{S})$ be the restriction to M of an analytic vector bundle $(\tilde{E},\tilde{\mathscr{S}})$ over $\tilde{M}$. Assume

(2.1) fiber dim. of S + complex dim. of $\tilde{M}$ = real dim. of M .

Then the restriction mapping of $\Gamma(\tilde{\mathscr{S}})$ into $\Gamma(\mathscr{S})$ is one-to-one. In fact the condition (2.1) implies that $T_p(\tilde{M})$ is the unique complex subspace of $T_p(\tilde{M})$ which contains $T_p(M)$ for every p of M , and vice versa. We say that M is *generally imbedded* if the condition (2.1) is satisfied. As an application of Theorem 1 we have

Theorem 4. Suppose that a complex manifold $\tilde{M}$ *has a compact generally imbedded submanifold* M *and that the pseudo-complex-structure induced by the inclusion map of* M *into* $\tilde{M}$ *satisfies the hypotheses of Theorem 1. Then the space of global sections of an analytic vector bundle over* $\tilde{M}$ *is always finite-dimensional. In particular, the holomorphic automorphism group of* $\tilde{M}$ *is a Lie transformation group over* $\tilde{M}$ *with respect to some natural topology.*

This theorem was suggested by N. Tanaka. Note that for any neighbourhood of M in $\tilde{M}$ the conclusion of this theorem holds. So one can easily construct a complex manifold M which is neither compact, nor pseudo-concave and for which the conclusion of Theorem 4 holds. (See Example 2).

Example 1. We give some examples which clarify what the validity of the conclusion would be in Theorems 1-3 without the total indefiniteness. Set

$$M_r : |z_0|^2 + \sum_{j=1}^{r} |z_j|^2 - \sum_{j=r+1}^{n-1} |z_j|^2 - |z_u|^2 = 0$$

where $(z_0, \cdots, z_n)$ is the homogeneous coordinate of $P^n(C)$. The pseudo-complex-structure (M_r, S_r) induced by the inclusion $M_r \subsetneq P^n(C)$ is regular, but not totally indefinite when $r = 0$, or $r = n-1$. Since M_0 (or M_{n-1}) can be imbedded into C^n and since C^n is a Stein manifold, global sections of an analytic vector bundle over C^n, hence also over (M_0, S_0) (or (M_{n-1}, S_{n-1})) form an infinite-dimensional vector space.

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But the automorphism group of (M_0, S_0) (or of (M_{n-1}, S_{n-1})) is still Lie transformation group. This follows from the result of N. Tanaka remarked after Theorem 3. However this is not true for $M_0 \times P^m(C)$ in $P^n(C) \times P^m(C)$ $(m \geq 1)$ because of the infinite-dimensionality of the space of (M_0, S_0)-analytic functions. But, when $1 \leq r < n-1$, the hypothesis of Theorems 1-3 holds for $M_r \times P^m(C)$ in $P^n(C) \times P^m(C)$, although the pseudo complex structure is degenerate in the sense of [9].

Example 2. (due to N. Tanaka) Put

$$G = GL(n,C)\ ,\ K = U(n)$$

$$H = \{(a_{ij}) \in G\ ;\ a_{ij} = 0 \ \text{ if } \ |i-j| \geq 2 \ \text{ or if } \ i\text{: even}\}$$

Then $M = K/K \cap H$ is imbedded generally into the complex manifold $\tilde{M} = G/H$. M is obviously compact and the pseudo-complex-structure induced by this imbedding is $(n-1)$-regular and totally indefinite.

3. In this paragraph we shall prove Lemma 2 and Theorems 1-2. Before proceeding we need an algebraic lemma which makes the meaning of total indefiniteness more clear. Let V^* be the dual space of a n-dimensional complex vector space V. The (real) vector space of Hermitian forms on V (resp. V^*) shall be denoted by F (resp. F^*). The notation F^* may be justified by the fact that F^* can be naturally identified with the dual space of F. In fact, we can define the bilinear form on $F \times F^*$ by setting

$$(f, g^*) = \sum_{j,k=1}^{n} f(e_j, e_k) g^*(e_k^*, e_j^*) \qquad f \in F, \; g^* \in F^*$$

where $\{e_1^*, \cdots, e_n^*\}$ is the dual base of a base $\{e_1, \cdots, e_n\}$ of V. Note that $(\;,\;)$ is independent of the choice of $\{e_1, \cdots, e_n\}$ and that $(f, g^*) = 0$ for any $g^* \in F^*$ implies $f = 0$. These facts gives us the desired identification.

Lemma 3. Notations being as above, for a subspace of L, *the following statements are equivalent.*

(i) L *contains no semi-definite element except* 0.

(ii) $L^\perp$ *contains a (positive) definite element, where* $L^\perp = \{g^* \in F^* : (f, g^*) = 0$ *for any* $f \in L\}$.

This lemma follows from a corresponding theorem for quadratic forms due to L. L. Dines [1], but we prefer to give a direct proof in the Appendix.

Now let (M,S) be a totally indefinite regular pseudo-complex-structure and let F_p (resp. F_p^*) denote the vector space of Hermitian forms on S_p (resp. S_p^*). Put

$$\eta^\circ(x,y) = \langle\eta,[\mathrm{Re}(x),\mathrm{Re}(iy)]_p^*\rangle + i\langle\eta,[\mathrm{Re}(x),\mathrm{Re}(y)]_p^*\rangle$$
$$= -\frac{i}{2}\langle\eta,[x,\bar{y}]\rangle \qquad x,y \in S_p \quad \eta \in (\mathscr{G}_p^2)^*$$

where $[\ ,\ ]_p^*$ is the bracket operation of the Levi-Tanaka form $m_p = \sum_{i=1}^{\mu} \mathscr{G}_p^i$ at p. Then η° is a usual Hermitian form on S_p. Set

$$L(p) = \{\eta^\circ : \eta \in (\mathscr{G}_p^2)^*\} .$$

The subspace $L(p)^\perp$ of F_p^* being as in Lemma 3 consider the vector bundle $L^\perp$ with its fiber $L(p)^\perp$ over p. Since $L(p)^\perp$ contains certainly a positive definite element by Lemma 3, and since the set of positive definite elements in $L(p)^\perp$ is convex, there exists $g \in \Gamma(L\ )$ such that the value $g(p)$ at p of g is positive definite for any $p \in M$. Thus if $\{X^k\}_{k=1}^m \subset \Gamma(\bar{S})$ is a frame of $\bar{S}$ (,that is, if $\{X_p^k\}_{k=1}^m$ is a base of $\bar{S}_p$ for $p \in M$), we have

$$(3.1) \qquad \sum_{j,k=1}^{m} g_{jk}[X^j, \bar{X}^k] \in \Gamma(S \oplus \bar{S})$$

where $g_{jk}(p) = g(p)(Y_p^j, Y_p^k)$ ($\{Y_p^k\}_{k=1}^m$ is the dual base of $\{X_p^k\}_{k=1}^m$).

Proof of Lemma 2. Since the validity of (2.1) is entirely a local property, we may assume that $X_p^1,\cdots,X_p^\rho$

are linearly independent for any $p \in M$ replacing M by some suitable open subset of M. g_{jk} being as above, we shall define three norms $\| \quad \|$, $\| \quad \|_1$, $\| \quad \|_2$ on $\underbrace{C_0^\infty \times \cdots \times C_0^\infty}_{\rho}$ by setting

$$\|U\|^2 = \sum_{j=1}^{\rho} \|u_j\|^2_{(0)}$$

$$\|U\|_1^2 = \sum_{j,k=1}^{\rho} (g_{jk}u_j, u_k)_{(0)} \qquad U = (u_1, \cdots, u_\rho)$$

$$\|U\|_2^2 = \sum_{j,k=1}^{\rho} (g_{jk}u_k, u_j)_{(0)}$$

where $(\; , \;)_{(0)}$ is the polar form of $\| \quad \|^2_{(0)}$. $\| \quad \|$, $\| \quad \|_1$, $\| \quad \|_2$ are equivalent on $C_0^\infty(K)^\rho$ for any compact subset K of M, since $(g_{jk}(p))$ is positive definite. Note that, for any $X \in \Gamma(T^{\mathbb{C}}(M))$, there exists $c \in C^\infty(M)$ such that

$$(Xu, v)_{(0)} + (u, \bar{X}v)_{(0)} = (cu, v)_{(0)} \qquad u, v \in C_0^\infty(M) .$$

Therefore it follows from (3.1) that there exists $X_0 \in \Gamma(S \oplus \bar{S})$ such that

$$\|\mathfrak{X}u\|_1^2 - \|\bar{\mathfrak{X}}u\|_2^2 = \sum_{j,k} ((g_{jk}X^j u, X^k u) - (g_{jk}\bar{X}^k u, \bar{X}^j u))$$

$$= (u, X_0 u) ,$$

where we have put $\mathfrak{X}u = (X^1 u, \cdots, X^\rho u)$, $\bar{\mathfrak{X}}u = (\bar{X}^1 u, \cdots, \bar{X}^\rho u)$.

This implies that, for any compact subset K there

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exists a positive constant C such that

$$\|\bar{\mathcal{X}} u\|_2^2 \leq \|\mathcal{X} u\|_1^2 + C\|u\|_{(0)}(\|\mathcal{X} u\| + \|\bar{\mathcal{X}} u\|) \qquad u \in C_0^\infty(K) .$$

Since $\| \quad \|$, $\| \quad \|_1$, $\| \quad \|_2$ are equivalent on $C_0^\infty(K)^\rho$, we obtain

$$\|\bar{\mathcal{X}} u\|^2 \leq C(\|\mathcal{X} u\|^2 + \|u\|_{(0)}(\|\mathcal{X} u\| + \|\bar{\mathcal{X}} u\|)) \qquad u \in C_0^\infty(K)$$

for some other $C > 0$.

Using the inequality $|ab| \leq \delta|a|^2 + \frac{1}{\delta}|b|^2$ for sufficiently small δ, we obtain

$$(2.3) \qquad \|\bar{\mathcal{X}} u\|^2 \leq C(\|\mathcal{X} u\|^2 + \|u\|_{(0)}^2)$$

for another $C > 0$. On the other hand, Lemma 1 implies

$$(2.4) \qquad \|u\|_{(\varepsilon)}^2 \leq C(\|\mathcal{X} u\|^2 + \|\bar{\mathcal{X}} u\|^2 + \|u\|_{(0)}^2 \qquad u \in C_0^\infty(K).$$

Combining (2.3) and (2.4) we conclude that there exists a positive constant C such that

$$\|u\|_{(\varepsilon)}^2 \leq C(\|\mathcal{X} u\|^2 + \|u\|^2) \qquad u \in C_0^\infty(K) .$$

Q.E.D.

Proof of Theorem 1. Let (M,S) be a regular compact pseudo-complex-structure and let $(E,\mathcal{S})$ is an analytic vector bundle over (M,S). First we shall introduce the Sobolev norms $\| \ \|_{(\sigma)}$ on $\Gamma(E)$ suitable for our purpose. Let $\{\Omega_\alpha\}$ be a finite convering of M such that there exist sections of $\mathcal{S}$ $s_\alpha^1,\cdots,s_\alpha^e$ satisfying (ii) in Definition 5 for any $p\in\Omega_\alpha$ and let $\{\varphi_\alpha\}$ is a partition of unity subordinate to $\{\Omega_\alpha\}$. Define

$$\|U\|^2_{(\sigma)} = \sum_\alpha \sum_{j=1}^{e} \|\varphi_\alpha u_\alpha^j\|^2_{(\sigma)} \qquad u\in\Gamma(E)$$

where $U = \sum_{j=1}^{e} u_\alpha^j s_\alpha^j$.

Recall that U is an element of $\Gamma(\mathcal{S})$ if and only if u_α^j are all analytic. Applying (2.2) to $\varphi_\alpha u_\alpha^j$, we obtain

$$\|\varphi_\alpha u_\alpha^j\|^2_{(\varepsilon)} \leq C\Big(\sum_{k=1}^{\rho} \|(X^k\varphi_\alpha)u_\alpha^j\|^2 + \|\varphi_\alpha u_\alpha^j\|^2\Big) \qquad U\in\Gamma(\mathcal{S})$$

since $X^k u_\alpha^j = 0$ by the (M,S)-analyticity of u_α^j.

Therefore there exists a positive constant C such that

$$\|U\|^2_{(\varepsilon)} = \sum_{\alpha,j} \|\varphi_\alpha u_\alpha^j\|^2_{(\varepsilon)} \leq C\|U\|^2_{(0)} \qquad U\in\Gamma(\mathcal{S}).$$

By the generalized Rellich lemma. $\Gamma(\mathcal{S})$ is finite-dimensional.

Q.E.D.

Proof of Theorem 1.6. Choosing the (complex-valued) 1-forms $\zeta^1,\cdots\zeta^\pi$ such that $\zeta_p^1,\cdots,\zeta_p^\pi$ span $\overline{S}_p^\perp$ where $\overline{S}^\perp$ is the bundle of annihilators in $T^{*\mathbb{C}}(M)$ of $\overline{S}$, we introduce Sobolev norms $\|\quad\|_{(\sigma)}$ on $\Gamma(T^{\mathbb{C}}(M)/\overline{S})$ by setting

$$\|s\|_{(\sigma)} = \sum_{j=1}^{\pi}\|\zeta^j(X)\|^2_{(\sigma)} \qquad s\in\Gamma(T^{\mathbb{C}}(M)/\overline{S})$$

where we have chosen a vector field X such that $\rho(X) = s$ (ρ: the canonical projection of $\Gamma(T^{\mathbb{C}})$ onto $\Gamma(T^{\mathbb{C}}/S)$.). The right hand side is independent of the choice of such a X, so $\|\quad\|_{(\sigma)}$ is well defined.

Suppose that $\rho(X)\in\Gamma(\mathcal{A})$ and that $X_p^1,\cdots,X_p^\rho$ where $X^j\in\Gamma(\overline{S})$ $(j = 1,2,\cdots,\rho)$ span $\overline{S}_p$ for any $p\in M$. Then

$$\zeta^j(X^k) = 0 \ .$$

Taking the Lie derivative of this with respect to X, we obtain

$$0 = \langle L_X(\zeta^j)|X^k\rangle + \langle\zeta^j|[X^k,X]\rangle$$

$$= \langle L_X(\zeta^j)|X^k\rangle$$

since $[X^k,X]\in\Gamma(\overline{S})$ by $\rho(X)\in\Gamma(\mathcal{A})$.

This can be rewritten in the form

(2.5) $\quad X^k(\zeta^j(X)) = \langle X^k \lrcorner d\zeta^j, X\rangle$.

From the complete integrability of $\overline{S}$, it follows

$$X^k \lrcorner d\zeta^j \in \Gamma(\overline{S}^{\perp}) .$$

This together with (2.5) implies that there exists a positive constant C such that

$$\sum_{k=1}^{\rho} \|X^k(\zeta^j(X))\|^2_{(0)} \le C\|s\|^2_{(0)}$$

where $s = \rho(X)$.

Applying Theorem 2.2 to this we obtain

$$\|\zeta^j(X)\|^2_{(\varepsilon)} \le C\|s\|^2_{(0)}$$

for some $C > 0$, and hence

$$\|s\|^2_{(\varepsilon)} = \sum_{j=1}^{\rho} \|\zeta^j(X)\|^2_{(\varepsilon)} \le C\|s\|^2_{(0)} \qquad s \in \Gamma(\mathcal{A})$$

for another $C > 0$. By the generalized Rellich lemma, $\Gamma(\mathcal{A})$ is finite-dimensional. Q.E.D.

Appendix

In this appendix we shall prove Lemma 3. Let $(\ , \ )$ be a (fixed) positive definite hermitian form on an n-dimensional complex vector space V. For $A \in \mathrm{Hom}_{\mathbb{C}}(V,V)$ we define $A^* \in \mathrm{Hom}_{\mathbb{C}}(V,V)$ by the following identity:

$$(Au,v) = (u,A^*v) .$$

A is called a hermitian endomorphism (with respect to $(\ , \ )$) if $A = A^*$. Given a Hermitian form f on V, there exists one and only one Hermitian endomorphism A_f such that

$$f(u,v) = (A_f(u),v) .$$

We denote by F_e the vector space of Hermitian endomorphisms. F_e can be then identified with F by the mapping $f \to A_f$. We also introduce an inner product $(\ , \ )$ of F_e by putting

$$(A,B) = Sp(A,B) .$$

Then Lemma 3 is equivalent to the following.

Lemma 3'. For a subspace L *of* F_e *the following conditions are equivalent.*

(i) L *contains no semi-definite element except* 0.

(ii) $L^{\perp}$ *contains a (positive) definite element.*

Here we have put $L^{\perp} = \{F_e \ni A \ ; \ (A,B) = 0 \quad {}^{\forall}B \in L\}$.

Proof of (i) $\Rightarrow$ (ii). Assume that $L^{\perp}$ contains no definite element. Note that the set of positive definite Hermitian endomorphism P is an open convex cone. Since the linear space $L^{\perp}$ does not interesect with P, there exists a hyperplane H of F_e containing $L^{\perp}$ such that H does not meet P, in view of a Theorem of Minkowsky [5]. Since $H^{\perp} \subseteq L$, a generator A of $H^{\perp}$ is not semi-definite. Let $e_1, \cdots, e_n$ be the unit eigen vectors of A and let $\lambda_1, \cdots, \lambda_n$ be the corresponding eigenvalues. We may assume $\lambda_1 \geq \cdots \geq \lambda_n$. Then $\lambda_1 > 0$, $\lambda_n < 0$. Therefore there exist positive numbers $\mu_1, \cdots, \mu_n$ such that $\lambda_1\mu_1 + \cdots + \lambda_n\mu_n = 0$. If we define a Hermitian endomorphism B by setting

$$Be_j = \mu_j e_j \qquad (j=1,2,\cdots,n) ,$$

then $(B,A) = 0$ and hence $B \in (H^{\perp})^{\perp} = H$. But the positive definiteness of B contradicts to $H \cap P = \phi$, thus (i) $\Rightarrow$ (ii) is proved.

Proof of (ii) $\Rightarrow$ (i). Let B be a positive definite element of $L^{\perp}$ and let A be a semi-definite element of L. Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be the eigenvalues of A and let $e_1, \cdots, e_n$ be the corresponding eigen vectors. If we set $(Be_j, e_j) = \mu_j$ $(j=1,2,\cdots,n)$, we obtain

$$\lambda_1\mu_1 + \cdots + \lambda_n\mu_n = 0 .$$

But this is impossible unless $\lambda_1 = \cdots = \lambda_n = 0$ since $\mu_j > 0$ by the positive definiteness of B. Thus $A = 0$ and (ii) $\Rightarrow$ (i) is proved.

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