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Laplace's calculations of length of the meter

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Laplace's calculations to derive the length of the meter are described. It is noted that any form of the method of least squares is not used in his determination of the length of the meter. Instead, he used two equations to solve for two unknowns that are necessary to determine the length of the meter. In Appendix 1, these two equations are derived as the asymptotic expansion of the elliptic integral derived by Bessel. Some estimation results obtained by the method of least squares, namely, results of calculations by Legendre and by Stigler, are summarized in Appendix 2.

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JEL Classification Numbers: C10, C13, C20

1 Introduction

The determination of the length of the meter was a major scientific endeavor after the French revolution. First, the meter was determined to be a ten-millionth of the meridian quadrant of the earth. Then, a part of the meridian arc between Dunkirk and Montjoui (Barcelona), which spans an angle of approximately ten degrees of latitudes was measured in the famous expedition by Delambre and Méchain; the expedition spanned seven years from 1792 (see Hellman (1936) and Alder (2002)). The actual meter was determined on the basis of the value calculated by Laplace (1829–1839) (p. 60, Stigler (1986)). Details are given in the volume two of Celestial Mechanics (Laplace 1829–1839). Other calculations include those performed by Gauss (1799), as translated by Stigler (1981, p. 466), and Legendre (1805). Both used the French data set. These two calculations lead to the famous priority dispute on the discovery of the method of least squares. See Stigler (1977, 1981) and Celmiņš (1998), who find flaws in the value of the ellipticity derived by Gauss. Appendix 2 summarizes the estimation results obtained by the method of least squares, including those of Legendre and Gauss.

Laplace's calculations related to the determination of the meter can be found from p. 417 to p. 468 in volume 2 (Legendre 1805). He analyzed the data sets by the three methods. One is the method of least absolute deviations (currently abbreviated as LAD). Laplace's method is different from the current LAD analysis since the constant term is adjusted so that the sum of deviations is zero. The second is the method of minimizing the maximum error of a regression equation. This is used to examine the validity of the hypothesis that the shape of the earth is ellipsoidal. The maximum error should be in a reasonable range if the earth is indeed ellipsoidal. Neither of these methods is used to determine the length of the meter. See Laplace (1799) for details on these two methods. The length of the meter is determined by the third method. A nonlinear function for the length of any meridian arc between two parallels of latitude is linearized, and a set of two equations is derived. The Bouguer data set obtained by the expedition to Peru (1734–1744) and the French data set were substituted in one equation, and the ellipticity of the earth was determined. The French data set and the estimated ellipticity were substituted in the second equation, and the length of the meridian quadrant was calculated. See Terrall (2006) and Trystram (1979) for details on the expedition to Peru. The French and Peruvian data sets are tabulated in Appendix 3.

The two basic equations used by Laplace are interpreted as the asymptotic expansion of the elliptic integral of Bessel (1837) in Appendix 1.

The derived ellipticity was $\frac{1}{334}$, and the derived meridian quadrant from the pole to the equator was 2565370 double toises, where toise is the French unit of time, and the meter, which is a ten-millionth of this length (Laplace (1799), p. 464 and p. 465.), is derived as follows:

$$meter = 0.2565370 \ double\ toises = 0.513074 \ toise. \ [2035] \qquad (1)$$

One double toise is approximately 3.9 m.

The equation and footnote numbers in brackets and parentheses, respectively, correspond to those in Volume 2 of Celestial Mechanics (Laplace 1829–1839). This includes many equations and footnotes added by Bowditch, who not only translated the book into English but also added intermediary steps to Laplace's calculations. Most of the symbols used are identical to those in the original, but some have been changed to new to avoid confusion.

2 Ellipticity

Assuming that a circumference of the path traced by the meridian around the earth is an ellipse, Laplace calculated the ellipticity and the major as well as minor radius of the earth using the French and Peruvian data sets.

The relationship between a radius of an ellipse and a latitude ψ of the radius is given as

$$1-\alpha(\sin\psi)^2$$

in the footnote (1480) on p. 462 (This equation follows from equation [1965] by setting $h=0$.). Then. the radius at the pole (minor radius) is

$$1-\alpha$$

and that at the equator is 1 (major radius). This implies that α is the ellipticity. In general,

$$\alpha=\frac{semimajor-semiminor}{semimajor}=\frac{k'-k}{k'} \tag{2}$$

where the length of the semimajor and semiminor axes of the ellipse are defined to be k' and k, respectively (See [1969d] in the book. Later, this ellipticity is alternatively defined as ε' in equation (12).).

2.1 The first equation (Footnotes (1480) on p. 462 and (1483) on p. 464)

The meridian arc length s' of a ν-degree interval with a mean latitude L is approximated as

$$\begin{aligned} s' &=\nu-\frac{\alpha\nu}{2}-\frac{3}{2}\alpha\nu\cos(2L) \\ &=\nu\left(1-\frac{\alpha}{2}\right)\left\{1-\frac{3}{2}\alpha\cos(2L)\right\}. \ [2032b] \end{aligned} \tag{3}$$

See the figure 1 where $\nu=\psi'-\psi$, and $\mathrm{L}=(\psi'+\psi)/2$. This is the first equation that is used to calculate the ellipticity (This equation is derived from [1966] by setting $h=0$. Equation [1966] includes $\cos^2 L$, but equation [1965k] should be referred to. See also equation (32) in Appendix 1 of this note; the equation is derived from the elliptic integral of Bessel (1837)).

The arc length of a degree interval is obtained from equation (3) as

$$1^\circ\left(1-\frac{1}{2}\alpha\right)\cdot\left(1-\frac{3}{2}\alpha\cdot\cos(2L)\right). \ [2033e] \tag{4}$$

By selecting the Peruvian length per degree (s'/ν, where s' *and* ν are listed in the table in Appendix 3) and setting the mean degree to 0°, this equation is modified as

$$25538.85=\left(1-\frac{1}{2}\alpha\right)\cdot\left(1-\frac{3}{2}\alpha\right). \ [2033f] \tag{5}$$

By selecting the French length per degree (s'/ν) and setting the mean degree to

46.19943°, we have

$$25658.28=\left(1-\frac{1}{2}\alpha\right)\cdot\left(1-\frac{3}{2}\alpha\cdot\cos\left(92.39886^{\circ}\right)\right). \tag{6}$$

The center angles of meridian arcs at Peru and France are 3.1170° and 9.6738°, respectively (pp. 443–444). From the ratio of these two equations, it followsthat

$$\alpha=\frac{1}{334} \tag{7}$$

which is the value derived by Laplace. The modern value is 1/ 298.257222101. The Peruvian data set is used only for this calculation.

3 The meridian quadrant

Bowditch used another function for the arc length between two latitudes to find the Laplace value of the meridian quadrant. He approximated the function first in terms of ε' by terms of order $O(\varepsilon')$ and then (confusingly) in terms of ε by terms of order $O(\varepsilon^2)$ (ε and ε' are alternative definitions of the ellipticity. ε' is the same as α in Section 2, and has been chosen to use the same notations as Laplace. Later, he also used ρ for the ellipticity.).

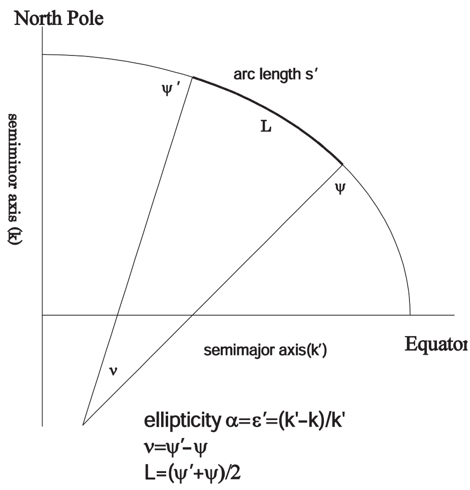

Figure 1 The meridian quadrant

3.1 The second equation (Footnote (1426) on pp. 417–421)

The equations necessary to calculate the arc length between two parallels of latitude are derived. Let the ellipticity be expressed *in parts of the polar radius k* ([1969l] in p. 418)

$$\varepsilon = \frac{\text{semimajor} - \text{semiminor}}{\text{semiminor}} = \frac{\mathrm{k}' - \mathrm{k}}{\mathrm{k}} = \frac{\mathrm{k}'}{\mathrm{k}} - 1,$$

where k' is the equatorial radius. The function for the arc length s' between two parallels of latitude ψ' and ψ is

$$s' = k\left\{\left(1 + \frac{1}{2}\varepsilon + \frac{1}{16}\varepsilon^2\right)(\psi' - \psi) - \frac{3}{4}\varepsilon\,(\sin(2\psi') - \sin(2\psi)) + \frac{15}{64}\varepsilon^2\,(\sin(4\psi') - \sin(4\psi))\right\}. \text{ [1969}o\text{]} \tag{8}$$

The meridian quadrant S is as follows. which is obtained by setting $\psi' = \pi/2$ and $\psi = 0$ in equation (8):

$$S = k\frac{1}{2}\pi\left(1 + \frac{1}{2}\varepsilon + \frac{1}{16}\varepsilon^2\right), \text{ [1969}u\text{]} \tag{9}$$

and the mean length per degree is

$$s = k\left(1 + \frac{1}{2}\varepsilon + \frac{1}{16}\varepsilon^2\right). \tag{10}$$

This equals the length of a degree at $\frac{\pi}{4}$ since $\sin(2\psi') - \sin(2\psi) = 0$ and similarly, $\sin(4\psi') - \sin(4\psi) = 0$. Neglecting terms of order $O(\varepsilon^3)$, equation(8) is expressed as

$$s' = s\left\{(\psi' - \psi) - \left(\frac{3}{4}\varepsilon - \frac{3}{8}\varepsilon^2\right)(\sin(2\psi') - \sin(2\psi)) + \frac{15}{64}\varepsilon^2(\sin(4\psi') - \sin(4\psi))\right\}. \text{ [2034d]} \tag{11}$$

This is the second equation of Laplace for determining the length of the meter and is given in the footnote (1984) of p. 465. This equation follows from equation (30) in Appendix 1 of this note.

An alternative definition of the ellipticity ε' (*oblateness in parts of the equatorial radius* as mentioned in the last line on p. 416) is

$$\varepsilon' = \frac{\textit{semimajor} - \textit{semiminor}}{\textit{semimajor}} = \frac{k' - k}{k'} = 1 - \frac{k}{k'} \tag{12}$$

which is the same as α of equation (2). This can be confirmed by referring to equation [1969p] on p. 419. Using this definition, equation (8) is arranged as

$$s' = k'\left\{\left(1-\frac{1}{2}\varepsilon'+\frac{1}{16}\varepsilon'^2\right)(\psi'-\psi)-\frac{3}{4}\varepsilon'(\sin(2\psi')-\sin(2\psi)) + \frac{15}{64}\varepsilon'^2(\sin(4\psi')-\sin(4\psi))\right\}, \text{ [1969}s\text{]} \tag{13}$$

the meridian quadrant S is obtained as follows, by setting $\psi'=\pi/2$ and $\psi=0$:

$$S=k'\frac{1}{2}\pi\left(1-\frac{1}{2}\varepsilon'+\frac{1}{16}\varepsilon'^2\right), \text{ [1969}v\text{]} \tag{14}$$

and the mean length per degree is

$$s=k'\left(1-\frac{1}{2}\varepsilon'+\frac{1}{16}\varepsilon'^2\right). \tag{15}$$

This is the length of a degree at $\frac{\pi}{4}$. Equation (13) is rearranged as

$$s' = s\left\{(\psi'-\psi)-\left(\frac{3}{4}\varepsilon'+\frac{3}{8}\varepsilon'^2\right)(\sin(2\psi')-\sin(2\psi)) + \frac{15}{64}\varepsilon'^2(\sin(4\psi')-\sin(4\psi))\right\}. \tag{16}$$

This is an alternative expression of the second equation. See equation (30) in Appendix 1 of this note.

3.2 Footnote (1484) on p. 465

The mean length s is calculated using terms of order $O(\varepsilon')$ in equation (16). The estimated ellipticity of $\frac{1}{334}$ is used hereafter. The Montjoui-Dunkirk equation is obtained using the measured distance s' in double toise, equation(15), the latitudes of Montjoui (41.36245°) and Dunkirk (51.03625°), and the radian adjustment as follows:

$$\frac{275792.36}{s}=9.6738°-\frac{180°}{\pi}\left(\frac{3}{4}\frac{1}{334}\right)[\sin(102.0725°)-\sin(82.7249°)]. \text{ [2022]} \tag{17}$$

This follows from equation (16), neglecting terms of order $O(\varepsilon'^2)$ and higher (This is the same equation as the lastequation in [2024] on p. 457, with $x^{(5)}=x^{(1)}=0$.

Bowditch did not use equation (11).). By solving this equation, the mean length s of a degree is obtained as 28503.88392. After multiplying by a factor of 90, the meridian quadrant is obtained as 2565349.553. This differs a little from Laplace's number of 2565370, and Bowditch continued the calculation up to terms of order $O(\varepsilon^2)$ by using (confusingly) equation (11).

3.3 Calculation up to terms of order $O(\varepsilon^2)$

Equation (11) is expressed as

$$s'=s\left\{(\psi'-\psi)-\frac{3}{4}\frac{180}{\pi}\varepsilon\,(\sin(2\psi')-\sin(2\psi))+\frac{3}{8}A\frac{180}{\pi}\varepsilon^2\right\}, \tag{18}$$

$$A=\sin 2\theta^{(5)}-\sin 2\theta^{(1)}+\frac{5}{8}\sin 4\theta^{(5)}-\frac{5}{8}\sin 4\theta^{(1)}.\ [2034\text{f}]$$

Setting $\psi'=\theta^{(5)}(51.03625°)$, $\psi=\theta^{(1)}(41.36245°)$, and the Monjoui-Dunkirk distance in double toise, we obtain

$$\frac{275792.36}{s}=9.675525339.\ [2034g] \tag{19}$$

The mean length of a degree s is 28504.12255, and the quadrant S is 2565371.0, [2034h] which is close to the Laplace value of 2565370 double toises. The length of a meter, which is given by equation (1), follows from this quadrant.

The ellipticity ε is also used in the footnote (1485) on p. 466 to derive the polar radius and equatorial radius. Using another notation for the ellipticity, $\rho=\frac{1}{334}$, and equation (9), the polar radius is

$$k=2565370/\left(\frac{\pi}{2}\right)/\left(1+\frac{1}{2}\rho+\frac{1}{16}\rho^2\right)=1630723.149\ [2035\text{a}] \tag{20}$$

double toises, 3261446 toises, or 6356678.175 m. The oblateness is

$$\frac{1}{334}k=\frac{1}{334}6356677=19031.96707.\ [2035\text{b}] \tag{21}$$

From the sum of the oblateness and k, the equatorial radius k' is calculated ass 6375709 m. Since k' is calculated as $k'=k+\frac{1}{334}k$ in this footnote, $\frac{1}{334}=\frac{k'-k}{k}$ is confirmed.

3.4 Calculation using ε' up to terms of order $O(\varepsilon'^2)$

The same sequence of steps as above can be applied to equation (16), where the ellipticity is ε', and s is defined by equation (15). Substituting the Montjoui and Dunkirk latitudes into equation (16), $s=28504.10996$, which is closer than the value obtained by Bowditch to the Laplace value. Laplace's calculation could be in terms of equation (16) and the ellipticity ε'. The meridian quadrant is 2565369.896, where the Laplace value is 2565370 double toises. The length of the meter is the same as that given in equation (1). The length of the semimajor axis k' is, by equation (14), 1635612.871 double toises, 3271225.742 toises, and 6375738.669 m. The oblateness is 19089.03793, and the length of the semiminor axis is 6356649.631 m. These values are around 30 m longer than the Bowditch values.

4 Conclusion

The calculations carried out by Laplace (1829–1839) to determine the length of the meter are summarized in this note. Laplace used a function for an arc length between two parallels of latitude to calculate the ellipticity of the earth by using the French and Peruvian data sets. The ellipticity is found to be 1/ 334. He also used another function for an arc length between two parallels of latitude to determine the length of the meridian quadrant by using the French data set and substituting the ellipticity value of 1/ 334. Therefore, the ellipticity and the meridian quadrant were determined from the two functions as two unknowns. Neither the method of least squares nor any form of statistical estimation is used.

Laplace's study is viewed in Appendix 1 as a calibration of the ellipticity and the meridian quadrant in the asymptotic expansion of the elliptic integral (Bessel (1837)). The least squares results obtained by Legendre (1805), Gauss(1799), and Stigler (1881) are summarized in Appendix 2. The data set is tabulated in Appendix 3.

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Appendix 1. Elliptic integral derived by Bessel and equations derived by Laplace

The elliptic integral for the arc length s' between two parallels of latitude ψ' and ψ is

$$s' = k'(1-e^2)\int_{\psi}^{\psi'} \sqrt{1-e^2\sin^2\theta}\,d\theta \tag{22}$$

by Bessel [2], where k' is the length of the semimajor axis, and $e=\sqrt{1-(k/k')^2}$ is the eccentricity. I use ε' of equation (12) for the ellipticity; then, $\varepsilon' = 1-\sqrt{1-e^2}$. This integral is developed as a sum of trigonometric functions with coefficients that are reproduced in Appendix of Stigler (1981). I approximate this equation as well as the coefficients of the order of ε' and derive the basic equations (3) and (16). These two basic equations in Laplace are found to be an expansion of the integral in (22).

First, two coefficients are approximated as

$$n = \frac{k'-k}{k'+k} \simeq \frac{\varepsilon'}{2} + \frac{\varepsilon'^2}{4} + O(\varepsilon'^3), \tag{23}$$

$$N = 1 + \frac{9}{4}n^2 + \cdots \simeq 1 + \frac{9}{16}\varepsilon'^2 + O(\varepsilon'^3)\ ; \tag{24}$$

then,

$$\alpha \simeq N^{-1}\left\{\frac{3}{2}n + O(\varepsilon'^3)\right\} = \frac{3}{4}\varepsilon' + \frac{3}{8}\varepsilon'^2 + O(\varepsilon'^3), \tag{25}$$

$$\alpha' \simeq N^{-1}\left\{\frac{15}{8}n^2 + O(\varepsilon'^3)\right\} = \frac{15}{32}\varepsilon'^2 + O(\varepsilon'^3), \tag{26}$$

and

$$\alpha'' \simeq N^{-1}\left\{\frac{105}{48}n^3 + O(\varepsilon'^4)\right\} = 0 + O(\varepsilon'^3). \tag{27}$$

The integral is approximated as

$$\begin{aligned} s' &= s\left\{(\psi'-\psi) - 2\frac{180}{\pi}\alpha\sin(\psi'-\psi)\cos(\psi'+\psi)\right. \\ &\quad \left. + \frac{180}{\pi}\alpha'\sin(2\psi'-2\psi)\cos(2\psi'+2\psi) + O(\varepsilon'^3)\right\} \qquad (28) \\ &\quad - \frac{2}{3}\frac{180}{\pi}\alpha''\sin(3\psi'-3\psi)\cos(3\psi'+3\psi) + \cdots \\ &= s\left\{(\psi'-\psi) - 2\frac{180}{\pi}\left(\frac{3}{4}\varepsilon' + \frac{3}{8}\varepsilon'^2\right)\sin(\psi'-\psi)\cos(\psi'+\psi)\right. \\ &\quad \left. + 2\frac{180}{\pi}\frac{15}{64}\varepsilon'^2\sin(2\psi'-2\psi)\cos(2\psi'+2\psi)\right\} + O(\varepsilon'^3) \qquad (29) \\ &= s\left\{(\psi'-\psi) - \frac{180}{\pi}\left(\frac{3}{4}\varepsilon' + \frac{3}{8}\varepsilon'^2\right)(\sin 2\psi' - \sin 2\psi)\right. \\ &\quad \left. + \frac{180}{\pi}\frac{15}{64}\varepsilon'^2(\sin 4\psi' - \sin 4\psi) + O(\varepsilon'^3)\right\}, \qquad (30) \end{aligned}$$

which is equation (16). Equation (3) follows by applying the Taylor expansion $\frac{180}{\pi}\sin(\psi'-\psi) \simeq (\psi'-\psi)$ in equation (28) and approximating it by terms of order $O(\varepsilon')$:

$$\begin{aligned} s' &= s\left\{(\psi'-\psi) - \frac{180}{\pi}\frac{3}{2}\varepsilon'\sin(\psi'-\psi)\cos(\psi'+\psi)\right\} + O(\varepsilon'^2) \qquad (31) \\ &= k'\left(1 - \frac{1}{2}\varepsilon'\right)(\psi'-\psi)\left\{1 - \frac{3}{2}\varepsilon'\cos(\psi'+\psi)\right\} + O(\varepsilon'^2). \qquad (32) \end{aligned}$$

This is equation (3), but the length of the semimajor axis k' is in the equation.

Both basic equations (3) and (16) in the Laplace calculation are derived from equation (29). Therefore, his calculations can be viewed as the process of solving s and ε' from the original equation (29). Neglecting terms of order $O(\varepsilon'^3)$, equation (29) is

$$88448.70 = s\left\{3.11697 - 2\frac{180}{\pi}\left(\frac{3}{4}\varepsilon' + \frac{3}{8}\varepsilon'^2\right)\sin(3.11697^\circ) + \frac{15}{32}\frac{180}{\pi}\varepsilon'^2\sin(6.23394^\circ)\right\} \quad (33)$$

for Peru since $\phi' + \phi = 0$, and is

$$275792.36 = s\left\{9.67380 - 2\frac{180}{\pi}\left(\frac{3}{4}\varepsilon' + \frac{3}{8}\varepsilon'^2\right)\sin(9.67380^\circ)\cos(92.39870^\circ) + \frac{15}{32}\frac{180}{\pi}\varepsilon'^2\sin(19.34759^\circ)\cos(184.79740^\circ)\right\} \quad (34)$$

for France. The ratio of these two quadratic equations gives ε' of 1.46011211 and $2.987422 \times 10^{-3} = 1/334.73677$. Substituting the latter value into the Peruvian equation, the meridian quadrant obtained is 2565370.862. See result (1). Laplace chose the French equation for calculating s, but the results obtained using the Peruvian equation were the same since the Peruvian and French equations form a two-equation system with two unknowns (Laplace did not use measurements obtained at the Cape of Good Hope, Pennsylvania, Italy, Austria, or Lapland 1829–1839, p. 444, 1799, p. 158.).

Appendix 2. Estimation by the method of the least squares

A regression equation follows from equation (31). Using terms of order $O(\varepsilon')$, equation (31) is transformed as

$$\frac{s'}{\phi' - \phi} = s - (s\varepsilon')\frac{3}{2}\frac{180}{\pi}\frac{\sin(\phi' - \phi)\cos(\phi' + \phi)}{\phi' - \phi} \quad (35)$$

$$\simeq s - (s\varepsilon')\frac{3}{2}\cos(\phi' + \phi) \quad (36)$$

$$\simeq s\left(1 - \frac{3}{2}\varepsilon'\right) + (s\varepsilon')3\sin^2\frac{\phi' + \phi}{2} \quad (37)$$

Stigler (1981) used equation (37) and the French data set and obtained an S of 2564801.46 and ε' of 1/ 157.95. He estimated the most likely variations of this equation, including weighted least squares.

Least squares estimation of equation (35) gives $s = 28497.795$ and $\varepsilon' = 1/157.918$, and the meridian quadrant S *is* 2564801.564; these values are approximately the same as the Stigler values. Adding the Peruvian measurement to this estimation, S is 2565397.694 and ε' is 1/ 320.738.

If the equation is expressed as

$$\phi' - \phi = \frac{s'}{s} + \varepsilon'\frac{3}{2}\frac{180}{\pi}\sin(\phi' - \phi)\cos(\phi' + \phi),$$

and the least squares estimation is applied, S is 2564771.04 and ε' is 1/ 150.675. Lagrange (1805, p. 78) applied this form to the French data set and obtained an S of 2564800.2 and ε' of 1/ 148. The values obtained by Gauss (1799) are 2565006 and 1/ 187, respectively, which are very different from others. See also Harter (1974, pp. 153–154) for other values of ellipticity.

I divide the whole French interval into two sub-intervals: one is the Dunkirk-Pantheon interval located north of Paris, and the other is the Pantheon-Montjoui interval located south of Paris. Equation (35) is

$$\frac{62472.59}{2.1891}=s-(s\varepsilon')\frac{3}{2}\frac{180}{\pi}\frac{\sin(2.1891^\circ)\cos(49.9417^\circ\times 2)}{2.1891}$$

for the Dunkirk-Pantheon interval, and

$$\frac{213319.77}{7.4847}=s-(s\varepsilon')\frac{3}{2}\frac{180}{\pi}\frac{\sin(7.4847^\circ)}{7.4847}\cos(45.105^\circ\times 2)$$

for the Pantheon-Montjoui interval. Then, ε' is solved as 1/ 192.8, and s is 28499.98. The meridian quadrant S is 2564998.0. I tried other combinations of sub-intervals as well as omitting intervals such as Pantheon-Evaux, but, I could not obtain results any closer to those of Gauss.

Appendix 3. Data set

The Laplace data set (1829–1839, p. 443 and p. 453) or equivalently, table 1.7 of Stigler (1986, p. 59) is shown here. s' and ν are the distance and an angle between two towns, respectively. The measurements of the last two rows were obtained from Laplace. Legendre (1805) used four pairs, namely, D–P, P–E, E–C, and C–M, when he applied the method of least squares. The unit of distance is double toise, which is approximately 3.89894 m.

town	*latitude* (ϕ)	*arc length*	*difference*	*mean latitude*
Dunkirk	51.03625°	s'	$\nu=\phi'-\phi$	$L=(\phi'+\phi)/2$
Pantheon	48.84715°	62472.59	2.18910°	49.94170°
Evaux	46.17847°	76145.74	2.66868°	47.51281°
Carcassonne	43.21511°	84424.55	2.96336°	44.69679°
Montjoui	41.36244°	52749.48	1.85266°	42.28878°
French Total		275793.15	9.67380°	46.19935°
Peru		88448.70	3.11697°	0°