Quantum Non-Escape Probability in Open Chaotic System
—Lyapunov Exponent versus Periodic Orbits—

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We calculate semiclassically the quantum non-escape probability in an open quantum system with chaotic classical limit. We shall show that for sufficiently large time, the decay of the non-escape probability at time "t" is described by the exponential decay characterized by classical Lyapunov exponent and the suppression of the first due periodic orbits with period 2t

我々は、古典的にカオティっクな開放量子系での滞在確率を半古典的に計算する。時間が十分経てば、次の一般的な結論を引けることができる：時刻 t での滞在確率は二つの項の競争となる：一つはリヤプノフ指数に特徴付けられている指数的な減少とその減少を抑える役割である周期が 2t の周期軌道和の項である。

1 Introduction

One of the measures of quantum decay in an open quantum system is the time evolution of an initially localized wave packet. In semiclassical regime, one should come to a "naive" guess that, the quantum decay will be a competition between the classical decay characterized by the Lyapunov exponent and the quantum localization due to the periodic orbits. The latter thus suppresses the former. In this paper, we shall elaborate this idea quantitatively, showing that the above intuition is correct.

We start by defining phenomenologically our spatially open quantum system as follows: First, an open space Ω which also defines the interaction region, is connected spatially through absorbing boundary to a large environment. The classical dynamics inside Ω is assumed to be chaotic. One then assumes that only paths that never leave the interaction region will contribute to the path integral representation of the time evolution operator. Thus, instead of a unitary time evolution operator, the presence of the environment forces one to use a non-unitary one. Similar idea has led Mensky[1] to his phenomenological "restricted path integral" model to the decoherence phenomena, without considering the details of the environment. To be more precious, an initial wave packet $|\psi_\Omega(0)\rangle$, which satisfies the normalization condition $\langle\psi_\Omega(0)|\psi_\Omega(0)\rangle = 1$, is put inside Ω. This initial wave packet is then evolved with the non-unitary time evolution operator $\hat{U}^t_\Omega$, which is obtained by ignoring all paths that pass through the absorbing boundary

$$\hat{\rho}_\Omega(t) = \hat{U}^t_\Omega \hat{\rho}(0)[\hat{U}^t_\Omega]^\dagger, \tag{1}$$

where $\rho(0) = |\psi_\Omega(0)\rangle\langle\psi_\Omega(0)|$. The quantum non-escape probability is then defined as the probability to find a particle in the interaction region Ω at time t. It is therefore nothing but the norm of the time-developed wave function:

$$P_\Omega(t) = \| \hat{U}^t_\Omega |\psi_\Omega(0)\rangle \|^2 = Tr[\hat{\rho}_\Omega(t)]. \tag{2}$$

The first semiclassical approximation proceeds as usual by replacing the propagator with its semiclassical version given as the summation over the classical trajectories starting from x and ending at z in time t, which is valid in the limit $\hbar \to 0$

$$\langle z|\hat{U}^t_\Omega|x\rangle \equiv K_\Omega(z,x;t) = \sum_{s\in\Omega}(2\pi i\hbar)^{-\frac{f}{2}}|C_s|^{\frac{1}{2}} \exp\left[\frac{i}{\hbar}R_s(z,x;t) - i\frac{\pi}{2}\mu_s\right], \tag{3}$$

where the summation over s is over trajectories that never leave Ω. After some simple manipulation, the quantum non-escape probability can be written as

$$P_\Omega(t) \approx \frac{1}{2\pi\hbar^f}\int_\Omega dzdy \int dp\, K^*_\Omega(z,x;t)K_\Omega(z,x;t)\rho^W_\Omega(p,x;0). \tag{4}$$

It is clear that the quantum non-escape probability in Ω at time t is given by the double summation of pairs of classical trajectories which start at the same point x, consume the same forward of time t and end at the same point z. In the semiclassical limit, due to the ergodicity of the chaotic system, each term of $\sum_{sk}$ will fluctuate wildly, such that in average most will cancel to each other except pairs of trajectories whose energy are the same $E_s = E_k$. The main contribution for the quantum non-escape probability is therefore given as

$$P_\Omega(t) = \sum_{s,k}\cdots \approx \sum_{E_s=E_k}\cdots. \tag{5}$$

Next, let us notice that there are two cases that satisfy the condition imposed by the summation, *i.e.*, $E_s = E_k$. First is the trivial diagonal case, *i.e.*, when the two classical trajectories are identical $s = k$, as illustrated in Fig. 1 (upper). In this case, all the phases become zero such that one has

$$P_\Omega(t) \sim \int_\Omega dzdx \int dp\, \rho^W_\Omega(p,x;0)\sum_s |C_s(z,x;t)|. \tag{6}$$

Following Jalabert and Patawski[2], notice that if the corresponding classical system is fully chaotic, $|C_s(z,x;t)|$ is independent of z and x, and can be written as $|C_s| \propto \exp[-\lambda t]$, where

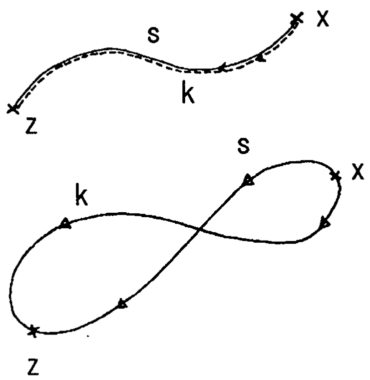

図 1: A pair of identical orbits (upper) and a periodic orbits of period $2t$ as a result of constructive interference of two different classical trajectories s and k belonging to the same energy hyperspace (lower).

λ is the average Lyapunov exponent of the corresponding classical chaotic system. Thus, in this case, keeping in mind the normalization condition for the initial wave packet, the survival probability is given as

$$P_{\Omega}^{Lya}(t) \sim V_{\Omega} \exp\left[-\lambda t\right], \tag{7}$$

where V_{Ω} is the spatial volume of the interaction region Ω. Similar results with different contexts are reported in Refs. [2].

Second, the condition of equal energy $E_s = E_k$ is also satisfied by pairs of different classical trajectories (off-diagonal part) which both comprise a periodic orbit of period $2t$ that passing through the point z and x as illustrated in Fig 1 (lower). Mathematically, the condition can be reduced as follows

$$p_s^{init}(x) = -p_k^{init}(x), \quad p_s^{fin}(z) = -p_k^{fin}(z). \tag{8}$$

In this case, the points z and x give partition to the periodic orbit such that, the trajectories s and k consume the same amount of time t. The periodic orbit thus emerges as the result of constructive interference between two specific different trajectories belonging to the same energy hyperspace. The contribution from these cases can therefore be written as

$$P_{\Omega}^{po}(t) \sim \sum_{po\in\Omega} \int_{\Omega} dzdx \int dp\, D_{po}(z,x;2t)\rho_{\Omega}^{W}(p,x;0)\cos\left(\frac{S_{po}(z,x;2t)}{\hbar} + \mu_{po}\frac{\pi}{2}\right), \tag{9}$$

where the amplitude is $D_{po}(z,x;2t) \equiv |C_s(z,x;t)|^{1/2}|C_k(z,x;t)|^{1/2}$, and the nontrivial phase is given by the classical action along the periodic orbit

$$S_{po}(z,x;2t) = \int_{x;s}^{z} dq\, p(q) + \int_{z;-k}^{x} dq\, p(q) = \oint_{po(x,z;2t)} dq\, p(q). \tag{10}$$

It is clear that the periodic orbits to be summed up are the ones that are trapped inside the interaction region Ω during their revolution. The summation spans over all the energy band of the initial wave packet and each periodic orbit is weighted by the probability density of the initial wave packet. This shows the initial condition dependence of the quantum decay[3]. Finally, writing all together, for $t > 0$, one obtains

$$P_{\Omega}(t) \sim \Omega \exp\left[-\lambda t\right] + \sum_{po \subset \Omega} \cdots . \tag{11}$$

The interpretation of the above formula is straightforward. The first term on the right hand side describes the classical diffusive decay due to the chaoticity of the corresponding classical system. This term will be dominant for small time. In other words, for small time, the quantum decay follows its classical counterpart. However if there are periodic orbits of period $2t$ that are trapped inside the interaction region Ω, the particles *remember* the way to go back to its original. This last picture appears in the second term on the right hand side, which suppress the exponential decay of the first. For small time, these periodic orbits will appear as a nontrivial fluctuation superimposed on the exponential decay. For large time, however, since for fully chaotic system the number of the periodic orbits grows exponentially (characterized by the topological entropy of the system), the fluctuations will cancel to each other, yet their amplitudes give significant suppression to the first term to induce an anomalous slow decay.

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