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Martingale Transforms and Linear controlled stochastic processes

Seigo Kanō
Res. Inst. of Fund. Inf. Sci.
Kyushu University

1. Introduction

The theory of linear controlled stochastic processes has been developed in the previous paper [1]. Linear controls are considered as a set of linear transformations of stochastic process which is given in advance, successively observed and desired to have some properties. In practice, we may decide the types of linear controls on the bases of the value of each obsevation.

In this paper, the theory of martingale transforms which was developed in [2] by Bruckholder, are generalized and applied to show the almost everywhere convergence of linear controlled stochastic process which is obtained from martingale or submartingale process.

It showld be emphasized that our aim of controls is to get the original stochastic process to take some given properties.

2. Martingale Transforms

Let $X=(x_1, x_2, \cdots)$ be random variables on probability space

(Ω, F, P). X is said to be martingale process if following conditions are satisfied.

1. There exist a sequence of σ-field such that

   $F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots \subset F$ and x_n is measurable with respect to F_n, $n=1, 2, \cdots$.

2. $E\{|x_n|\}<\infty$, $n=1, 2, \cdots$.

3. $E\{x_n | x_1, \cdots, x_{n-1}\}=x_{n-1}$. $n=2, 3, \cdots$.

If the last condition is replaced by

$$E\{x_n | x_1, \cdots, x_{n-1}\} \geqq x_{n-1}, \quad n=2, 3, \cdots,$$

then X is called submartingale process.

In Burkholder [2], martingale transform is defined as follows.

For any martingale process X and sequence of random variables $(v_1, v_2, \cdots)$ in which v_n is measurable with respect to F_{n-1}, $n=1, 2, \cdots$, we put

$$d_1=x_1, \quad d_2=x_2-x_1, \quad \cdots, \quad d_n=x_n-x_{n-1},$$

and $$Y_n=\sum_{k=1}^{n} v_k d_k.$$

This the martingale transform from X to $Y=(y_1, y_2, \cdots)$. In general, Y need not be martingale process but if $E\{|y_n|\}<\infty$, $n=1, 2, \ldots$, then Y becomes martingale process. Transforms of submartingales may be defined similarly. Following theorem for convergence of Y holds.

Theorem 2.1. (Burkholder)

If X is L_1 bounded martingale process then Y converges almost *everywhere on the set* $\{\omega;\ \sup_n |v_n(\omega)| < \infty\}$.

This result implies the corresponding result for submartingale process.

Now we extends above martingale transforms to the case in which double sequences of random variables $V=(v_{ij})$ are applied instead of $(v_1,\ v_2,\ \cdots)$.

Let $V=(v_{ij})$ be matrix of random variables v_{ij}, $j=1,\ 2,\ \cdots,\ i$, $i=1,\ 2,\ \cdots$, where $v_{ii}=1$, $i=1,\ 2,\ \cdots$, and $v_{ij}=0$, $j>i$.

In the following we assume that v_{ij} are bounded and measurable with respect to F_i, $j=1,\ 2,\cdots,i$, $i=1,\ 2,\ \cdots$.

In this case, martingale transforms from X to Y is defined as

(2.1) $$Y_m=\sum_{k=1}^{m} V_{mk}d_k.$$

We obtain the following,

Theorem 2.2.

Let X be L_1 bounded martingale. We assume that

(2.2) $$E\{V_{mk}|V_{ij},\ i=1,2,\cdots,\ m-1,\ j=1,2,\cdots,i\}=V_{m-1,k},$$
$$k=1,\ 2,\cdots,\ m-1,$$

and

(2.3) $$E\{(V_{mk}-V_{m-1,k})^2\ |V_{ij},\ i=1,2,\cdots,m-1,\ j=1,2,\cdots,i\}$$
$$=O\left(\frac{1}{m^{4+\delta}}\right),\quad k=1,\ 2,\cdots,m-1$$

where δ is any positive number.

Then Y converges almost everywhere.

To prove this theorem, we need two lemmas.

Lemma 1.

Let X be L_2 bounded martingale. If V satisfies the conditions of Theorem 2.2, then Y converges almost everywhere.

Proof.

We begin with the calculation of $E\{Y_{m+1} \mid Y_1, \cdots, Y_m\}$, which is reduced to

$$E\{\sum_{k=1}^{m+1} V_{m+1,k}(X_k - X_{k-1}) \mid Y_1, \cdots, Y_m\}$$
$$= E\{(X_{m+1} - X_m) \mid X_1, \cdots, X_m\} + E\{\sum_{k=1}^{m} V_{m+1,k}(X_k - X_{k-1}) \mid X_1, \cdots, X_m\}$$
$$= E\{X_{m+1} \mid X_1, \cdots, X_m\} - X_m + \sum_{k=1}^{m} E\{V_{m+1,k} \mid X_1, \cdots, X_m\}(X_k - X_{k-1})$$
$$= \sum_{k=1}^{m} V_{m,k}(X_k - X_{k-1}) = Y_m$$

Using this relation, we obtain

$$E\{(Y_i - Y_{i-1})(Y_j - Y_{j-1})\} = E\{E\{Y_i - Y_{i-1} \mid Y_1, \cdots, Y_{i-1}\}(Y_j - Y_{j-1})\}$$
$$= 0, \qquad i > j.$$

Hence, if we put $Y_0 = 0$ then

$$E\{Y_n^2\} = E\{(\sum_{k=1}^{n} (Y_k - Y_{k-1}))^2\}$$
$$= \sum_{k=1}^{n} E\{(Y_k - Y_{k-1})^2\}$$
$$= \sum_{k=1}^{n} E\{(\sum_{i=1}^{k} V_{ki} d_i - \sum_{i=1}^{k-1} V_{k-1,i} d_i)^2\}$$

$$= \sum_{k=1}^{n} E\{(d_k + \sum_{i=1}^{k-1} (V_{ki} - V_{k-1,i}) d_i)^2\}$$

$$= E\{(\sum_{k=1}^{n} d_k)^2\} + \sum_{k=2}^{n} E\{2d_k \sum_{i=1}^{k-1} (V_{ki} - V_{k-1,i}) d_i\}$$

$$+ \sum_{k=2}^{n} E\{(\sum_{i=1}^{k-1} (V_{ki} - V_{k-1,i}) d_i)^2\}.$$

In order to show $E\{Y_n^2\}$ bounded, each term of the last expression, say I_1, I_2 and I_3 respectively, is evaluated.

From the assumption of lemma 1,

$$I_1 = E\{X_n^2\} < M.$$

where M is a constant.

For the second therm, we have

$$|I_2| \leqq 2 \sum_{k=2}^{n} \sum_{i=1}^{k-1} | E\{d_k (V_{ki} - V_{k-1,i}) d_i\}|$$

$$\leqq 2 \sum_{k=2}^{n} \sum_{i=1}^{k-1} \left[E\{d_k^2\} E\{(V_{ki} - V_{k-1,i})^2 d_i^2\}\right]^{\frac{1}{2}}$$

$$\leqq 2\sqrt{2M} \sum_{k=2}^{n} \sum_{i=1}^{k-1} \left[E\{E\{(V_{ki} - V_{k-1,i})^2 | d_1, \cdots, d_{k-1}\} d_i^2\}\right]^{\frac{1}{2}}.$$

From the assumption on V, this is less than

$$4M \sum_{k=2}^{n} \sum_{i=1}^{k-1} O(k^{-(2+\frac{\delta}{2})})$$

which is bounded.

Now we consider I_3.

$$|I_3| \leqq \sum_{k=2}^{n} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} | E\{(V_{ki} - V_{k-1,i})(V_{kj} - V_{k-1,j}) d_i d_j\}|$$

$$\leqq \sum_{k=2}^{n} \sum_{i,j=1}^{k-1} E\left[\{E\{(V_{ki} - V_{k-1,i})^2 d_i^2 \mid d_1, \cdots, d_{k-1}\} \times\right.$$

$$\times E\{(V_{kj}-V_{k-1,j})^2 d_j^2 \mid d_1,\cdots,d_{k-1}\}\}^{\frac{1}{2}}\Big)$$

$$\leqq \sum_{k=2}^{n} \sum_{i,j=1}^{k-1} O(k^{-(4+\delta)}) E\{|d_i d_j|\}$$

$$\leqq 2M \sum_{k=2}^{n} O(k^{-(2+\delta)})$$

which is again bounded.

Therefore we conclude that

$$\sup_n E\{Y_n^2\} < \infty.$$

Hence Y converges almost everywhere.

Lemma 2.

Let X be uniformly bounded submartingale. If V satisfies the conditins of Theorem 3.1, then Y converges almost everywhere.

Proof.

Since X is uniformly bounded, we may assume $X \geqq 0$ by adding certain constant to each term of X.

So that,

$$E\{X_{n-1} d_n\} = E\{X_{n-1} E\{d_n \mid X_1,\cdots,X_{n-1}\}\} \geqq 0, \tag{2.4}$$

where

$$E\{d_n \mid X_1,\cdots,X_{n-1}\} = E\{X_n - X_{n-1} \mid X_1,\cdots,X_{n-1}\}$$
$$= E\{X_n \mid X_1,\cdots,X_{n-1}\} - X_{n-1} \geqq 0.$$

From (2.4) we obtain,

$$E\{X_n^2\} = E\{(X_{n-1}+d_n)^2\} \geqq E\{X_{n-1}^2\} + E\{d_n^2\}.$$

So that,

$$E\{X_n^2\} - E\{X_{n-1}^2\} \geqq E\{d_n^2\}$$

and

$$E\{X_n^2\} \geqq \sum_{k=1}^{n} E\{d_k^2\}, \quad n \geqq 1.$$

Now we define new martingale process $\hat{X}$ and its martingale transforms $\hat{Y}$, using $\hat{d}_n$ which is given by

$$\hat{d}_1 = d_1, \quad \hat{d}_n = d_n - E\{d_n \mid X_1, \cdots, X_{n-1}\}.$$

Then it follows that

$$\hat{X}_n = \sum_{k=1}^{n} \hat{d}_k$$

is a martingale process and

$$\hat{Y}_n = \sum_{k=1}^{n} v_{nk} \hat{d}_k$$

is martingale transforms of $\hat{X}$.

Since

$$\begin{aligned} E\{\hat{d}_n^2\} &= E\{(d_n - E\{d_n \mid X_1, \cdots, X_{n-1}\})^2\} \\ &= E\{d_n^2\} - E\{(E\{d_n \mid X_1, \cdots, X_{n-1}\})^2\} \\ &\leqq E\{d_n^2\}, \quad n \geqq 2, \end{aligned}$$

we have

$$E\{\hat{X}_n^2\} = \sum_{k=1}^{n} E\{\hat{d}_k^{\,2}\} \leqq \sum_{k=1}^{n} E\{d_k^2\} \leqq E\{X_n^2\}, \quad n \geqq 1.$$

It follows that $\hat{X}$ is L_2 bounded and $\hat{Y}$ converges almost everywhere (Lemma. 1.)

On the other hand we get

$$\hat{X}_n = \sum_{k=1}^{n} \hat{d}_k = \sum_{k=1}^{n} \{d_k - E\{d_k \mid X_1, \cdots, X_{k-1}\}\}$$
$$= \sum_{k=1}^{n} d_k - \sum_{k=2}^{n} E\{d_k \mid X_1, \cdots, X_{k-1}\}$$
$$= X_n - \sum_{k=2}^{n} E\{d_k \mid X_1, \cdots, X_{k-1}\}.$$

Therefore convergence of X and $\hat{X}$ implies convergence of

$$\sum_{k=2}^{n} E\{d_k \mid X_1, \cdots, X_{k-1}\}.$$

Since V is bounded, convergence of

$$\sum_{k=2}^{n} E\{d_k \mid X_1, \cdots, X_{k-1}\}$$

conclud the convergence of

$$\sum_{k=2}^{n} V_{n,k} E\{d_k \mid X_1, \cdots, X_{k-1}\}.$$

Now $\hat{Y}_n$ is expressed as follows.

$$\hat{Y}_n = \sum_{k=1}^{n} V_{nk} \hat{d}_k = \sum_{k=1}^{n} V_{nk} (d_k - E\{d_k \mid X_1, \cdots, X_{k-1}\})$$
$$= \sum_{k=1}^{n} V_{nk} d_k - \sum_{k=1}^{n} V_{nk} E\{d_k \mid X_1, \cdots, X_{k-1}\}$$
$$= Y_n - \sum_{k=1}^{n} V_{nk} E\{d_k \mid X_1, \cdots, X_{k-1}\}.$$

In this last equation, convergence of $\hat{Y}$ and

$$\sum_{k=1}^{n} V_{nk} E\{d_k \mid X_1, \cdots, X_{k-1}\}$$

implies the convergence of Y which was to be proved.

Proof of Theorem 2.2.

Any martingale process is expressed as the difference of two non-negative martingale processes, by a result due to Krickeberg, [4]. Then we may assume $X \geq 0$ without any loss of generality.

Let $c>0$. Then $\hat{X}_n = -\min(X_n, c)$, $n \geq 1$, defines a uniformly bounded submartingale $\hat{X}$. Let $\hat{Y}$ be the transform of $\hat{X}$ under $\{-V_{nk}\}$. By lemma 2, $\hat{Y}$ conveges almost everywhere.

Since $Y = \hat{Y}$ if $\sup_n |X_n(\omega)| < c$, Y converges almost everywhere on the set $\{\omega; \sup_n |X_n(\omega)| < c\}$. Now X is L_1 bounded martingale process and $P\{\sup_n |X_n(\omega)| < c\} = 1$.

Therefore Y converges almost everywhere and proof of Theorem 2.2 is complete.

3. Convergence of linear controlled stochastic processes.

Let $X=(x_1, x_2, \cdots)$ be a discrete parameter stochastic process. We generalize the definition of linear controlled stochastic process $Y=(y_1, y_2, \cdots)$ given in [1]. Let $A=(\xi_1, \xi_2, \cdots)$ be target stochastic process. We put $A_n=(\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots)$.

Let $c_n=(c_{n1},c_{n2},\cdots, c_{nn}, 0, 0,\cdots)^T$ be n-th stage control vector where c_{ni}, $i=1, \cdots, n$ are random variables dependent on the values $x_1, y_{11}, \cdots, y_{n-1,n-1}$.
We put $C_n= I + (\underbrace{0, 0,\cdots 0}_{n} \; c_n, c_n, \cdots)^T$ where I is unite infinite matrix and 0 is infinite zero vector. Now we define sequence of stochastic processes $Y_n=(y_{n1},y_{n2},\cdots)$ as follows. We take $Y_1=X$. If the value of y_{mm} is observed then $Y_{m+1}=(Y_m-A_m)\, C_m^T + A_m$. And so on. Finally we put $Y=(y_{11},y_{22},\cdots)$.

Before proceeding to the main theorem, we state the theorem given in [1], where we were concerned only with the real matrix C and real vector A. The theorem gives the general form of linear controlled stochastic process in non-random case and enables us to calculate v_{ij}, $i>j$ from c_{ij}, $i\geq j$ in random case.

Theorem 3.1.

The linear controlled stochastic process $Y=(y_1,y_2,\cdots)$, *obtained from original stochastic process* X, *is given by*

$$y_{m+1} = x_{m+1} + \sum_{k=1}^{m} e_{m,k}(\xi_k-x_k),$$

where $e_{m,k}$, $k=1,2,\cdots,m$ *satisfy the following equations,*

$$e_{m,m} = c_{m,m}$$

$$e_{m,m-1} = \sum_{k=m-1}^{m} c_{k,m-1} - e_{m,m}c_{m-1,m-1}$$

$$e_{m,m-2} = \sum_{k=m-2}^{m} c_{k,m-2} - e_{m,m} \sum_{k=m-2}^{m-1} c_{k,m-2} - e_{m,m-1} c_{m-2,m-2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$e_{m,1} = \sum_{k=1}^{m} c_{k,1} - e_{m,m} \sum_{k=1}^{m-1} c_{k,1} - \cdots - e_{m,2} c_{1,1} .$$

If, as stated in the biginning of this section, the elements of C are random variables and A is a stochastic process, then our main theorem is obtained.

Theorem 3.2.

Let stochastic process X-A be L_1 bounded martingale. If the elements v_{ij}, $j \leq i$, $i=1,2,\cdots$, of matrix V are given by

$$v_{m+1,k} = 1 - e_{mm} - e_{m,m-1} - \cdots - e_{m,k+1} - e_{m,k},$$

$$v_{m+1,m+1} = 1 \qquad k=1,2,\cdots,m, \quad m=1,2,\cdots,$$

and satisfy (2.2) and (2.3).

Then stochastic process Y-A converges almost everywhere, where Y is linear controlled stochastic process obtained from X.

Proof.

Stochastic process Y-A is reduced to

$$Y_{m+1}-\xi_{m+1} = (x_{m+1}-\xi_{m+1}) - \sum_{k=1}^{m} e_{m,k}(x_k-\xi_k)$$

$$= \sum_{k=1}^{m+1} (1-e_{mm}-\cdots-e_{m,k})\{(x_k-\xi_k)-(x_{k-1}-\xi_{k-1})\}$$

(3.1)

$$= \sum_{k=1}^{m+1} v_{m+1,k}\{(x_k-\xi_k)-(x_{k-1}-\xi_{k-1})\}$$

where we have put $x_0=\xi_0=0$. Then, applying Theorem 2.2 to (3.1), our result of the theorem is immediatly obtained.

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