Quantum anomaly and effective field description of a quantum chaotic billiard

Institute of Physics, University of Tsukuba　Nobuhiko Taniguchi [1]

カオス的量子ビリヤード系の有効場理論を等エネルギー面上の場の理論の観点から考察する。準位の離散性を量子異常の観点から解釈することにより、拡散・バリスティック領域の有効場理論として Kac-Moody 代数を導出でき、それが量子カオス系の Wigner-Dyson 相関に相応していることを示す。

Introduction

In this work, we investigate the effective field description of a quantum chaotic billiard from a novel perspective — quantum anomalies in spectra. The relevance of the anomaly to a quantum billiard is most easily understood by noting different spectral structures between classical and quantum theories. Whereas classical dynamics has continuous symmetry along the energy, *e.g.*, by changing the momentum continuously without altering the orbit in space, such continuous symmetry is absent quantum mechanically since discrete energy levels are formed. By examining the algebraic structure,on the energy shell, we will show the presence of anomalous part, *i.e.*, the Schwinger-type term. Its presence enables one to construct effective fields as phase variables *without* any additional coarse-graining nor ensemble averaging, while the spectral Husimi function acts as the amplitude degree of freedom. Some technical details are reported elsewhere [1].

Effective field theories of quantum dots Effective field theories of quantum dots by the supermatrix nonlinear-sigma (NL-σ) model have been successful in describing disordered metals with *diffusive* dynamics [2]. Not only describing quantum interference phenomena, it has provided a *direct proof* of the BGS conjecture [3] stating that the level correlations of quantum chaotic systems in general obey the Wigner-Dyson statistics. By seeing the success of effective field description in diffusive systems, the same zero-dimensional model has been anticipated in a *ballistic* chaotic system and a great effort has been put forth to extend the framework toward it [4].

[1] E-mail: taniguch@cm.ph.tsukuba.ac.jp

The validity of these ballistic NL-σ models, however, is not so transparent unlike the diffusive counterpart. Soon after initial derivations, it has been recognized that such a theory has some (unphysical) zero mode vertical to the energy shell and nothing suppresses those fluctuations [4]. As a result, how to attain the necessary "mode-locking" has been questioned. Here we will find an explanation by the anomaly carried by each level.

Sepctral QFT (QFT with the energy coordinate)

Our purpose is to derive the classical-quantum correspondence for the time scale longer than the Ehrenfest time. We are particularly concerned with "classicalization" of the quantum theory in a quantum chaotic billiard. We will show that the existence of the Wigner-Dyson correlator is closely related to the generic symmetry on the energy shell.

Let us consider a *generic* quantum chaotic billiard, by which we mean no special symmetry in spectra. We begin with the full quantum theory, $\mathcal{H}\phi_\alpha(\boldsymbol{r}) = \varepsilon_\alpha\phi_\alpha(\boldsymbol{r})$ with eigen energies ε_α and eigen functions $\phi_\alpha(\boldsymbol{r})$. When we attach independent fermionic creation/annihilation operators $\psi_\alpha^\dagger$ and ψ_α to each level, the field operator on a certain energy shell ε is defined by $\psi(\boldsymbol{r}t) = \int_{-\infty}^{\infty} \psi(\boldsymbol{r}\varepsilon)\, e^{-i\varepsilon t/\hbar} d\varepsilon$ and $\psi(\boldsymbol{r}\varepsilon)$ satisfies

$$\left\{\psi(\boldsymbol{r}_1\varepsilon_1), \psi^\dagger(\boldsymbol{r}_2\varepsilon_2)\right\} = \delta(\varepsilon_1 - \varepsilon_2)\, \langle \boldsymbol{r}_1 |\, \delta(\varepsilon_2 - \mathcal{H})\, | \boldsymbol{r}_2 \rangle\,. \tag{1}$$

Interestingly the system may rather be viewed as the one-dimensional system along the "ε-axis" with some internal degrees of freedom attached.

The field $\psi(\boldsymbol{r}\varepsilon)$ is subtle. It exists only at ε coinciding with one of eigen energy levels, when it diverges due to discrete spectrum. This intrinsic divergence must be regularized usually by introducing the Dirac delta function with some broadening η. However, the prescription is incomplete for composite fields such as currents. Correctly, we need to define the current operator by using the normal-ordering :: as $j(\boldsymbol{r}, \boldsymbol{r}'; \varepsilon) = :\psi^\dagger(\boldsymbol{r}'\varepsilon)\psi(\boldsymbol{r}\varepsilon):$. An appropriate definition of :: is given by decomposing ψ into $(\pm)$ parts by

$$\psi_\pm(\boldsymbol{r}\varepsilon) = \frac{\pm i}{2\pi}\int_{-\infty}^{\infty} \frac{\psi(\boldsymbol{r}\varepsilon')}{\varepsilon - \varepsilon' \pm i\eta}\, d\varepsilon'; \qquad \psi_+|0\rangle = \psi_-^\dagger|0\rangle = 0. \tag{2}$$

$$\{\psi_\pm(\boldsymbol{r}_1\varepsilon_1), \psi_\pm^\dagger(\boldsymbol{r}_2\varepsilon_2)\} = \frac{(\pm i/2\pi)}{\varepsilon_1 - \varepsilon_2 \pm i\eta}\, \langle \boldsymbol{r}_1 | \delta(\varepsilon_2 - \mathcal{H}) | \boldsymbol{r}_2 \rangle. \tag{3}$$

Noting $2\pi\delta'(z) = (z - i\eta)^{-2} - (z + i\eta)^{-2}$, we readily evaluate the current commutator as

$$\begin{aligned}[j(\boldsymbol{r}_1, \boldsymbol{r}_1'; \varepsilon_1), j(\boldsymbol{r}_2, \boldsymbol{r}_2'; \varepsilon_2)] &= \delta(\varepsilon_1 - \varepsilon_2)\left[\langle \boldsymbol{r}_1|\delta(\varepsilon_2 - \mathcal{H})|\boldsymbol{r}_2'\rangle j(\boldsymbol{r}_1', \boldsymbol{r}_2; \varepsilon_2) - (1 \leftrightarrow 2)\right] \\ &+ \frac{i}{2\pi}\langle \boldsymbol{r}_2|\delta(\varepsilon_1 - \mathcal{H})|\boldsymbol{r}_1'\rangle\langle \boldsymbol{r}_1|\delta(\varepsilon_2 - \mathcal{H})|\boldsymbol{r}_2'\rangle\delta'(\varepsilon_1 - \varepsilon_2),\end{aligned} \tag{4}$$

where we clearly confirm the anomalous contribution as the last term.

Spectral QFT on the phase space Though the above commutator determines the current algebra completely, working on the bilocal operator $j(\boldsymbol{r}, \boldsymbol{r}')$ is unpleasant. A way to circumvent it is to recast j onto the classical phase space. Surprisingly, in spite of prevalent uses of the Wigner representation, we find that the Husimi representation (the wave-packet representation) is needed to identify the exact symmetry of the algebra. The Husimi representation of j is defined as $\langle \boldsymbol{x}|j(\varepsilon)|\boldsymbol{x}\rangle$ (still an operator) by is the coherent state $|\boldsymbol{x}\rangle$ centered at $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})$. The definition can be equally rewritten in terms of the field operator for the wave-packet $\psi(\boldsymbol{x}) = \int d\boldsymbol{r} \langle \boldsymbol{x}|\boldsymbol{r}\rangle \psi(\boldsymbol{r})$ as $j(\boldsymbol{x};\varepsilon) = :\psi^\dagger(\boldsymbol{x};\varepsilon)\psi(\boldsymbol{x};\varepsilon):$. The operator $\psi(\boldsymbol{x};\varepsilon)$ obeys $\{\psi(\boldsymbol{x};\varepsilon_1), \psi^\dagger(\boldsymbol{x};\varepsilon_2)\} = \delta(\varepsilon_1 - \varepsilon_2)H(\boldsymbol{x})$ where $H(\boldsymbol{x};\varepsilon)$ is the spectral Husimi function. Hence we can write Eq. (4) as

$$[j(\boldsymbol{x};\varepsilon_1), j(\boldsymbol{x};\varepsilon_2)] = \frac{i}{2\pi} H(\boldsymbol{x};\varepsilon_1)H(\boldsymbol{x};\varepsilon_2)\delta'(\varepsilon_1 - \varepsilon_2). \tag{5}$$

This reveals that $j(\boldsymbol{x},\varepsilon)/H(\boldsymbol{x},\varepsilon)$ satisfies the Abelian Kac-Moody algebra *exactly* at each $\boldsymbol{x}$. We can complete bosonization by introducing chiral boson fields $\varphi(\boldsymbol{x};\varepsilon)$ by $j(\boldsymbol{x};\varepsilon) = H(\boldsymbol{x};\varepsilon)\,\partial_\varepsilon \varphi(\boldsymbol{x};\varepsilon)$. Note the dual field φ is meaningful only when the "amplitude" $H(\boldsymbol{x};\varepsilon)$ does not vanish. Therefore the mode-locking is fulfilled automatically.

Supersymmetric extension In considering the n-point spectral correlation, it is necessary to evaluate the n-fold ratio of the determinant correlator $\prod_{i=1}^n \det(\varepsilon_{fi} - \mathcal{H})/\det(\varepsilon_{bi} - \mathcal{H})$. As a result, the relevant symmetry is enlarged to the general linear superalgebra $\mathfrak{g} = \mathfrak{gl}(n|n)$ in the simplest (unitary) case. The preceding treatment can be extended straightforwardly by using the superbracket $[\![\cdot,\cdot]\!]$ and the corresponding current operator

$$j_a(\boldsymbol{x},;\varepsilon) = :\psi^\dagger(\boldsymbol{x};\varepsilon)X_a\psi(\boldsymbol{x};\varepsilon): \qquad X_a \in i\mathfrak{g}. \tag{6}$$

One can check the current commutator as before to find that $j_a(\boldsymbol{x};\varepsilon)/H(\boldsymbol{x};\varepsilon)$ satisfies exactly the Kac-Moody algebra of a corresponding Lie superalgebra at each $\boldsymbol{x}$. Hence the effective field theory is described by the (chiral) WZNW model defined by the field $g(\boldsymbol{x};\varepsilon)$ on the corresponding Lie supergroup, and the current can be expressed as

$$j(\boldsymbol{x};\varepsilon) = H(\boldsymbol{x};\varepsilon)\; g^{-1}\partial_\varepsilon g(\boldsymbol{x};\varepsilon). \tag{7}$$

Connection with the standard NL-σ formulation By recognizing the convoluted function $(\varepsilon - \varepsilon' \pm i\eta)^{-1}$ in Eq. (2) is a Fourier transform of the step function, the projection onto $(\pm)$ may be regarded as the decomposition into the retarded (R) and advanced (A) components. We can make the correspondence explicit by writing

$$\psi_+(\boldsymbol{x};\varepsilon) = \begin{pmatrix} b_R \\ f_R \end{pmatrix}; \quad \psi_-(\boldsymbol{x};\varepsilon) = \begin{pmatrix} -b_A^\dagger \\ f_A^\dagger \end{pmatrix}, \tag{8}$$

where $b_{R,A}$ ($f_{R,A}$) are bosonic (fermionic) fields to generate the retarded/advanced Green functions. With the source term proportional to the energy difference, one can construct the color-flavor transformation [5] at each at each x to obtain the supermatrix NL-σ model. The only modification crucial for the mode-locking problem is the presence of the spectral Husimi function instead of the average DOS. In this way, the supermatrix NL-σ model with the exact DOS can be derived in a ballistic system.

Husimi vs. Wigner representations The difference between the Husimi and the Wigner representations is worth mentioning. In the former, the symmetry can be identified *exactly* as the Kac-Moody type. In the latter, a similar algebra can be obtained *approximately* by a semiclassical expansion, which is genrally hard to justify by wild oscillations of the Wigner functions. The necessity of using the Husimi representation enforces one to consider *coarse-grained* semiclassical dynamics (wave-packet dynamics) in evaluating the nonuniversal deviation.

Concluding remark

An extension to the time-reversal symmetric system is also possible, where the relevant symmetry is enlarged to $\mathfrak{osp}(2n|2n)$. Any additional symmetry is accommodated similarly.

The usefulness of the present construction is not limited to quantum chaos and disordered systems. It seems possible to extend the present approach even to interacting electrons, where $U(1)$ symmetry no longer exists to each level, yet we have one global $U(1)$ phase. Hence the bosonization is possible at E_F. Apparently, the present scheme is closely connected with the Luther-Haldane bosonization in arbitrary dimension, but a direct comparison remains to be seen at present.

References

[1] N. Taniguchi, `cond-mat/0407802`.

[2] See, *e.g.*, K. B. Efetov, *Supersymmetry in Disorder and Chaos* (CUP, 1997).

[3] O. Bohigas et al, PRL **52** (1984), 1.

[4] See reviews by Altland et al. and Zirnbauer in *Supersymmetry and Trace Formulae*, NATO ASI **B370** (1999, Kluwer, I. V. Lerner et al. eds.) and references therein.

[5] M. R. Zirnbauer, J. Phys. **A29** (1996), 7113.